INVARIANT MEANS ON LOCALLY COMPACT GROUPS[1]

BY

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Introduction

Let G be locally compact group and $LUC(G)$ be the space of real left uniformly continuous bounded functions on G with usual sup norm (so $f \in LUC(G)$ iff f is bounded and whenever $a_\alpha \to a$, a_α, $\alpha \in G$ then

$$\lim_\alpha \sup_t |f(a_\alpha t) - f(at)| = 0).$$

If $f \in LUC(G)$, $a, s \in G$ define $f_a(s) = f(as), f^a(s) = f(sa), l_a f = f_a, r_a f = f^a$. Let $\mathcal{R}(f) = \{r_a f; a \in G\}$, $\mathcal{L}(f) = \{l_a f; a \in G\}$ be the right and left orbits of f and denote by τ_c the topology of uniform convergence on compacta on $LUC(G)$. Denote by Co A the convex hull of the set A.

One of the purposes of the present paper is to give the following characterization of locally compact amenable groups:

THEOREM. *If G is locally compact then $LUC(G)$ admits a left invariant mean (LIM) iff for any $f \in LUC(G)$ and any $a \in G$ the τ_c closure of* Co $\mathcal{R}(f - f_a)$ *contains the zero function. This is the case iff for any $f \in LUC(G)$ the τ_c closure of* Co $\mathcal{R}(f)$ *contains some constant function. In this case the set of all such constants coincides with $\{\varphi(f); \varphi$ a LIM on $LUC(G)\}$.*

It is interesting to note that for *any* topological semigroup the *uniform* closure of Co $\mathcal{L}(f - f_a)$ contains 0 (Note that $\| (1/n) \sum_1^n l_{a^i}(f - f_a) \| \leq (2/n) \|f\|$).

In fact we prove much more than this theorem. We prove a theorem in abstract setting which when applied to $LUC(S)$ for *any* topological semigroup S with only separately continuous multiplication yields both cases of the following theorem and unifies their proof.

THEOREM. *Let S be a topological semigroup with separately continuous multiplication. Then $LUC(S)$ admits a [multiplicative] LIM if and only if [the τ_c closure of $\mathcal{R}(f - f_a)$] the τ_c closure of* Co $\mathcal{R}(f - f_a)$ *contains the 0 function for any $f \in LUC(S)$ and $a \in S$. This holds if and only if [the τ_c closure of $\mathcal{R}(f)$] the τ_c closure of* Co $\mathcal{R}(f)$, *contains a constant function for any $f \in LUC(S)$ and in this case the set of all such constants coincides with $\{\varphi(f); \varphi$ a [multiplicative] LIM on $LUC(S)\}$.*

The amenable case of this theorem (i.e. $LUC(S)$ admits a LIM) for *discrete* semigroups S, is due to T. Mitchell [11] while the extremely amenable case (i.e. $LUC(S)$ admits a multiplicative LIM) for *discrete* S is due to Granirer

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[4, p. 97]. The proofs given in [11] and [4] are entirely different and we find some merit in the fact that we provide a unified proof for both. We also note that our theorem yields something new even for locally compact groups G. The topic of locally compact amenable groups arouse much recent interest and we refer the interested reader to the recent works by F. Greenleaf [5], J. Gilbert [2], H. Leptin [9] etc. and the bibliography given in these works.

In the last part of this paper we prove a theorem a particular case of which implies that if $S \neq \{e\}$ is any subsemigroup of any locally compact group G then $LUC(S)$ does not admit a multiplicative LIM. This result is proved for *subgroups* S of locally compact *abelian* groups (and for several other cases) in [4] p. 103 and has been improved by T. Mitchell (written communication) to include all subsemigroups of locally compact *abelian* groups. The commutativity is though heavily used in both proofs. This result is also known to hold for all *discrete* right cancellation semigroups (see [4] and compare with Mitchell [12]). The proof for this case makes though essential use of the discreteness of S.

Some notations. Let S be a topological semigroup $m(S)$ $[C(S)]$ the space of all [continuous] bounded real functions on S with norm

$$\|f\| = \sup\{|f(s)|;\, s \in S\}.$$

If $f \in m(S)$, $a, s \in S$ let $f_a(s) = f(as)$, $f^a(s) = f(sa)$, $l_a f = f_a$, $r_a f = f^a$, $\mathcal{R}(f) = \{r_a f; a \in S\}$, $\mathcal{L}(f) = \{l_a f; a \in S\}$.

