THE RING OF POLAR PRESERVING ENDOMORPHISMS OF AN ABELIAN LATTICE-ORDERED GROUP

BY

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1. Introduction

Let G be an abelian lattice-ordered group (l-group). We investigate the ring $P(G)$ generated by the semiring $P^+(G)$ of all group endomorphisms α of G such that for $x, y \in G$

$$x \wedge y = 0 \quad \text{implies} \quad x \wedge y\alpha = 0.$$

$P(G)$ is a po subring of the ring $B(G)$ of all order-bounded endomorphisms of G with $P^+(G)$ as its positive cone. If A is any ring of l-endomorphisms of G that contains the identity automorphism I, then $A \subseteq P(G)$. Thus $P(G)$ is the largest such ring. We show (Theorem 3.4) that the class

$$\{P(G) : G \text{ is an archimedean } l\text{-group}\}$$

is identical with the class of archimedean f-rings with identity. This allows us to derive many useful properties of $P^+(G)$.

For an archimedean l-group G, the largest f-ring of $B(G)$ that contains the identity is $P(G)$. Let G be an archimedean l-group with a weak order unit e. Then there is at most one multiplication on G so that G is an f-ring with identity e, and such a multiplication exists if and only if $\{e\alpha : \alpha \in P^+(G)\} = G^+$.

The elements in $P^+(G)$ preserve minimal prime subgroups. In Section 6 we investigate those group endomorphisms of G which preserve all the prime subgroups. In Section 7 we apply our theory to solve a problem posed by G. Birkhoff.

Notation and terminology. If G is an l-group, then we denote its positive cone by $G^+ = \{g \in G : g \geqq 0\}$. An *l-subgroup* of G is a subgroup K which is also a sublattice. If, in addition, $0 < x < k \in K$ implies $x \in K$, then we say that K is a *convex l-subgroup*. An *l-ideal* is a normal convex l-subgroup. A *prime* subgroup is a convex l-subgroup M such that $x \wedge y \in M$ implies $x \in M$ or $y \in M$. Various other characterizations of prime subgroups are given in [4] and [9]. An *l-endomorphism* of G is a group endomorphism that also preserves the lattice operations. Thus an endomorphism α of G is an l-endomorphism if and only if $x \wedge y = 0$ implies $x\alpha \wedge y\alpha = 0$ [7].

If X is a subset of G, then

$$X' = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called the *polar* of X. X' is a convex l-subgroup of G and the set $p(G)$ of

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all polars of G forms a complete Boolean algebra [11]. Note that $P^+(G)$ consists of those group endomorphisms of G that preserve each polar.

2. Polar preserving endomorphisms of a lattice-ordered group

Let G be a lattice-ordered group (l-group). Then a *polar preserving endomorphism* or *p-endomorphism* of G is a group endomorphism α of G such that for all $x, y \in G$

$$x \wedge y = 0 \quad \text{implies} \quad x \wedge y\alpha = 0.$$

Note that α is an l-endomorphism and hence $G^+\alpha \subseteq G^+$. For if $x \wedge y = 0$, then $x \wedge y\alpha = 0$ and hence $x\alpha \wedge y\alpha = 0$.

In the first part of this section we give several characterizations of p-endomorphisms. Then we show that the semiring of all p-endomorphisms of an abelian l-group G is the unique maximal subsemiring of the ring of group endomorphisms of G which contains the identity automorphism I and which consists of lattice endomorphisms. Finally we investigate this semiring in the case where G is a subdirect sum of subgroups of the reals.

2.1 PROPOSITION. *Let α be a group endomorphism of the l-group G. Then the following are equivalent.*

(a) *α is a p-endomorphism.*

(b) *$G^+\alpha \subseteq G^+$ and $M\alpha \subseteq M$ for each minimal prime subgroup M of G.*

(c) *$G^+\alpha \subseteq G^+$ and $P\alpha \subseteq P$ for each polar P of G.*

(d) *$G^+\alpha \subseteq G^+$ and $x' \subseteq (x\alpha)'$ for each $x \in G^+$.*

Moreover if G is abelian, then (a) *is equivalent to*

(e) *$G^+\alpha \subseteq G^+$ and $\alpha + I$ is an l-endomorphism.*

Proof. (a) implies (b). If $0 < x \in M$, then there is a $y \notin M$ such that $x \wedge y = 0$ (see [6] or [9]). But then $x\alpha \wedge y = 0$, so that $x\alpha \in M$ since $y \notin M$ and M is prime. It follows that $M\alpha \subseteq M$.

(b) implies (c). This follows from the fact that a polar of G is the intersection of minimal prime subgroups of G [4].

(c) implies (d). Let $y \in x'$ be positive. Then $x \wedge y = 0$ and so $x \in y'$. Thus $x\alpha \in y'\alpha \subseteq y'$. But $x\alpha \geq 0$, so that $x\alpha \wedge y = 0$. Hence $y \in (x\alpha)'$, and it follows that $x' \subseteq (x\alpha)'$.

