AVERAGING PROJECTIONS

BY

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Introduction

An averaging operator was defined by G. Birkhoff to be a linear operator L on a Banach algebra A satisfying the condition that $L(fLg) = Lf \cdot Lg$ for f and g in A. In this paper it is shown that if A is a subalgebra of $C(X)$, any projection of norm one onto a subalgebra of A is an averaging operator. This theorem was first established by J. L. Kelley under the additional assumptions that the projection was positive and A was the algebra of all real continuous functions that vanish on some point of X. The main tool used in our proof is a generalization of a theorem due to G. Seever. We replace a hypothesis in Professor Seever's theorem that a projection be positive by a weaker condition on the range of the projection.

Notation and definitions

Let X denote a compact Hausdorff space. We denote the Banach space of all real-valued continuous functions on X, topologized with the sup norm by $C_r(X)$.[2] If B is a linear subspace of $C(X)$, we can define an equivalence relation on X by saying that two points x and y in X are equivalent if $f(x) = f(y)$ for every f in B. The partition of X into closed subsets corresponding to this equivalence relation will be called the *B-partition* of X. The dual of a normed linear space P will be written P^*. If F is in $C(X)^*$, supp (F) will denote the support of the unique Radon measure on X which represents F. For x in X, $\hat{x}$ is the linear functional defined on $C(X)$ by $\hat{x}(f) = f(x)$ for all f in $C(X)$. Such a linear functional is termed a point evaluation functional. If P is a normed linear space, $S(P)$ is the set of all points in P with norm less than or equal to one. If K is a convex subset of a normed linear space, ext K is the set of extreme points of K. The composite of two functions f and g is written $f \circ g$. A projection L on a normed linear space is an idempotent ($L \circ L = L$) linear mapping of the space into itself. If f is a function defined on a set X, the restriction of f to a subset Y of X will be written f_Y.

Let P be a linear subspace of $C(X)$. We say that P has a *weakly separating quotient* if for every two distinct points x and y in X and for each scalar $t \neq 1$ such that $\hat{x}_P = t\hat{y}_p$ we we have that $\hat{x}_P$ is not in the set ext $S(P^*)$. In particular the range, P, of a positive projection, L, defined on a sublinear-lattice of $C_r(X)$ has a weakly separating quotient. For suppose x is a point in X such that $\hat{x}_P$ is in ext $S(P^*)$. Then there is a function p in P such that $p(x) \neq 0$.

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[2] When we do not wish to distinguish the real from the complex case, we write $C(X)$.

Since $p = p \vee 0 + p \wedge 0$, either $L(p \vee 0)(x) \neq 0$ or $L(p \wedge 0)(x) \neq 0$. If $L(p \vee 0)(x) \neq 0$ then since L is a positive operator, there is no y in X such that $\hat{y}_P = \hat{x}_P$. The analogous argument is valid for the case in which $L(p \wedge 0)(x) \neq 0$.

Main results

The following theorem was proved by G. Seever [6] in the case where A is the subalgebra of $C_r(X)$ of all functions which vanish at a given point, and L is a positive projection of norm one. See also S. P. Lloyd [4, Theorem 2].

Let A denote a subalgebra of $C(X)$, and let B be a linear subspace of A which has a weakly separating quotient.

1. THEOREM. *If L is a projection of A onto B which has norm one, then $L(fLg) = L(Lf \cdot Lg)$ for all f and g in A.*

Proof. Let K denote the B-partition of X, and let q represent the quotient mapping of X onto X/K.

Let H denote the subalgebra of $C(X)$ of all functions which are constant on the members of K. The map, J, which carries each function f in $C(X/K)$ onto $f \circ q$ is an isometric linear isomorphism of $C(X/K)$ onto H.

Every extreme point p of $S(B^*)$ has an extension to a point p' of $\operatorname{ext} S(C(X)^*)$ i.e. p' restricted to B agrees with p. Clearly then the set

$$Q = \{x \text{ in } X : \hat{x}_B \text{ is in } \operatorname{ext} S(B^*)\}$$

is nonempty, and every extreme point of $S(B^*)$ or its multiple is contained in $\{\hat{x}_B : x \text{ in } Q\}$. We continue the proof by establishing the following lemma.

2. LEMMA. *If x is in Q and F is any linear functional in $C(X)^*$ of norm one that agrees with $\hat{x}$ on B, then F is positive and* supp (F) *is contained in $q^{-1}q(x)$.*

Proof of lemma. We show first that for such a functional F, $F_H = \hat{x}_H$.

Let $B' = J(B)$. Since J is an isometric isomorphism of H onto $C(X/K)$, $\hat{x} \circ J_{B'}^{-1}$ is an extreme point of $S(B'^*)$. It is also clear from the definition of J and the assumptions on F that the linear functionals $F \circ J^{-1}$, $\hat{x} \circ J^{-1}$, and $(q(x))^\wedge$ all agree on B'. Since B' separates the points of X/K no other point evaluation functional $\hat{y}$, (y in X/K) can agree with $(q(x))^\wedge$ on B'. Moreover since B has a weakly separating quotient, there does not exist a point y in X/K such that $t\hat{y}$ agrees with $(q(x))^\wedge$ on B' for any t with $|t| = 1$.