If $X \subset m(S)$ is a linear subspace then X is *invariant* if $l_t X \subset X$ and $r_t X \subset X$ for all $t \in S$. X is *introverted* if for any $\varphi \in X^*$ (the conjugate Banach space of X) and any $f \in X$, the function of s, $\varphi(f_s)$ belongs to X. In this case X^*, with the multiplication given by $\psi \odot \varphi(f) = \psi(h)$ where $h(s) = \varphi(f_s)$ becomes a Banach algebra. (This is readily checked; see for example Day [1, p. 527] or [3, p. 103] or Namioka [14, p. 72].)

For $a \in S$ let $p_a \in m(S)^*$ be the point measure at a, i.e. $p_a f = f(a)$ for all $f \in m(S)^*$. Any element in Co $\{p_a\,;\, a \in S\}$ is said to be a finite mean (on $m(S)$). $\varphi \in X^*$ is said to be a finite mean on X iff φ is the restriction to X of some *finite mean on* $m(S)$.

In all that follows $X \subset m(S)$ will stand for a linear invariant subspace containing 1. $\varphi \in X^*$ is a mean on X if $\varphi(1) = 1$ and $\varphi(f) \geq 0$ for $f \in X$ with $f \geq 0$. The finite means on X are $\sigma(X^*, X)$ (i.e. w^*) dense in the w^* compact convex set of means on X. (This holds for $X = m(S)$ as stated in Day [1, p. 513] and readily follows from it.) $\varphi \in X^*$ is a left invariant mean LIM on X iff φ is a mean and $\varphi(f_a) = \varphi(f)$ for all $f \in X$ and $a \in S$. If $\varphi = \sum_1^n \alpha_i p_{s_i}$ is a finite mean on $m(S)$ let

$$r_\varphi,\, l_\varphi : m(S) \to m(S)$$

be defined by $r_\varphi = \sum_1^n \alpha_i r_{s_i}$ and $l_\varphi = \sum_1^n \alpha_i l_{s_i}$. It is clear that $\varphi(f_t) = (r_\varphi f)(t)$ for all $f \in X$ and $t \in S$. The restriction of r_φ to X is also denoted by r_φ.

A triple (S, X, Σ) will always mean in what follows that S is a semigroup, X is an invariant *introverted* linear subspace of $m(S)$ with $1 \in X$ (hence X^* is a Banach algebra with respect to $\odot$) and $\Sigma \subset X^*$ is a $\sigma(X^*, X)$ closed subsemigroup of means (with respect to $\odot$) for which the set of finite means in Σ is w^* (i.e. $\sigma(X^*, X)$) dense in Σ.

Main example. Let S be a semigroup with a topology in which multiplication is separately continuous. Let $LUC(S)$ be the space of all $f \in C(S)$ for which whenever $s_\alpha \to s\ s_\alpha, s \in S$ then $\lim_\alpha \| f_{s_\alpha} - f_s \| = 0$. Then $X = LUC(S)$ is an invariant Banach subalgebra of $m(S)$ with $1 \in X$, which is in addition introverted as easily seen (see Namioka [14, pp. 64, 68, 72]).

As Σ we can take either all multiplicative means on $LUC(S)$ or all means on $LUC(S)$. In both cases (S, X, Σ) are triples as readily seen. In particular if S is discrete then $LUC(S) = m(S)$ so $(S, m(S), \beta S)$ or $(S, m(S), M(S))$ where $M(S)$ is the set of all means on $m(S)$, are triples. βS is the Stone Cěch compactification of S.

LEMMA 1. *Let (S, X, Σ) be a triple. Then there is some $\varphi \in \Sigma$ which is a LIM on X if and only if $(*)$ for any $f \in X$, $a \in S$ there is some $\mu \in \Sigma$ such that $\mu[l_s(f - f_a)] = 0$ for all $s \in S$.*

Proof. Any LIM μ on X satisfies $(*)$. Conversely assume $(*)$ and let

$$K(f, a) = \{\varphi \in \Sigma; \varphi[l_s(f - f_a)] = 0 \text{ for all } s \text{ in } S\}.$$

We show that

$$\mathcal{K} = \{K(f, a); f \in X, a \in S\}$$

has the finite intersection property. The fact that each $K(f, a)$ and Σ are $\sigma(X^*, X)$ compact will imply then that there is some φ in Σ such that $\varphi(l_t(f - f_a)) = 0$ for all $t, a \in S$. $\mu = \varphi \odot \varphi \in \Sigma$ will be a LIM on X.