(d) implies (a). If $x \wedge y = 0$, then $y \in x' \subseteq (x\alpha)'$ and $x\alpha \in G^+$. Thus $x\alpha \wedge y = 0$ and hence α is a p-endomorphism.

Finally suppose that G is abelian and let α be a p-endomorphism of G. Then $x \wedge y = 0$ implies $x\alpha \wedge y = 0$ and hence $(x\alpha + x) \wedge y = 0$. Therefore $\alpha + I$ is a p-endomorphism and hence an l-endomorphism. Conversely if $\alpha + I$ is an l-endomorphism and $G^+\alpha \subseteq G^+$ and $x \wedge y = 0$, then

$$0 = x(\alpha + I) \wedge y(\alpha + I) = (x\alpha + x) \wedge (y\alpha + y) \geq x\alpha \wedge y \geq 0.$$

Thus $0 = x\alpha \wedge y$ and so α is a p-endomorphism.

We now characterize p-endomorphisms in the case where G is a subdirect sum of totally ordered groups (o-groups). In particular, this characterization holds for abelian l-groups. Recall that an l-group is *representable* if there is an l-isomorphism σ of G into a cardinal sum πG_λ of a family $\{G_\lambda : \lambda \in \Gamma\}$ of o-groups. In general, the intersection of all minimal prime subgroups of G is zero, and Byrd has shown that G is representable if and only if each minimal prime subgroup is normal [3].

2.2 Proposition. *Let G be a representable l-group and let α be a group endomorphism of G such that $G^+\alpha \subseteq G^+$. Then the following are equivalent.*

(a) *α is a p-endomorphism.*

(b) *There is a set $\mathfrak{N}$ of normal prime subgroups of G such that $\cap\mathfrak{N} = 0$ and $N\alpha \subseteq N$ for each $N \in \mathfrak{N}$.*

Proof. (a) implies (b). Let $\mathfrak{N}$ be the set of all minimal prime subgroups of G.

(b) implies (a). Since $N\alpha \subseteq N$, α induces an o-endomorphism on the o-group G/N and so it induces an o-endomorphism $\bar{\alpha}$ of $\pi\{G/N : N \in \mathfrak{N}\}$. Let σ be the natural map of G into $\pi G/N$:

$$g\sigma = (\cdots, g + N, \cdots).$$

Then it is clear that $\sigma\bar{\alpha} = \alpha\sigma$ and for $x \in \pi G/N$ we have that the support of $x\bar{\alpha}$ is contained in the support of x (since $N\alpha \subseteq N$ for each $N \in \mathfrak{N}$). Thus $\bar{\alpha}$ is a p-endomorphism of $\pi G/N$ and so α is a p-endomorphism of G.

Let G be an abelian l-group and let $E(G)$ denote the endomorphism ring of G. We make $E(G)$ into a po ring by setting

$$E(G)^+ = \{\alpha \in E(G) : G^+\alpha \subseteq G^+\}.$$

Elements of $E(G)^+$ are called *o-endomorphisms* of G. In general $E(G)$ is not directed under this partial order and so we define $B(G) = E(G)^+ - E(G^+)$. Then $B(G)$, *the ring of order bounded endomorphisms of* G, is a po ring with positive cone $B(G)^+ = E(G)^+$.

2.3. *If G is an abelian l-group, then the set $P^+(G)$ of all p-endomorphisms of G is a subsemiring of $E(G)^+$ and*

$$P(G) = \{\alpha - \beta : \alpha, \beta \in P^+(G)\}$$

is a po subring of $B(G)$ with positive cone

$$P(G) \cap E(G)^+ = P^+(G).$$

Proof. Consider α, $\beta \in P^+(G)$ and suppose that $x \wedge y = 0$. Then $x\alpha \wedge y = 0$ and hence $x\alpha\beta \wedge y = 0$, so that $\alpha\beta \in P^+(G)$. Also $x\beta \wedge y = 0$ and hence $(x\alpha + x\beta) \wedge y = 0$ so that $\alpha + \beta \in P^+(G)$. Thus $P^+(G)$ is a subsemiring of $E(G)^+$ and so $P(G)$ is a po subring of $B(G)$.

Clearly $P(G) \cap E(G)^+ \supseteq P^+(G)$. Consider $\alpha - \beta \in P(G) \cap E(G)^+$ and

$x \in G^+$. Then $x\alpha \geq x\alpha - x\beta = x(\alpha - \beta) \geq 0$ and so if $x \wedge y = 0$, then

$$0 = x\alpha \wedge y \geq x(\alpha - \beta) \wedge y \geq 0.$$

Therefore $\alpha - \beta \in P^+(G)$.

