SERIES EXPANSIONS AND INTEGRAL REPRESENTATIONS OF GENERALIZED TEMPERATURES

BY

DEBORAH TEPPER HAIMO[1]

1. Introduction

The generalized heat equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\nu}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \tag{1.1}$$

ν a fixed positive number. The fundamental solution of (1.1) is the function $G(x; t) = G(x, 0; t)$, where

$$\begin{aligned} G(x, y; t) &= \int_0^\infty e^{-u^2 t} \mathcal{J}(xu)\mathcal{J}(yu)\, d\mu(u), \quad t > 0, \\ &= \left(\frac{1}{2t}\right)^{\nu+1/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \mathcal{I}\left(\frac{xy}{2t}\right), \end{aligned} \tag{1.2}$$

with

$$d\mu(u) = \frac{2^{1/2-\nu}}{\Gamma(\nu + \frac{1}{2})} u^{2\nu}\, du,$$

$$\mathcal{J}(z) = 2^{\nu-1/2}\Gamma(\nu + \tfrac{1}{2}) z^{1/2-\nu} J_{\nu-1/2}(z), \quad \mathcal{I}(z) = 2^{\nu-1/2}\Gamma(\nu + \tfrac{1}{2}) z^{1/2-\nu} I_{\nu-1/2}(z),$$

$J_\alpha(z)$ being the ordinary Bessel function of order α and $I_\alpha(z)$ the Bessel function of imaginary argument. It is well known (see [5]) that if $u(x, t)$ is a solution of (1.1), so is its Appell transform $u^A(x, t)$ defined by

$$u^A(x, t) = G(x; t)u(x/t, -1/t). \tag{1.3}$$

The Poisson-Hankel transform of a function φ is given by

$$\int_0^\infty G(x, y; t)\varphi(y)\, d\mu(y), \qquad t > 0, \tag{1.4}$$

whenever the integral exists. Taking $\varphi(x) = x^\gamma$, we set

$$S_{\gamma,\nu}(x, t) = \int_0^\infty y^\gamma G(x, y; t)\, d\mu(y), \qquad \gamma > -2\nu. \tag{1.5}$$

$S_{\gamma,\nu}(x, t)$ satisfies equation (1.1), and in particular, if $\gamma = 2n$, $S_{\gamma,\nu}(x, t)$ is the generalized heat polynomial $P_{n,\nu}(x, t)$ studied in [5]. In that paper, those solutions of (1.1) were characterized which have representations in series of $P_{n,\nu}(x, t)$ and of their Appell transforms $W_{n,\nu}(x, t)$. It is the present

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goal to consider the functions $Q_{n,\nu}(x, t) = S_{2n+1,\nu}(x, t)$ and their Appell transforms $V_{n,\nu}(x, t)$, and to derive conditions for expansions of generalized temperatures in series of the forms

$$\sum_{n=0}^{\infty} [a_n P_{n,\nu}(x, t) + c_n Q_{n,\nu}(x, t)] \quad \text{and} \quad \sum_{n=0}^{\infty} [b_n W_{n,\nu}(x, t) + d_n V_{n,\nu}(x, t)].$$

2. Definitions and preliminary results

By evaluating the integral

$$S_{\gamma,\nu}(x, t) = \int_0^\infty y^\gamma G(x, y; t)\, d\mu(y), \qquad t > 0, \tag{2.1}$$

(see [2, p. 30]), we find that

$$\begin{aligned} &S_{\gamma,\nu}(x, t) \\ &\quad = 2^\gamma t^{\gamma/2}(\Gamma(\gamma/2 + \nu + \tfrac{1}{2})/\Gamma(\nu + \tfrac{1}{2}))\,{}_1F_1(-\gamma/2; \nu + \tfrac{1}{2}; -x^2/4t). \end{aligned} \tag{2.2}$$

From this, it follows that

$$S_{\gamma,\nu}(x, -t) = (-1)^{\gamma/2} S_{\gamma,\nu}(ix, t), \tag{2.3}$$

and we have

$$S_{\gamma,\nu}(x, -t) = (-1)^{\gamma/2} \int_0^\infty y^\gamma G(ix, y; t)\, d\mu(y), \qquad t > 0. \tag{2.4}$$