So assume that $\mu \in \bigcap_{f \in Y, a \in A} K(f, a) = K(Y, A)$ for subsets $Y \subset X$, $A \subset S$ and let $g \in X$, $b \in S$.

Define $g'(s) = \mu(l_s g)$, pick $\nu \in K(g', b)$ and let $\lambda = \nu \odot \mu \in \Sigma$. We show that $\lambda \in K(Y, A) \cap K(g, b)$. If $f \in Y$, $a \in A$ and $s \in S$ are fixed then $\nu \odot \mu[l_s(f - f_a)] = \nu(h)$ where

$$h(t) = \mu[l_t\, l_s(f - f_a)] = \mu[l_{st}(f - f_a)] = 0$$

since $\mu \in K(Y, A)$. Thus $\lambda \in K(Y, A)$. Now $\nu \odot \mu(g - g_b) = \nu(h)$ where

$$h(t) = \mu[l_t(g - l_b\, g)] = \mu(l_t\, g) - \mu(l_{bt}\, g) = g'(t) - g'_b(t).$$

Thus $\nu(h) = 0$ since $\nu \in K(g', b)$ which finishes the proof.

We use in what follows the following notation: If $\Sigma \subset X^*$ is a set of means then $F_\Sigma \subset m(S)^*$ will denote the set of all finite means on $m(S)$ whose restrictions to X belong to Σ. For any $Y \subset m(S)$, p cl Y will denote the pointwise closure of Y in $m(S)$.

THEOREM 1. *Let* (S, X, Σ) *be a triple.*

(a) *If* φ *is a LIM on* X *and* φ_α *is a net of finite means such that* $\varphi_\alpha(f) \to \varphi(f)$ *for all* f *in* X *then* $(r_{\varphi_\alpha} f)(t) \to \varphi(f)$ *for all* t *in* S. *Thus* $r_{\varphi_\alpha}(f - f_a)(t) \to 0$ *for all* $a, t \in S, f \in X$.

(b) *If for all* $f \in X$, $a \in S$, p cl $\{r_\varphi(f - f_a); \varphi \in F_\Sigma\}$ *contains the zero function then there exists* $\varphi \in \Sigma$ *which is a LIM on* X.

(c) *If there is some* $\varphi \in \Sigma$ *which is a LIM on* X *then for any* $f \in X$,

$$\{\varphi(f); \varphi \in \Sigma, \varphi \text{ a } LIM \text{ on } X\} = \{c; c1 \in \text{p cl } \{r_\varphi(f), \varphi \in F_\Sigma\}\}.^3$$

Proof. (a) $(r_{\varphi_\alpha} f(t) = \varphi_\alpha(l_t f) \to \varphi(l_t f) = \varphi(f)$ for all $t \in S$.

(b) Let $f \in X$, $a \in S$ be fixed and choose $\varphi_\alpha \in F_\Sigma$ such that $\varphi_\alpha[l_t(f - f_a)] = r_{\varphi_\alpha}(f - f_a)(t) \to 0$. Let $\varphi \in \Sigma$ be a $\sigma(X^*, X)$ limit of a subnet of the restriction of the φ_α's to X. Then $\varphi[l_t(f - f_a)] = 0$ for all $t \in S$. By lemma 1 there is some $\varphi_0 \in \Sigma$ which is a *LIM* on X.

(c) *If* $\varphi(f) = c$ then by (a), $c1 \in$ p cl $\{r\varphi\, f; \varphi \in F_\Sigma\}$. Conversely let $\varphi_\alpha \in F_\Sigma$ satisfy $\varphi_\alpha(l_t f) = (r_{\varphi_\alpha} f)(t) \to c$ for all $t \in S$. If $\varphi \in \Sigma$ is a $\sigma(X^*, X)$ limit point of a subnet of φ_α then $\varphi(l_t f) = c$ for all $t \in S$. If $\mu \in \Sigma$ is a *LIM* then $\mu \odot \varphi(f) = \mu[\varphi l_t f] = c$ and $\mu \odot \varphi \in \Sigma$ is a *LIM* on X (see Day [1, p.529]) since $\mu \odot \varphi(g) = \mu(h)$ where $h(t) = \varphi(l_t g)$ and $\mu \odot \varphi(g_a) = \mu(h')$ where $h'(t) = \varphi(l_t l_a g) = \varphi(l_{at} g) = h_a(t)$. Thus $\mu(h) = \mu(h')$.