Now let S be a subsemiring of $E(G)^+$ and let $\tilde{S} = \{\alpha - \beta : \alpha, \beta \in S\}$ be the subring of $B(G)$ that is generated by S. If each element of S is an l-endomorphism, we say that $\tilde{S}$ is a *ring of l-endomorphisms* of G.

2.4 THEOREM. *Let $\tilde{S}$ be a ring of l-endomorphisms of an abelian l-group G which satisfies* (*) *for each $0 < g \in G$ there is a $\beta \in \tilde{S}$ such that $g\beta \geq g$. Then $S \subseteq P^+(G)$ and hence $\tilde{S} \subseteq P(G)$.*

Proof. Let $\alpha \in S$ and let $x, y \in G$ be such that $x \wedge y = 0$. Now choose $\beta \in \tilde{S}$ such that $x\beta \geq x$ and write $\beta = \beta_1 - \beta_2$ where $\beta_1, \beta_2 \in S$. Then $\beta + \beta_2 = \beta_1 \in S$ so that $\beta + \beta_2 + \alpha \in S$. But then

$$\begin{aligned} 0 &= x(\beta + \beta_2 + \alpha) \wedge y(\beta + \beta_2 + \alpha) \\ &= (x\beta + x\beta_2 + x\alpha) \wedge (y\beta + y\beta_2 + y\alpha) \\ &\geq x\beta \wedge y\alpha \geq x \wedge y\alpha \geq 0. \end{aligned}$$

Thus $x \wedge y\alpha = 0$, so that $\alpha \in P^+(G)$. The result now follows.

2.5 COROLLARY. *If $\tilde{S}$ is a ring of l-endomorphisms of an abelian l-group G, which contains I, $S \subseteq P^+(G)$.*

Remarks. (i) The proof of 2.4 actually shows that if S is an additive subsemigroup of l-endomorphisms of G, where G is an abelian l-group, and if $I \in S$, then $S \subseteq P^+(G)$.

The examples referred to in the next three remarks can be found in §8.

(ii) A ring of l-endomorphisms of an abelian l-group G need not be contained in $P(G)$ (Example 1).

(iii) If $\tilde{S}$ is a ring of l-endomorphisms of an abelian l-group G and if $I \in \tilde{S}$, then I need not belong to S (Example 4).

(iv) If $\tilde{S}$ is a ring of l-endomorphisms of an abelian l-group G and if $I \in \tilde{S}$, then $(\tilde{S})^+$ need not be contained in S (Example 4).

We now identify the ring $P(G)$ in case G is a subdirect sum of subgroups of the reals. First recall that if G is an o-group, then the set of convex subgroups of G is totally ordered. Moreover, if G has a largest convex subgroup M, then M is normal and G/M is o-isomorphic to a subgroup of the reals R [7]. Now let π be an o-endomorphism of G, and let $x \in M$ with $x > 0$ be such that $x\pi \notin M$. Then since G is an o-group and $x\pi \notin M$, we have that $x\pi \geq nx$ for all $n \geq 0$. Thus $x\pi^2 \geq nx\pi$ for all $n \geq 0$, so that $x\pi^2 + M \geq n(x\pi + M)$ for all $n \geq 0$. This contradicts the fact that G/M is archimedean and hence $x\pi \in M$, so that $M\pi \subseteq M$. Thus we have shown the following.

2.6. *Let G be an o-group with a largest convex subgroup M. Then for each o-endomorphism π* of G, $M\pi \subseteq M$.

2.7. LEMMA. *Let G be a representable l-group, let $\alpha \in P^+(G)$, and suppose that M is a maximal convex l-subgroup of G. Then $M\alpha \subseteq M$.*

Proof. Since M is a maximal convex l-subgroup of G, M is prime. Let P be a minimal prime convex l-subgroup contained in M [4]. Then G/P is an o-group and M/P is the largest convex subgroup of G/P. Now by 2.1, $P\alpha \subseteq P$ so that α induces an o-endomorphism $\bar{\alpha}$ defined by $(x + P)\bar{\alpha} = x\alpha + P$ of G/P. But by 2.6, $(M/P)\bar{\alpha} \subseteq M/P$, so that $M\alpha \subseteq M$.

Remark. Recall (2.1) that p-endomorphisms leave invariant all minimal prime subgroups. Moreover, by 2.7, p-endomorphisms leave invariant maximal convex l-subgroups in representable l-groups. However, even in an archimedean l-group a prime subgroup need not be left invariant by a p-endomorphism (See Example 9 of §8). In Section 6 we examine those o-endomorphisms which leave fixed all the prime subgroups of G.