Further, the Appell transform $T_{\gamma,\nu}(x, t)$ of $S_{\gamma,\nu}(x, t)$ is given by

$$\begin{aligned} T_{\gamma,\nu}(x, t) &= S^A_{\gamma,\nu}(x, t) \\ &= G(x; t) S_{\gamma,\nu}(x/t, -1/t) \\ &= 2(-1)^{\gamma/2} \int_0^\infty y^\gamma e^{-y^2 t} \mathcal{G}(xy)\, d\mu(y). \end{aligned} \tag{2.5}$$

As a consequence of the elementary inequality

$$x^k e^{-x^2\alpha} \le \left(\frac{k}{2\alpha e}\right)^{k/2}, \qquad k > 0, \tag{2.6}$$

and the identity

$$G(x, y; t) = e^{x^2/4\delta} e^{-y^2/4(t+\delta)} (1 + t/\delta)^{\nu+1/2} G(x(t + \delta)/\delta, y; t(t + \delta)/\delta) \tag{2.7}$$

which holds for $t > 0$ and any $\delta > 0$, we may readily establish the estimates

$$\begin{aligned} &| S_{\gamma,\nu}(x, t) | \le e^{x^2/4\delta}(1 + t/\delta)^{\nu+1/2}[2\gamma(t + \delta)/e]^{\gamma/2}, \\ &\qquad\qquad t > 0, \quad \delta > 0, \quad 0 \le x < \infty, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &| S_{\gamma,\nu}(x, -t) | \le e^{x^2/4\delta} 2^\gamma (\Gamma(\gamma/2 + \nu + \tfrac{1}{2})/\Gamma(\nu + \tfrac{1}{2})) t^{\gamma/2}, \\ &\qquad\qquad t > 0, \quad \delta > 0, \quad 0 \le x < \infty. \end{aligned} \tag{2.9}$$

In addition, from (2.1) and the fact that $\mathcal{G}(x) \ge 1$, we also find that

(2.10) $S_{\gamma,\nu}(x, t) \geq e^{-x^2/4t}2^{\gamma}(\Gamma(\gamma/2 + \nu + \frac{1}{2})/\Gamma(\nu + \frac{1}{2}))t^{\gamma/2}, \quad t > 0.$

A class of entire functions needed in our development is described as follows.

DEFINITION 2.1. An entire function

(2.11) $$f(x) = \sum_{n=0}^{\infty} \alpha_k x^k$$

belongs to class (ρ, r) or has growth (ρ, r) iff

(2.12) $$\limsup_{k\to\infty} k \mid \alpha_k \mid^{\rho/k} \leq e\rho r.$$

3. Regions of convergence

In [5], we proved the following result.

THEOREM 3.1. *If*

(3.1) $$\lim_{n\to\infty} n \mid a_n \mid^{1/n} = e/4\sigma < \infty,$$

then the series

(3.2) $$\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$$

converges absolutely in the strip $\mid t \mid < \sigma$.

Using the inequalities (2.8) and (2.9), with $\gamma = 2\nu + 1$, we may establish in a similar way that the following holds.

THEOREM 3.2. *If*

(3.3) $$\limsup_{n\to\infty} n \mid c_n \mid^{1/n} = e/4\sigma < \infty,$$

then the series

(3.9) $$\sum_{n=0}^{\infty} c_n Q_{n,\nu}(x, t)$$

converges absolutely for $0 < \mid t \mid < \sigma$.

For the series of Appell transforms, this region of convergence is a half plane. Indeed, in [5] we established the following result.

THEOREM 3.3. *If*

(3.5) $$\limsup_{n\to\infty} n \mid b_n \mid^{1/n} = e\sigma/4 < \infty,$$

then the series

(3.6) $$\sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t)$$

converges absolutely for $t > \sigma \geq 0$.

Similarly, we have the corresponding theorem for $V_{n,\nu}(x, t)$, the Appell transform of $Q_{n,\nu}(x, t)$.

THEOREM 3.4. *If*

$$\limsup_{n\to\infty} n \mid d_n \mid^{1/n} = e\sigma/4 < \infty,$$

then the series

$$\sum_{n=0}^{\infty} d_n\, V_{n,\nu}(x, t)$$

converges absolutely for $t > \sigma \geq 0$.