Since $(q(x))^\wedge$ is the only extreme point of $S(C(X/K)^*)$ that agrees with $F \circ J^{-1}$ on B', it in fact is the only member of $S(C(X/K)^*)$ that agrees with $F \circ J^{-1}$ on B'. For if

$$D = \{M \text{ in } S(C(X/K)^*) : M_{B'} = F \circ J_{B'}^{-1}\}$$

then, with the weak topology induced by $C(X/K)$, D is a compact convex set. It therefore is the closed convex hull of its extreme points. Now every extreme point of D is an extreme point, d, of $S(C(X/K)^*)$. For if p and q are in $S(C(X/K)^*)$ and $\frac{1}{2}p + \frac{1}{2}q = d$, then since $d_{B'}$ is an extreme point of $S(B'^*)$ both p and q agree with d on B'; thus p and q are in D. Since d is an extreme point of D, $p = q = d$. Since $(q(x))^\wedge$ is the only extreme point of D it must be the only member of D. Finally since the norm of $F \circ J^{-1}$ is one, $F \circ J^{-1}$ agrees with $(q(x))^\wedge$ on all of $C(X/K)$.

Hence for each h in H, $F(h)$ is the constant value that h assumes on $q^{-1}q(x)$. Since the constant function 1 is in H, the last statement implies that $F(1) = 1$. Since F has norm one this implies that F is positive.

It remains to show that $\operatorname{supp}(F)$ is contained in $q^{-1}q(x)$. Suppose that y is a point of X which is not contained in $q^{-1}q(x)$. To show that y is not in the support of F it will suffice to exhibit a nonnegative continuous function h such that $F(h) = 0$, and $h(y) = 1$. Let k be a function in $C(X/K)$ that is nonnegative, vanishes at $q(x)$ and is one at $q(y)$. Then $h = k \circ q$ is the desired function.

This completes the proof of the lemma.

Proof of theorem continued. It follows from the lemma that for x in Q, $\hat{x} \circ L$ is the restriction to A of a positive linear functional whose support is contained in $q^{-1}q(x)$. From this we observe that if f is in A and f is constant on $q^{-1}q(x)$ (where x is in Q), then Lf attains precisely the same constant value on this set.

Now consider $(\hat{x} \circ L)(fLg)$ for x in Q. Since Lg is constant on $q^{-1}q(x)$, $(\hat{x} \circ L)(fLg) = Lf(x) \cdot Lg(x)$. Thus for each x in Q, $L(fLg)$ agrees with $LfLg$ on $q^{-1}q(x)$, and $LfLg$ agrees with $L(LfLg)$ on each of these sets. This however implies that $L(fLg)$ agrees with $L(LfLg)$ on each extreme point of $S(B^*)$. Hence they must be equal. Our proof is completed.

3. Corollary. *A projection, L, of norm one from a subalgebra of $C(X)$ onto a further subalgebra, B, is an averaging operator.*

Proof. Since the square of every function in B is also in B, B has a weakly separating quotient. Since B is an algebra, $L(LfLg) = LfLg$. Hence L is an averaging operator.

4. Corollary. *A projection, L, of norm one from a subalgebra of $C_r(X)$ onto a further subalgebra, B, is a positive projection.*

Proof. Since B is a subalgebra of $C_r(X)$ it must in fact be dense in the subalgebra of $C_r(X)$ of all functions which are constant on each member of the B-partition of X, and which vanish on the common zeros (if any) of members of B. It follows that for each x in X, the restriction of x to B is either an extreme point of $S(B^*)$ or the zero functional. By the lemma $\hat{x} \circ L$ is either positive or the zero functional. Since this is true for every x in X, L itself must be positive.

Remarks. In the theorem and corollaries we assumed that the projection L was defined on a subalgebra A of $C(X)$. In Theorem 1 and Corollary 3 this was done to guarantee that if f and g were in A, then also fLf and $LfLg$ were in A. In Corollary 4 it would suffice to assume that A were a linear subspace of $C(X)$ which contained the subalgebra of $C(X)$, B.

The assumption in Theorem 1 that B has a weakly separating quotient can not arbitrarily be dropped.

5. *Example.* Let $A = C[-1, 1]$, let $f(x) = x$ and let $B = \{rf : r \text{ a real number}\}$. For g in $C[-1, 2]$ let $Lg = \frac{1}{2}(g(1) - g(-1))f$. Now L is a projection of norm one of $C[-1, 1]$ onto B. However if H represents the constant function 1,

$$L(hLf) = f \neq 0 = L(Lh \cdot Lf).$$

Now suppose $A = C_r(X)$. The range of any averaging operator must be an algebra. If in addition the operator is a projection of norm one, Corollary 4 implies that the operator is positive. It is also known that a positive averaging projection has norm one [3, Remark 2.3, p. 219]. Hence a projection onto a subalgebra of $C_r(X)$ is a positive averaging operator if and only if it has norm one. It is not true however that every positive projection onto a subalgebra of $C_r(X)$ is an averaging operator.

6. *Example.* Let $X = [0, 1] \cup \{2\}$ have the topology it inherits from the real line. Let B be the subalgebra of $C_r(X)$ of all functions that vanish at the point 2. Let h be the function which is identically one on $[0, 1]$, and vanishes at 2. For f in $C_r(X)$ let $Lf = f \cdot h + f(2)h$.

Clearly L is a positive projection onto B. However if g is the function which takes the constant value one on X, then $L(gLg) = 2h$, but $LgLg = 4h$.

A functional representation has recently been given [10] for the spaces which are the range of a norm one projection on $C(X)$. Also some related results on averaging operators have been announced by Dhombres [19].

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