Now suppose that G is an l-group which is a subdirect sum $\pi_S\{R_\lambda : \lambda \in \Gamma\}$ of a family $\{R_\lambda : \lambda \in \Gamma\}$ of subgroups of the reals. Then $\pi_S\{R_\lambda : \lambda \in \Gamma\}$ is an l-subgroup of the l-group R^Γ of all functions from Γ into R where addition and the partial order are defined pointwise. Moreover, pointwise multiplication on R^Γ makes the l-group R^Γ into a lattice-ordered ring (l-ring) which is an f-ring.[2] We are now ready to prove

2.8 THEOREM. *Let G be an l-group which is a subdirect sum $\pi_S\{R_\lambda : \lambda \in \Gamma\}$ of a family $\{R_\lambda : \lambda \in \Gamma\}$ of subgroups of the reals R and let α be an o-endomorphism of G. Then*

(i) *$\alpha \in P^+(G)$ if and only if $M\alpha \subseteq M$ for all maximal l-ideals of G;*

(ii) *if $\alpha \in P^+(G)$, then there is a unique extension of α to a p-endomorphism $\bar{\alpha}$ of R^Γ; and*

(iii) *$P(G)$ is isomorphic to the subring $\{f \in R^\Gamma : fg \in G$ for all $g \in G\}$ of R^Γ.*

Proof. Let $\mathfrak{M}(G)$ denote the set of maximal l-ideals of G. Then since G is a subdirect sum of subgroups of the reals, we have that $\cap\mathfrak{M}(G) = 0$. Thus if $M\alpha \subseteq M$ for all $M \in \mathfrak{M}(G)$, then $\alpha \in P^+(G)$ by 2.2. The converse implication in (i) is immediate from 2.7.

To see (ii) let $x \in G$ and let x_λ denote the value of x, considered as function on Γ, at $\lambda \in \Gamma$. Let $G_\lambda = \{x \in G : x_\lambda = 0\}$. Then $G_\lambda \in \mathfrak{M}(G)$ and hence by (i), $G_\lambda\alpha \subseteq G_\lambda$. Thus α induces an o-endomorphism α^* on G/G_λ. Since G/G_λ is a subgroup of the reals R, there is a real number α_λ such that

$$(g + G_\lambda)\alpha^* = \alpha_\lambda(g + G_\lambda).$$

It follows that there is a unique extension of α to $\bar{\alpha} : R^\Gamma \rightarrow R^\Gamma$. Moreover, it is now clear how to prove (iii).

[2] An f-ring is an l-ring in which $a \wedge b = 0$ and $c \geq 0$ imply $ca \wedge b = ac \wedge b = 0$.

Remark. In the next section we extend the above result to show that $P(G)$ is an archimedean f-ring whenever G is an archimedian l-group.

2.9 COROLLARY. *Let X be a topological space and let $C(X)$ denote the l-group of continuous real-valued functions on X. Then $P(C(X))$ is isomorphic to $C(X)$.*

To close this section we remark that almost all the results in the sequel are valid only for archimedean l-groups. Consequently, we shall restrict our attention to the class of archimedean l-groups. The final result of this section shows that our results will apply to archimedean vector lattices.

2.10 PROPOSITION. *Suppose that G is a vector lattice. Then*
(i) *each l-ideal of the l-group G is a subspace and*
(ii) *If $\alpha : G \to G$ is an o-endomorphism of the l-group G, then α is linear provided that G is archimedean.*

Proof. (i) Let C be an l-ideal of G, let $c \in C$, and let $r \in R^+$. Then there is a positive integer $n > r$ such that $0 \leq rc^+ < nc^+ \in C$ and $0 \leq rc^- < nc^- \in C$. Thus since C is convex, both rc^+ and rc^- are in C. Hence $rc = rc^+ - rc^- \in C$.
(ii) For $g \in G$ and k a rational number, we have that $(kg)\alpha = k(g\alpha)$ for any o-endomorphism α of the l-group G. Now suppose that $g \in G^+$ and $r \in R$ is strictly positive. Then for rationals r_1 and r_2 such that $0 < r_1 \leq r \leq r_2$, we have, since α is an o-endomorphism, that $0 \leq r_1(g\alpha) = (r_1 g)\alpha \leq (rg)\alpha \leq (r_2 g)\alpha = r_2(g\alpha)$. Moreover, we have that $r_1(g\alpha) \leq r(g\alpha) \leq r_2(g\alpha)$. Thus

$$(rg)\alpha - r(g\alpha) \leq r_2(g\alpha) - r_1(g\alpha) = (r_2 - r_1)(g\alpha)$$

and

$$r(g\alpha) - (rg)\alpha \leq r_2(g\alpha) - r_1(g\alpha) = (r_2 - r_1)(g\alpha),$$

so that $|(rg)\alpha - r(g\alpha)| \leq (r_2 - r_1)(g\alpha)$. It follows that

$$|(rg)\alpha - r(g\alpha)| \leq s(g\alpha)$$

for all positive rationals s. Thus since G is archimedean, $|(rg)\alpha - r(g\alpha)| = 0$ and so $(rg)\alpha = r(g\alpha)$.