Remarks. 1. Take $X = m(S)$, Σ the set of all means of $m(S)$. Theorem 1 implies Theorem 3 and Theorem 4 (a) and (b) of Mitchell [11, p. 253].

2. Take $X = m(S)$, $\Sigma = \beta S$ all multiplicative means. Theorem 1 implies Theorem 1 of Granirer [4, p. 97].

The following lemma seems to be of independent interest.

Let $X \subset m(S)$ be an invariant linear subspace with $1 \in X$. As usual p cl A denotes the pointwise closure *in* $m(S)$ of $A \subset m(S)$.

LEMMA 2. *X is introverted if and only if for any* $f \in X$, p cl Co $\mathcal{R}(f) \subseteq X^{\cdot}$

Proof. Assume that X is introverted. Any $g \in$ Co $\mathcal{R}(f)$ can be written as $g = r_\varphi f$ for some finite mean $\varphi \in m(S)^*$. Assume that $(r_{\varphi_\alpha} f)(t) \to f_0(t)$ for all $t \in S$ where φ_α are finite means in $m(S)^*$. Then a subnet φ_β of the φ_α's will converge w^* in $m(S)^*$ to some mean $\varphi \in m(S)^*$. Thus $(r_{\varphi_\beta} f)(t) = \varphi_\beta(f_t) \to \varphi(f_t)$ for all $t \in S$; thus $f_0(t) = \varphi(f_t)$ and since X is introverted, $f_0 \in X$, i.e. p cl Co $\mathcal{R}(f) \subset X$.

Conversely assume that p cl Co $\mathcal{R}(f) \subset X$ for all $f \in X$. Let $\varphi \in m(S)^*$ be a mean and $\varphi_\alpha \in m(S)^*$ be a net of finite means such that $\varphi_\alpha(h) \to \varphi(h)$ for all $h \in m(S)$. Now $(r_{\varphi_\alpha} f)(t) = \varphi_\alpha(f_t) \to \varphi(f_t)$ so $\varphi(f_t) \in$ p cl Co $\mathcal{R}(f) \subset X$ for any mean φ on $m(S)$ and any $f \in X$. If $\varphi \in X^*$ is arbitrary let $\varphi_0 \in m(S)^*$ be an extension of φ. Then $\varphi_0 = \alpha\varphi_1 - \beta\varphi_2$ for some $\alpha, \beta \geq 0$ and means

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[3] Here as well as in Lemma 1, X need only be Σ-introverted that is; $\varphi(f_t) \in X$ for all $\varphi \in \Sigma$ and all $f \in \Sigma$.

φ_1, φ_2 in $m(S)^*$. If $f \in X$ then since $\varphi_j(f_t) \in X$ it follows that $\varphi(f_t) = \varphi_0(f_t) \in X$ for all $\varphi \in X^*$ and $f \in X$ so X is introverted.

Remark. The closed invariant subalgebra $X \subset m(S)$ with $1 \in X$ is said to to be m-introverted if $\varphi(f_t) \in X$ for any *multiplicative* $\varphi \in X^*$ and any $f \in X$. This concept is due to T. Mitchell [13, p. 121] and has applications to fixed point theorems [13]. For any multiplicative $0 \neq \varphi \in X^*$ there is a net of point measures p_{s_α} such that $p_{s_\alpha} f = f(s_\alpha) \to \varphi(f)$ for all $f \in X$. This remark together with a trivial adaptation of the above proof yields the following proposition (not needed in the sequel) which is of independent interest:

PROPOSITION. *The closed invariant subalgebra $X \subset m(S)$ with $1 \in X$ is m-introverted if and only if* p cl $\mathcal{R}(f) \subset X$ *for all* $f \in X$.

Let now S be a semigroup with a topology in which multiplication is separately continuous.

LEMMA 3. *For any $f \in LUC(S)$,* p cl Co $\mathcal{R}(f) \subset LUC(S)$ *and* p cl Co $\mathcal{R}(f)$ *coincides with the τ_c closure in $LUC(S)$ of* Co $\mathcal{R}(f)$. *Furthermore τ_c coincides with the pointwise topology on* p cl Co $\mathcal{R}(f)$.