4. Expansions in terms of $P_{n,\nu}(x, t)$ and $Q_{n,\nu}(x, t)$

We now establish our principal expansion theorems.

THEOREM 4.1. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} a_n\, P_{n,\nu}(x, t), \qquad 0 \leq x < \infty, \tag{4.1}$$

the series converging for $|t| < \sigma$ *is that*

$$\begin{aligned} u(x, t) &= \int_0^{\infty} G(ix, y; -t)\varphi(iy)\, d\mu(y), && -\sigma < t < 0, \\ &= \varphi(x), && t = 0 \\ &= \int_0^{\infty} G(x, y; t)\varphi(y)\, d\mu(y), && 0 < t < \sigma \end{aligned} \tag{4.2}$$

where φ *is an even entire function of growth* $(2, 1/4\sigma)$ *and*

$$a_n = \varphi^{(2n)}(0)/(2n)\,!. \tag{4.3}$$

Proof. To prove sufficiency, assume that (4.2) holds with φ as described. Then we have

$$\varphi(y) = \sum_{n=0}^{\infty} \alpha_n\, y^{2n}, \qquad \alpha_n = \varphi^{(2n)}(0)/(2n)\,!, \tag{4.4}$$

and substituting in (4.2), we find that

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} \alpha_n(-1)^n \int_0^{\infty} y^{2n}G(ix, y; -t)\, d\mu(y), && -\sigma < t < 0 \\ &= \sum_{n=0}^{\infty} \alpha_n\, x^{2n}, && t = 0, \\ &= \sum_{n=0}^{\infty} \alpha_n \int_0^{\infty} y^{2n}G(x, y; t)\, d\mu(y), && 0 < t < \sigma, \end{aligned} \tag{4.5}$$

provided termwise integration is valid. By Definitions (2.1) and (2.4) with $\gamma = 2n$ and the fact that $P_{n,\nu}(x, 0) = x^{2n}$, it follows that

$$u(x, t) = \sum_{n=0}^{\infty} \alpha_n\, P_{n,\nu}(x, t), \qquad |t| < \sigma. \tag{4.6}$$

Taking $\alpha_n = a_n$, we establish the result.

The validity of termwise integration in (4.5) is a consequence of the growth behavior of φ, which implies that $\sum_{n=0}^{\infty} |\alpha_n|\, y^{2n}$ also belongs to class $(2, 1/4\sigma)$. This means that for any $\varepsilon > 0$ and some constant K,

$$\sum_{n=0}^{\infty} |\alpha_n|\, y^{2n} < K \exp\,(1/4\sigma + \varepsilon)y^2 \tag{4.7}$$

Hence

$$(4.8)\qquad \begin{aligned} &\int_0^\infty |G(ix, y; -t)|\, d\mu(y) \sum_{n=0}^\infty |\alpha_n|\, y^{2n}, && -\sigma < t < 0, \\ &\int_0^\infty G(x, y; t)\, d\mu(y) \sum_{n=0}^\infty |\alpha_n|\, y^{2n}, && 0 < t < \sigma \end{aligned}$$

are both finite, and sufficiency is proved.

Conversely, if (4.1) holds, then by the definition of $P_{n,\nu}(x, t)$, we have

$$(4.9)\qquad \begin{aligned} u(x, t) &= \int_0^\infty G(ix, y; -t)\, d\mu(y) \sum_{n=0}^\infty (-1)^n a_n y^{2n}, && -\sigma < t < 0, \\ &= \sum a_n x^{2n}, && t = 0 \\ &= \int_0^\infty G(x, y; t)\, d\mu(y) \sum_{n=0}^\infty a_n y^{2n}, && 0 < t < \sigma, \end{aligned}$$

or

$$(4.10)\qquad \begin{aligned} u(x, t) &= \int_0^\infty G(ix, y; -t)\varphi(iy)\, d\mu(y), && -\sigma < t < 0, \\ &= \varphi(x), && t = 0, \\ &= \int_0^\infty G(x, y; t)\varphi(y)\, d\mu(y), && 0 < t < \sigma, \end{aligned}$$

where

$$(4.11)\qquad \varphi(x) = \sum_{n=0}^\infty a_n\, x^{2n},$$

provided that

$$(4.12)\qquad \begin{aligned} &\int_0^\infty |G(ix, y; -t)|\, |\varphi(iy)|\, d\mu(y), && -\sigma < t < 0, \\ &\int_0^\infty G(x, y; t)\, |\varphi(y)|\, d\mu(y), && 0 < t < \sigma, \end{aligned}$$