Remark. Example 10 of §8 shows that (ii) of 2.10 need not hold in the non-archimedean case.

3. p-endomorphisms of archimedean l-groups

The main result of this section provides a generalization of 2.8 in case the l-group G is archimedean but not necessarily a subdirect sum of subgroups of the reals. Our techniques require a functional representation of the archimedean l-group. We use Bernau's representation [1] of an archimedean l-group as an l-subgroup of a vector lattice $D(X)$ of almost finite continuous functions on a Stone space X. We now collect the pertinent facts about this representation.

Let G be an archimedean l-group, and let $\mathcal{P}(G)$ denote the set of polars of G. Then $\mathcal{P}(G)$ is a complete Boolean algebra [11], and so the associated Stone space X is extremely disconnected, Hausdorff and compact. Let $D(X)$ be the collection of almost finite continuous functions from X into $R \cup \{\pm\infty\}$ (i.e., $D(X) = \{f : X \to R \cup \{\pm\infty\} : f$ is continuous and $\{x \in X : f(x) \in R\}$ is dense$\}$). Then $D(X)$ is a complete vector lattice and Bernau established the following result.

THEOREM. *Let G be an archimedean l-group. Then there is an l-isomorphism σ of G in $D(X)$ (where X is the Stone space of the complete boolean algebra of polars of G) which preserves all existing infima and suprema and such that if $d \in D(X)$ is strictly positive, then there is a $g \in G$ and an integer n such that $0 < g < nd$. Moreover, if $\{e_\lambda : \lambda \in \Gamma\}$ is a maximal disjoint subset of G^+, then σ can be chosen so that each $e_\lambda\sigma$ is the characteristic function of a subset X_λ of X where the family $\{X_\lambda : \lambda \in \Gamma\}$ is a collection of compact open subsets of X whose union is dense in X.*

Now let G be an archimedean l-group and let $\{e_\lambda : \lambda \in \Gamma\}$ be maximal disjoint subset of G. Let σ be an l-isomorphism of G into $D(X)$with the properties of Bernau's Theorem. We shall identify G with $G\sigma$. Thus G is an l-subgroup of $D(X)$ and all infima and suprema in G agree with those in $D(X)$.

For each $x \in X$, let $G_x = \{g \in G : g(x) = 0\}$. Then for each $h \in G$ with $h(x) > 0$ and real, we have that G_x is a l-ideal maximal with respect to not containing h. Then if G^x is the intersection of all l-ideals of G that properly contain G_x, then $h \in G^x \setminus G_x$ and G^x/G_x is (isomorphic to) a subgroup of the reals R. Conversely, if $g \in G^x \setminus G_x$, then $g(x)$ is real. We are now ready to prove

3.1 LEMMA. *Let α be a p-endomorphism of G, let $e_\lambda \in \{e_\lambda : \lambda \in \Gamma\}$ a maximal disjoint subset of G^+, let X_λ be the compact open subset of X associated with e_λ, and let $x \in X_\lambda$. Then $e_\lambda\alpha(x) = +\infty$ if either $G^x\alpha \nsubseteq G^x$ or $G_x\alpha \nsubseteq G_x$.*

Proof. First suppose that $G^x\alpha \nsubseteq G^x$. Then there is a strictly positive $g \in G^x$ such that $g\alpha \notin G^x$. Since $e_\lambda(x) = 1$, we have that $e_\lambda \in G^x \setminus G_x$ and hence there is an $n > 0$ such that $G_x + g < G_x + ne_\lambda$ since G^x/G_x is archimedean. Since G_x is a prime l-ideal, there is a minimal prime l-ideal N contained in G_x. Thus, by 2.1, $N\alpha \subseteq N$ so that α induces an o-endomorphisms on the o-group G/N. If $N + g \geq N + ne_\lambda$, then $G_x + g \geq G_x + ne_\lambda$. Hence $N + g < N + ne_\lambda$. Thus $N + g\alpha \leq N + n(e_\lambda\alpha)$, so that $G_x + g\alpha \leq G_x + n(e_\lambda\alpha)$. Then $G^x < G^x + g\alpha \leq G^x + n(e_\lambda\alpha)$, so that $n(e_\lambda\alpha) \notin G^x$ and hence $e_\lambda\alpha \notin G^x$. Thus $e_\lambda\alpha(x)$ is not real, so that $e_\lambda\alpha(x) = +\infty$.

Now suppose that $G_x\alpha \nsubseteq G_x$. Then there is a strictly positive $g \in G_x$ such that $g\alpha \notin G_x$. Thus for all positive integers n we have that $G_x + ng = G_x < G_x + e_\lambda$. Hence $N + ng < N + e_\lambda$ where N is a minimal prime l-ideal contained in G_x. Thus $N + n(g\alpha) \leq N + e_\lambda\alpha$, and hence $G_x < G_x + n(g\alpha) \leq G_x + e_\lambda\alpha$. But G^x/G_x is archimedean so that $e_\lambda\alpha \notin G^x$ and thus $e_\lambda\alpha(x) = +\infty$.