Proof. Let $f \in LUC(S) \subset C(S)$. Then Co $\mathcal{R}(f)$ is an equicontinuous set of functions. Since if $a \in S$ there is some nieghborhood V_a of a such that $\|f_v - f_a\| < \varepsilon$ if $v \in V_a$, hence

$$|(r_t f)(v) - (r_t f)(a)| = |f(vt) - f(at)| < \varepsilon$$

for all $t \in S$ and $v \in V_a$. If $\alpha_i \geq 0$, $\sum_1^n \alpha_i = 1$, $t_i \in S$ then

$$\left|\sum_1^n \alpha_i (r_{t_i} f)(v) - \sum_1^n \alpha_i (r_{t_i} f)(a)\right| \leq \max_{1 \leq i \leq n} |(r_{t_i} f)(v) - (r_{t_i} f)(a)| < \varepsilon \quad \text{if } v \in V_a.$$

By Kelley [8, p. 232, Theorems 14, 15], p cl Co $\mathcal{R}(f)$ is equicontinuous and τ_c coincides with the pointwise topology on this set. That p cl Co $\mathcal{R}(f) \subset LUC(S)$ follows from the previous lemma. If now $f_\alpha \in$ Co $\mathcal{R}(f)$ and $f_\alpha \to f$ pointwise then $f_\alpha \to f$ in τ_c hence p cl Co $\mathcal{R}(f)$ coincides with the τ_c closure in $LUC(S)$ of Co $\mathcal{R}(f)$. τ_c cl A will denote the τ_c closure of $A \subset LUC(S)$ in $LUC(S)$.

THEOREM 2. *Let S be a topological semigroup.*

(1) *If $LUC(S)$ has a [multiplicative] LIM φ and $\varphi_\alpha \in m(S)^*$ are finite means such that $\varphi_\alpha(f) \to \varphi(f)$ for all $f \in LUC(S)$ then $r_{\varphi_\alpha} f \to \varphi(f)1$ in τ_c.*

(2) *If for any $f \in LUC(S)$ and $a \in S$,* p cl Co $\mathcal{R}(f - f_a)$ [p cl $\mathcal{R}(f - f_a)$] *contains the 0 function then $LUC(S)$ admits a [multiplicative] LIM. In this case* τ_c cl Co $\mathcal{R}(f)$ [τ_c cl $\mathcal{R}(f)$] *contains some constant function for any $f \in LUC(S)$ and (for fixed $f \in LUC(S)$) the set of all such constants coincides with*

$$\{\varphi(f);\ \varphi \text{ a [multiplicative] } LIM \text{ on } LUC(S)\}.$$

Remarks. (a) We note that by Lemma 3, p cl Co $\mathcal{R}(f)$ or p cl $\mathcal{R}(f)$ coincides with τ_c cl Co $\mathcal{R}(f)$, τ_c cl. $\mathcal{R}(f)$ resp. for any $f \in LUC(S)$.

(b) (1) clearly implies that τ_c cl Co $\mathcal{R}(f - f_a)$ [τ_c cl $\mathcal{R}(f - f_a)$] contains the 0 constant function for all $f \in LUC(S)$ and $a \in S$.

Proof. Combine the example preceding Lemma 1 with Theorem 1 and Lemma 3. Note that if $\varphi \in LUC(S)^*$ is multiplicative LIM we can take in (1) point measures, $\varphi_\alpha = p_{s_\alpha}$, and then $r_{s_\alpha} f \to \varphi(f)$ in τ_c.

For locally compact groups G a stronger statement is true, namely:

PROPOSITION. *Let G be a locally compact group, $UC(G)$ the bounded real left and right uniformly continuous functions on G. If* p cl Co $\mathcal{R}(f - f_a)$ *contains the zero function for each $a \in G$ and $f \in UC(G)$ then $LUC(G)$ admits a LIM.*

Proof. Let $f \in UC(G)$, $a \in G$ be fixed and φ_α a net of finite means in $m(G)^*$ such that $r_{\varphi_\alpha}(f - f_a)(t) = \varphi_\alpha(l_t(f - f_a)) \to 0$ for all t in G. By possibly passing to subnets we can assume that $\varphi_\alpha(h) \to \varphi(h)$ for all $h \in m(G)$ to some mean φ on $m(G)$. Thus $\varphi[l_t(f - f_a)] = 0$ for all t in G. Hence for any $a \in G, f \in UC(G)$ there is a mean φ on $UC(G)$ such that $\varphi(l_t(f - f_a)) = 0$ for all $t \in G$. Let now E be a compact symmetric neighborhood of the identity ($E = E^{-1}$) and $f \in LUC(G)$, $a \in G$ be fixed. Let φ_E be the normalized characteristic function of E. Then $f * \varphi_E \in UC(G)$ [5, Lemma 2.1.2] and $(f - f_a) * \varphi_E = f * \varphi_E - l_a(f * \varphi_E)$ (see Hewitt-Ross, *Abstract harmonic analysis*, Springer, 1963, p. 292). Hence there is a mean m on $UC(G)$ such that $m(l_s[f * \varphi_E - l_a(f * \varphi_E)]) = 0$ for all $s \in G$. Define now the mean m' on $LUC(G)$ by $m'(h) = m(h * \varphi_E)$. Then