are finite. But, by the convergence of the series (4.1) for $|t| < \sigma$, we have, in particular, the convergence of the series

$$(4.13)\qquad \sum_{n=0}^\infty a_n\, P_{n,\nu}(0, t) = \sum_{n=0}^\infty a_n\, 2^{2n}(\Gamma(\nu + \tfrac{1}{2} + n)/\Gamma(\nu + \tfrac{1}{2}))t^n$$

for $|t| < \sigma$. Hence $\sigma \le$ the radius of convergence $1/\mu$, where

$$\mu = \limsup_{n\to\infty} 4\, |\Gamma(\nu + \tfrac{1}{2} + n)a_n|^{1/n} = \limsup_{n\to\infty}(4n/e)\, |a_n|^{1/n},$$

or

$$(4.14)\qquad \limsup_{n\to\infty} n\, |a_n|^{1/n} \le e/4\sigma.$$

Hence $\varphi \in (2, 1/4\sigma)$, and the integrals (4.12) are finite. Thus, $u(x, t)$ has the required representation (4.10) and the proof is complete.

We may derive, in an analogous way, the corresponding theorem for series in terms of $Q_{n,\nu}(x, t)$.

THEOREM 4.2. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} c_n Q_{n,\nu}(x, t), \qquad 0 \leq x < \infty,$$

the series converging for $0 < |t| < \sigma$, is that

$$u(x, t) = \int_0^\infty G(ix, y; -t)\varphi(iy)\, d\mu(y), \quad -\sigma < t < 0,$$

$$= \int_0^\infty G(x, y; t)\varphi(y)\, d\mu(y), \qquad 0 < t < \sigma,$$

where φ is an odd entire function of growth $(2, 1/4\sigma)$. Here

$$c_n = \varphi^{(2n+1)}(0)/(2n + 1)\,!.$$

By combining the preceding two theorems, we have our first principal result.

THEOREM 4.3. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} [a_n P_{n,\nu}(x, t) + c_n Q_{n,\nu}(x, t)],$$

the series converging for $0 < |t| < \sigma$, is that

$$u(x, t) = \int_0^\infty G(ix, y; -t)\varphi(iy)\, d\mu(y), \quad -\sigma < t < 0,$$

$$= \int_0^\infty G(x, y; t)\varphi(y)\, d\mu(y), \qquad 0 < t < \sigma,$$

where φ is an entire function of growth $(2, 1/4\sigma)$. Here

$$a_n = \varphi^{(2n)}(0)/(2n)\,!, \qquad c_n = \varphi^{(2n+1)}(0)/(2n + 1)\,!.$$

An example illustrating the theorem is given by

$$u(x, t) = \sum_{n=0}^{\infty} (-1)^n/(2^{2n} n\,!)[P_{n,\nu}(x, t) + Q_{n,\nu}(x, t)]$$

the series converging for $0 < |t| < 1$. The integral representation of $u(x, t)$ is

$$u(x, t) = \int_0^\infty G(ix, y; -t)[G(iy; 1)(1 + iy)]\, d\mu(y), \quad -1 < t < 0,$$

$$= \int_0^\infty G(x, y; t)[G(y; 1)(1 + y)]\, d\mu(y), \qquad 0 < t < 1.$$

Here $\varphi(y) = G(y; 1)(1 + y)$, a function of growth $(2, 1/4)$. Note that the integral for $t > 0$ actually converges in a larger region than predicted by the theorem.

5. Expansions in terms of $W_{n,\nu}(x, t)$ and $V_{n,\nu}(x, t)$

In [5], we proved the following series representation theorem.

THEOREM 5.1. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t), \tag{5.1}$$

the series converging for $0 \leq \sigma < t$, *is that*

$$u(x, t) = \int_0^\infty \mathcal{G}(x, y)e^{-ty^2}\varphi(y)\, d\mu(y), \tag{5.2}$$

where φ *is an even entire function of growth* $(2, \sigma)$ *and*

$$b_n = (-1)^n\varphi^{(2n)}(0)/(2^{2n}(2n)\,!) \tag{5.3}$$

The corresponding result for the Appell transforms $V_{n,\nu}(x, t)$ of $Q_{n,\nu}(x, t)$ may be similarly established.