3.2 THEOREM. *Let G be an archimedean l-group, and let $\{e_\lambda : \lambda \in \Gamma\}$ be a maximal disjoint subset of G. Consider G as a l-subgroup of $D(X)$ as in Bernau's Theorem. Then each p-endomorphism of G is a multiplication of G by an element in $D(X)^+$; and conversely if $d \in D(X)^+$ and $Gd \subseteq G$, then the map $g \to gd$ $(g \in G)$ is a p-endomorphism of G.*

Proof. For each $g \in G$, the closure $S(g)$ of the set $\{x \in X : g(x) \neq 0\}$ is the *support* of g. Now let α be a p-endomorphism of G and let $\lambda \in \Gamma$. Then by 2.1,

$$e_\lambda \alpha \in (e_\lambda \alpha)'' \subset e'' = \{g \in G : S(g) \subseteq S(e_\lambda) = X_\lambda\}.$$

Thus $e_\lambda \alpha$ is zero outside of X_λ and is finite on a dense open subset of X_λ. Thus by 3.1 the set $D_\lambda = \{x \in X_\lambda : G^x\alpha \subseteq G^x$ and $G_x \alpha \subseteq G_x\}$ is dense and open. Now for each $x \in D_\lambda$, α induces an o-endomorphism on G^x/G_x. But G^x/G_x is a subgroup of R and hence there is a real number $\alpha_x > 0$ such that if $g \in G$ and $g(x)$ is real, then $g\alpha(x) = g(x)\alpha_x$. Moreover, $e_\lambda \alpha(x) = \alpha_x$ so that map $x \to \alpha_x$ is a continuous real-valued function defined on D_λ. But this holds for each $\lambda \in \Gamma$, and hence it follows that there is a continuous real-valued function $x \to \alpha_x$ defined on a dense open subset of X such that $x \in D$, $(g\alpha)(x) = g(x)\alpha_x$ if $g(x)$ is real. But then the map $x \to \alpha_x$ has a unique extension to an element of $D(X)^+$, say $\bar{\alpha}$; and it follows that $g\alpha = g\bar{\alpha}$ where $g\bar{\alpha}$ denotes the f-ring multiplication in $D(X)$.

Finally, the converse is clear since $D(X)$ is an f-ring.

We remark that a routine computation shows that the map $P(G)$ such that $\alpha \to \bar{\alpha} \in D(X)$ is an l-isomorphism of l-rings. Hence we have

3.3. COROLLARY. *For archimedean l-groups G and the associated Bernau representation in $D(X)$ we have that $P(G)$ is an f-ring with identity and*

$$P(G) = \{d \in D(X) : Gd \subseteq G\}.$$

Now let F be an archimedean f-ring with identity 1. Then by Bernau [1], F is a subring of $D(X)$ (1 is the identity of $D(X)$) where X is the Stone space associated with l-group $(F, +)$. Moreover, $P(F) = \{d \in D(X) : Fd \subseteq F\}$ and since $1 \in F$ it follows that $P(F) = F$. This together with 3.3 proves

3.4 THEOREM. *The class $\{P(G) : G$ is an archimedean l-group$\}$ is identical with the class of archimedean f-rings with identity.*

We now turn our attention to some consequences of 3.2. We shall, of course, assume that G is an archimedean l-group.

3.5. (i) *For $\alpha, \beta \in P^+(G)$ and $g \in G^+$, $g(\alpha \vee \beta) = g\alpha \vee g\beta$ and $g(\alpha \wedge \beta) = g\alpha \wedge g\beta$.*

(ii) *For $\alpha, \beta \in P^+(G)$ and $g \in G^+\backslash\{0\}$, $g\alpha\beta = 0$ if and only if $g\alpha \wedge g\beta = 0$; and hence $\alpha\beta = 0$ if and only if $\alpha \wedge \beta = 0$.*

(iii) *For $\alpha \in P^+(G)$ and $x \in G\backslash\{0\}$, $x\alpha^n = 0$ if and only if $x\alpha = 0$; and hence $\alpha^n = 0$ implies $\alpha = 0$.*

3.6. *If* $\alpha \in P^+(G)$ *and* α *is onto, then* α *is one-to-one and* $\alpha^{-1} \in P^+(G)$.

Proof. Let $y \in G$ be non-zero. Then since α is onto, there is an $x \in G$ such that $x\alpha = y \neq 0$. But then by 3.5 (iii) $0 \neq x\alpha^2 = y\alpha$ so that α is one-to-one. The rest is routine.