$$m'[l_s(f - f_a)] = m[(l_s(f - f_a)) * \varphi_E] = m(l_s[(f - f_a) * \varphi_E])$$
$$= m(l_s(f * \varphi_E - l_a(f * \varphi_E))] = 0$$

for all s in G. Lemma 1 now implies that $LUC(G)$ admits a LIM.

Our purpose in what follows is to show that the only subsemigroups S of a locally compact group G for which $LUC(S)$ admits a LIM φ which belongs to Co (M) where M is the set of all multiplicative $\varphi \in LUC(S)^*$ are the finite subgroups of G. We prove this result in a sequence of lemmas.

LEMMA 4. *Let G be a locally compact group. If $LUC(G)$ admits a multiplicative LIM then G = consists of identity e only.*

Proof. Assume that $G \neq \{e\}$ and that $LUC(G)$ admits a multiplicative LIM. Let $V \neq \{e\}$ be an open symmetric neighborhood of e with compact closure. Let $t_0 \in V$, $t_0 \neq e$. G is completely regular so there exists $f \in C(G)$, $0 \leq f \leq 1$ such that $f(e) = 1, f(\{t_0\} \cup \{G - V\}) = 0$ $(G - V = \{g \in G; g \notin V\})$. f has compact support (included in $\bar{V}$) and by Theorem 28B in [10, p. 109], f is left (and right) uniformly continuous. Applying Zorn's lemma one easily selects a family $\{t_\beta, \beta \in I\}$ of elements of G maximal with respect to the property $Vt_\alpha \cap Vt_\beta = \emptyset$ if $\alpha \neq \beta$. Let $g(t) = \sup_\beta f(tt_\beta^{-1})$. $g(t)$ restricted to Vt_β coincides with $r_{t_\beta^{-1}} f$ and it is readily checked that $g \in LUC(G)$, $g(t_\beta) = 1$ and $g(t_0 t_\beta) = 0$ for all β. By Theorem 2 there is a net r_{s_α} and a real c, for which

$r_{s_\alpha} g \to c1$ uniformly on compacta. Hence for some α_0, $|g(ts_{\alpha_0}) - c| < \frac{1}{2}$ for all $t \in V^3$. By the maximality of $\{t_\beta\}$, $Vs_{\alpha_0} \cap Vt_{\beta_0} \neq 0$ for some β_0. Thus $t_{\beta_0} \in V^2 s_{\alpha_0}$ and $Vt_{\beta_0} \subset V^3 s_{\alpha_0}$. Hence $t_{\beta_0} = v_1 s_{\alpha_0}$ and $t_0 t_{\beta_0} = v_2 s_{\alpha_0}$ for some $v_1, v_2 \in V^3$. Thus $g(v_1 s_{\alpha_0}) = 1$, $g(v_2 s_{\alpha_0}) = 0$, so $|1 - c| < \frac{1}{2}$ and $|c| < \frac{1}{2}$ which cannot be. Hence $G = \{e\}$.

Remark. If $H \subset G$ then $1_H \in m(G)$ is the function which is one on H and zero outside H.

LEMMA 5. *Let G be any topological group and assume that $LUC(G)$ admits a LIM φ of type $\varphi = \sum_1^n \alpha_i \varphi_i$ where $0 \neq \varphi_i \in LUC(G)^*$ are multiplicative, $\alpha_i > 0$, for all $1 \leq i \leq n$ $\sum_1^n \alpha_i = 1$, and $\varphi_i \neq \varphi_j$ if $i \neq j$. Then there is some open and closed normal subgroup $N \subset G$ such that G/N is a finite group and $LUC(N)$ admits a multiplicative LIM.*

Proof. As known, if $\psi_1, \cdots, \psi_k \in LUC(G)^*$ are nonzero multiplicative and $\psi_i \neq \psi_j$ if $i \neq j$ then $\{\psi_1, \cdots, \psi_k\}$ is linearly independent. In fact if M is the set of all multiplicative nonzero elements of $LUC(G)^*$ with the w^* topology then the map $F : LUC(G) \to C(M)$ defined by $(Ff)\varphi = \varphi(f)$ is an isometry and algebraic isomorphism of $LUC(G)$ onto $C(M)$. M is compact hausdorff then $\psi_1, \cdots, \psi_k$ become different point measures on $C(M)$ which as readily seen are linearly independent on $C(M)$ and hence on $LUC(G)$.