THEOREM 5.2. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} d_n V_{n,\nu}(x, t), \tag{5.4}$$

the series converging for $t > \sigma \geq 0$, *is that*

$$u(x, t) = \int_0^\infty e^{-ty^2}\mathcal{G}(xy)\varphi(y)\, d\mu(y), \qquad t > \sigma \geq 0, \tag{5.5}$$

where φ *is an odd entire function of growth* $(2, \sigma)$, *and*

$$d_n = (-1)^{n+3/2}\varphi^{(2n+1)}(0)/(2^{2n+1}(2n + 1)\,!). \tag{5.6}$$

Proof. To prove sufficiency, assume that (5.5) holds with

$$\varphi(y) = \sum_{n=0}^{\infty} \beta_n y^{2n+1}, \qquad \beta_n = \varphi^{(2n+1)}(0)/(2n + 1)\,!,$$

and $\varphi \in (2, \sigma)$ so that

$$|\varphi| \leq Ke^{(\sigma+\varepsilon)y^2}.$$

Then, by (2.5),

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} \beta_n \int_0^\infty e^{-ty^2}\mathcal{G}(xy)y^{2n+1}\, d\mu(y) \\ &= \sum_{n=0}^{\infty} \beta_n((-1)^{n+3/2}/2^{2n+1})V_{n,\nu}(x, t) \end{aligned}$$

with termwise integration valid since

$$\begin{aligned} \int_0^\infty e^{-ty^2}\, d\mu(y) \sum_{n=0}^{\infty} |\beta_n|\, y^{2n+1} &\leq K \int_0^\infty e^{-ty^2}e^{(\sigma+\varepsilon)y^2}\, d\mu(y) \\ &< \infty \qquad \text{for } t > \sigma. \end{aligned}$$

Taking $d_n = ((-1)^{n+3/2}/2^{2n+1})\beta_n$, we have established that the condition is sufficient.

On the other hand, assume that for $t > \sigma \geq 0$ (5.4) holds. Then by (2.5),

$$u(x, t) = \int_0^\infty e^{-ty^2} \mathcal{G}(xy)\, d\mu(y) \sum_{n=0}^\infty (-1)^{n+1/2} 2^{2n+1} d_n y^{2n+1}$$

$$= \int_0^\infty e^{-ty^2} \mathcal{G}(xy) \varphi(y)\, d\mu(y),$$

where

$$\varphi(y) = \textstyle\sum_{n=0}^\infty (-1)^{n+1/2} 2^{2n+1} d_n y^{2n+1}.$$

Since the convergence of the series (5.4) implies that

$$\limsup_{n\to\infty} n \mid d_n \mid^{1/n} \leq \sigma e/4 \quad \text{or} \quad \limsup_{n\to\infty} n \mid 2^{2n+1} d_n \mid^{1/n} \leq \sigma e,$$

$\varphi(y) \in (2, \sigma)$ and termwise integration is valid. The theorem is thus proved.

The two preceding theorems may be combined to give the following result.

THEOREM 5.3. *A necessary and sufficient condition that*

$$u(x, t) = \textstyle\sum_{n=0}^\infty [b_n W_{n,\nu}(x, t) + d_n V_{n,\nu}(x, t)],$$

the series converging for $t > \sigma \geq 0$, is that

$$u(x, t) = \int_0^\infty e^{-ty^2} \mathcal{G}(xy) \varphi(y)\, d\mu(y), \qquad t > \sigma \geq 0,$$

where φ is an entire function of growth $(2, \sigma)$.

The theorem may be illustrated by

$$u(x, t) = \textstyle\sum_{n=0}^\infty ((-1)^n / 2^{2n} n\,!) [W_{n,\nu}(x, t) + V_{n,\nu}(x, t)],$$

the series converging for $t > 1$. Here we have

$$u(x, t) = \int_0^\infty e^{-ty^2} \mathcal{G}(xy) [e^{y^2}(1 + 2iy)]\, d\mu(y), \qquad t > 1,$$

where $\varphi(y) = e^{y^2}(1 + 2iy)$ is of growth (2, 1).

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University of Missouri
St. Louis, Missouri