3.7. *If* $\alpha \in P^+(G)$, *then the subgroup of* G *generated by* Ker (α) *and* $G\alpha$ *is the cardinal sum* Ker $(\alpha) \oplus G\alpha$ *and it is an* l*-subgroup of* G. *In particular, if* $\alpha^2 = \alpha$, *then* $G =$ Ker $(\alpha) \oplus G\alpha$.

Proof. First we show that the sum of an l-ideal B and an l-subgroup C of G is an l-subgroup of G. To this end let $x = b + c \in B + C$. Then $B + x = B + c$ and so

$$B + (x \vee 0) = (B + x) \vee B = (B + c) \vee B = (B + c) \vee B = B + c \vee 0$$

since B is an l-ideal of G. Thus $x \vee 0 = b' + c \vee 0$ for some $b' \in B$. Thus $x \vee 0 \in B + C$, and $B + C$ is an l-subgroup of G.

Now $G\alpha$ is an l-subgroup of G and Ker (α) is an l-ideal of G, so that Ker $(\alpha) + G\alpha$ is an l-subgroup of G. Now let $y \in$ Ker $(\alpha) \cap G\alpha$, and let $x \in G$ be such that $y = x\alpha$. Then $0 = y\alpha = x\alpha^2$ so that $x\alpha = 0$ by 3.5 (iii) and hence $y = 0$. Thus Ker $(\alpha) + G\alpha =$ Ker $(\alpha) \oplus G\alpha$. To show that Ker $(\alpha) \oplus G\alpha =$ Ker $(\alpha) \oplus G\alpha$ it suffices to show that $G\alpha$ is convex in Ker $(\alpha) \oplus G\alpha$. To this end suppose that $0 < x < y\alpha$ where $x = a + b\alpha$ with $a \in$ Ker (α) and $b \in G$. Suppose that $a \neq 0$. Then $-b\alpha < a < (y - b)\alpha$, so that

$$((-b) \vee 0)\alpha \leq a \vee 0 \leq ((y - b) \vee 0)\alpha$$

and

$$((-b) \wedge 0)\alpha \leq a \wedge 0 \leq ((y - b) \wedge 0)\alpha.$$

Thus, since $a \vee 0$ and $a \wedge 0$ are in Ker (α), there is a $c \in$ Ker (α) and an $h \in G^+$ such that $0 < c < h\alpha$. If M is a minimal prime l-ideal of G such that $c \notin M$, then $M + c \leq M + h\alpha$ so that $h\alpha \notin M$. Now since G is archimedean, there is a positive integer n such that $nc \nleq h$. Hence there is a minimal prime l-ideal N of G such that $N + nc > N + h$. Now α induces an o-endomorphism $\bar{\alpha}$ of G/N and we have that $N + nc\alpha \geq N + h\alpha$ or $N \geq N + h\alpha$ since $c \in$ Ker (α). Thus $h\alpha \in N$ since $N \geq N + h\alpha \geq N$. But $c \notin N$ since $N + nc > N + h$ and $h \geq 0$. This contradicts the fact that any minimal prime l-ideal not containing c also does not contain $h\alpha$, and we are done.

3.8. *If* $\alpha \in P^+(G)$, *then* α *preserves all existing infima and suprema.*

Proof. This follows from the fact that in any archimedean f-ring F, $y = \wedge \{y_\gamma : \gamma \in \Gamma\}$ and $\alpha \in F^+$ imply $y\alpha = \wedge \{y_\gamma \alpha : \gamma \in \Gamma\}$. (See [1] or [7].)

3.9. *If* $\alpha \in P^+(G)$, *then* Ker (α) *is a polar and* $G/$Ker (α) *is archimedean.*

Proof. By 3.8 it is clear that Ker (α) is closed. Hence Ker (α) is a polar [6] and so $G/$Ker (α) is archimedean [5].

3.10. *If G is complete, then $P(G)$ is complete.*

Proof. Let $\{\alpha_\gamma : \gamma \in \Gamma\}$ be an upward directed subset of $P^+(G)$ which is bounded above by $\alpha \in P^+(G)$. Then for $g \in G^+$, $g\alpha_\gamma \leq g\alpha$ for all $\gamma \in \Gamma$. Thus we set $g\beta' = \bigvee \{g\alpha_\gamma : \gamma \in \Gamma\}$. But then for $g, h \in G^+$, it follows since $\{\alpha_\gamma : \gamma \in \Gamma\}$ is directed, that $(g + h)\beta' = g\beta' + h\beta'$, and hence there is an o-endomorphism β of G such that $g\beta = g\beta'$ for $g \in G^+$. Now if $g \wedge h = 0$, then

$$g \wedge h\beta = g \wedge (\bigvee \{h\alpha_\gamma : \gamma \in \Gamma\} = \bigvee \{g \wedge h\alpha_\gamma : \gamma \in \Gamma\} = 0$$

and it follows that $\beta \in P^+(G)$ and hence $\beta = \bigvee \{\alpha_\gamma : \gamma \in \Gamma\}$. Similarly, the infimum of a downward directed subset of $P^+(G)$ has an infimum, and it follows that $P(G)$ is complete.