If $L_a = l_a^*$, $a \in G$ then $\sum_1^n \alpha_i L_a \varphi_i = \sum_1^n \alpha_i \varphi_i = \varphi$. L_a is one to one and $L_a \varphi_i \in M$. The linear independence implies that

$$\{L_a \varphi_1, \cdots, L_a \varphi_n\} = \{\varphi_1, \cdots, \varphi_n\}$$

for all $a \in G$. Denote by L_a' the restriction of L_a to $\{\varphi_1, \cdots, \varphi_n\}$. Then $a \to L_a'$ is a homomorphism of G onto a subgroup G' of the finite group of all one to one maps of $\{\varphi_1, \cdots, \varphi_n\}$ into itself. If

$$N = \{a \in G; L_a' \varphi_i = \varphi_i, 1 \leq i \leq n\}$$

then G/N is finite, as being (algebraically) isomorphic to G'.

Now N is closed since $a \to L_a \varphi(f)$ is continuous for all $f \in LUC(G)$ and $\varphi \in LUC(G)^*$. If $a_1, \cdots, a_k \in G$ are different representatives of G/N with $a_1 = e$ the identity then $G = \bigcup_1^k a_i N$, $a_i N \cap a_j N = \emptyset$ if $i \neq j$, and $\bigcup_2^k a_i N$ is closed. Thus N is open too. Now $1_N \in LUC(G)$. Since if $a_\alpha \in G$ and $a_\alpha \to e$ then if $\alpha \geq \alpha_0$, $a_\alpha \in N$, for some α_0. Then $l_{a_\alpha} 1_N = 1_N$ if $\alpha \geq \alpha_0$.

Denote now $a_i^{-1} = b_i$, $1 \leq i \leq k$. Then

$$1 = \varphi(1) = \sum_1^k L_{b_i} \varphi(1_N) = \sum_{i=1}^n \sum_{j=1}^k \alpha_i \varphi_i(l_{b_j} 1_N).$$

Hence $L_{b_j} \varphi_i(1_N) = \varphi_i(l_{b_j} 1_N) > 0$ for some i, j and $L_{b_j} \varphi_i \in \{\varphi_1, \cdots, \varphi_n\}$. Hence $\varphi_m(1_N) > 0$ for some $1 \leq m \leq n$ and so $\varphi_m(1_N) = 1$ since φ_m is multiplicative. Moreover, $L_a \varphi_i = \varphi_i$ for all $1 \leq i \leq n$, $a \in N$, and so $L_a \varphi_m = \varphi_m$ for all $a \in N$.

If $f \in LUC(N)$ let $\pi(f) \in LUC(G)$ be defined by $\pi f(a) = f(a)$ for $a \in N$ and

$\pi f(a) = 0$ if $a \notin N$. That $\pi f \in LUC(G)$ is readily checked: If $g_\alpha \to e$, $g_\alpha \in G$ then $g_\alpha \in N$, if $\alpha \geq \alpha_0$ for some α_0. But for any $a \in N$ and $f \in LUC(N)$ $l_a(\pi f) = \pi(l_a^0 f)$ (where l_a^0 is the restriction of l_a to $LUC(N)$). Since $\lim_\alpha \| l_{g_\alpha}^0 f - f \| = 0$ it follows that

$$\lim_\alpha \| l_{g_\alpha} \pi f - \pi f \| = \lim_\alpha \| \pi(l_{g_\alpha}^0 f - f) \| = \lim_\alpha \| l_{g_\alpha}^0 f - f \| = 0.$$

We show now that $\pi^* \varphi_m \in LUC(N)^*$ is a multiplicative LIM on $LUC(N)$. $\pi^* \varphi_m$ is multiplicative since φ_m is so and $\pi(f_1 f_2) = \pi f_1 \pi f_2$ for $f_1, f_2 \in LUC(N)$. If $a \in N$ then

$$(\pi^* \varphi_m)(l_a^0 f) = \varphi_m(\pi l_a^0 f) = \varphi_m(l_a(\pi f)) = \varphi_m(\pi f) = (\pi^* \varphi_m) f,$$

which finishes this proof.

COROLLARY 1. *If G is locally compact and $LUC(G)$ admits a LIM φ of type $\varphi = \sum_1^n \alpha_i \varphi_i$ where $\varphi_i \in LUC(G)^*$ are multiplicative, $\alpha_i > 0$, $1 \leq i \leq n$, $\varphi_i \neq \zeta_j$ if $i \neq j$ and $\sum_1^n \alpha_i = 1$, then G is a finite group of order n.*

Proof. By Lemma 5, G contains an open and closed normal subgroup N for which G/N is finite and $LUC(N)$ admits a multiplicative LIM. By Lemma 4, $N = \{e\}$ so G is finite. But as known and easily shown the unique LIM ψ on a finite group G is given by

$$\psi(f) = (1/k) \sum_1^k f(g_i) \quad \text{where } G = \{g_1, \cdots, g_k\}.$$

Thus $\psi = \varphi = \sum_1^n \alpha_i \varphi_i$ so $k = n$.

THEOREM 3. *Let S be a subsemigroup of the locally compact group G and assume that $LUC(S)$ admits a LIM φ of type $\varphi = \sum_1^n \alpha_i \varphi_i$ where $\varphi_i \in LUC(S)^*$ are multiplicative, $\varphi_i \neq \varphi_j$ if $i \neq j$, $\alpha_i > 0$ for $1 \leq i \leq n$ and $\sum_1^n \alpha_i = 1$. Then S is a finite subgroup of G of order n.*

Proof. Let G_0 be the group generated by S and $\bar{G}_0$ be its closure in G. Let A be the algebra of all restrictions to S of functions in $LUC(\bar{G}_0)$. Define $\varphi_0 \in LUC(\bar{G}_0)^*$ by $\varphi_0(f) = \varphi(Pf)$ where $Pf \in A$ is the restriction, $(Pf)(s) = f(s)$ if $s \in S$. Denote by l_a^0 $[l_a]$ the left translation operator by a in $LUC(\bar{G}_0)$ $[A]$. Then $l_s\, Pf = P(l_s^0 f)$ for any $s \in S$ and $f \in LUC(\bar{G}_0)$. If $s \in S$ then $\varphi_0(l_s^0 f) = \varphi(P l_s^0 f) = \varphi(l_s(Pf)) = \varphi_0(f)$ and $\varphi_0(l_{s^{-1}}^0 f) = \varphi_0\, l_s^0(l_{s^{-1}}^0 \zeta) = J_0(f)$ for any $f \in LUC(\bar{G}_0)$. Since any $a \in G_0$ is a product of elements of s and s^{-1} it follows that $\varphi_0(l_a^0 f) = \varphi_0(f)$ for all $a \in G_0$ and $f \in LUC(\bar{G}_0)$. It is clear that $\varphi_0 \in LUC(\bar{G}_0)^*$ (since φ_0 is a mean). Now for fixed $f \in LUC(\bar{G}_0)$, the function $\varphi_0(l_g f)$ is continuous on $\bar{G}_0$ and equals the constant $\varphi_0(f)$ on the dense set G_0 of $\bar{G}_0$. Hence φ_0 is a LIM on $LUC(\bar{G}_0)$. But $\varphi_0 = P^* \varphi = \sum_1^n \alpha_i P^* \varphi_i$ and $P^* \varphi_i \in LUC(\bar{G}_0)$ are multiplicative. Assemble now the equal $P^* \varphi_i$'s and write $\varphi_0 = \sum_1^k \beta_j \psi_j$ with different multiplicative ψ_j's. The previous corollary implies now that $\bar{G}_0$ is a finite group of order k. Thus $S = \bar{G}_0$

and $LUC(S) = LUC(G_0)$. Since $\varphi = \sum_1^n \alpha_i \varphi_i$, with all $\alpha_i > 0$ and different multiplicative $\varphi_i \neq 0$ it will follow that S is a group of order n.[4]

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[4] The need for the algebra A here arises from the following remark of T. Mitchell (written communication). Let S be the additive semigroup $\{x; 0 < x < \infty\}$ with the usual topology. Then the function $\sin 1/x$ belongs to $LUC(S)$ but is *not* the restriction to S of any uniformly continuous function on G (the additive reals with usual topology) to S. Thanks are due to T. Mitchell for sending us this interesting remark. Our original proof contained a minor mistake at this point.