Remark. If $P(G)$ is complete, G need not be complete. See Example 5 of §8.

We close this section with some results about relationship between $P(G)$ and $P(\hat{G})$ where $\hat{G}$ is the completion of the archimedean l-group G. First we prove

3.11. *Let $\hat{G}$ denote the completion of the archimedean l-group G, and let α be an endomorphism of G which preserves all existing infima and suprema. Then there is a unique extension of α to an endomorphism $\hat{\alpha} : \hat{G} \to \hat{G}$ which preserves all existing infima and suprema.*

Proof. Let $h \in \hat{G}$. Then $h = \bigvee \{k \in G : k \leq h\} = \bigwedge \{k \in k : h \leq k\}$. Define $h\bar{\alpha} = \bigvee \{k\alpha : k \in G \text{ and } k \leq h\}$. This supremum exists since α preserves order and the set $\{k \in G : k \leq h\}$ is bounded above by any element of G larger than h. Now let $h' \in \hat{G}$. Then

$$\begin{aligned} h\hat{\alpha} + h'\hat{\alpha} &= \bigvee \{k\alpha : k \in G \text{ and } k \leq h\} + \bigvee \{k'\alpha : k' \in G \text{ and } k' \leq h'\} \\ &= \bigvee \{(k + k')\alpha : k, k' \in G, k \leq h \text{ and } k' \leq h\} \leq (h + h')\alpha. \end{aligned}$$

To see the reverse inequality, let $x \in G$ be such that

$$x \leq h + h' = \bigvee \{k : k \in G \text{ and } k \leq h\} + \bigvee \{k' : k' \in G \text{ and } k' \leq h'\}.$$

Then

$$x = \bigvee \{(k + k') \wedge x : h, k' \in G, k \leq h, \text{ and } k' \leq h'\},$$

and

$$\begin{aligned} x\alpha &= \bigvee \{(k\alpha + k'\alpha) \wedge x\alpha : k, k' \in G, k \leq h, \text{ and } k' \leq h'\} \\ &\leq \bigvee \{k\alpha + k'\alpha : k, k''G, k \leq h, \text{ and } k' \leq h'\} = h\hat{\alpha} + h'\hat{\alpha}. \end{aligned}$$

Since $h + h'$ is the supremum of all such x's, we have $(h + h')\hat{\alpha} \leq h\hat{\alpha} + h'\hat{\alpha}$. Thus $\hat{\alpha}$ is an endomorphism of $\hat{G}$. The rest is straightforward.

3.12 Proposition. *Let G be an archimedean l-group, and let $\alpha \in P^+(G)$. Then there is a unique extension of α to $\hat{\alpha} \in P^+(\hat{G})$. Moreover, α is one-to-one if and only if $\hat{\alpha}$ is one-to-one; and $\hat{\alpha}$ is onto if α is onto.*

Proof. Let $\alpha \in P^+(G)$. Then by 3.8 and 3.11, there is an extension of α to $\hat{\alpha} : \hat{G} \to \hat{G}$. Now let $h, h' \in \hat{G}$ with $h \wedge h' = 0$. Then

$$h\hat{\alpha} = \bigvee \{k\alpha : k \in G^+ \text{ and } k \leq h\},$$

so that

$$h\hat{\alpha} \wedge h' = \bigvee \{k\alpha : k \in G^+ \text{ and } k \leq h\} \wedge h' = \bigvee \{k\alpha \wedge h' : k \in G^+ \text{ and } k \leq h\}$$
$$= \bigvee \{k\alpha \wedge k' : k, k' \in G^+, k \leq h, \text{ and } k' \leq h'\} = 0.$$

Thus $\hat{\alpha} \in P^+(\hat{G})$.

Now suppose that α is one-to-one on G, and let $g \in \hat{G}^+ \backslash \{0\}$ be such that $g\hat{\alpha} = 0$. Then there is a $g' \in G^+ \backslash \{0\}$ such that $0 < g' < g$. But then $0 \leq g'\hat{\alpha} \leq g\hat{\alpha} = 0$ so that $g'\alpha = 0$. Thus $\hat{\alpha}$ is one-to-one on $\hat{G}$.

Finally suppose that α is onto, and let $h \in \hat{G}$ with

$$h = \bigvee \{k \in G : k \leq h\} \leq g$$

where $g \in G$. Since α is onto, α is one-to-one by 3.6 and hence α is an l-automorphism of G. Now $k \leq g$ implies $k\alpha^{-1} \leq g\alpha^{-1}$, so that

$$\bigvee \{k\alpha^{-1} : k \in G \text{ and } k \leq h\} \in G.$$

Thus

$$[\bigvee \{k\alpha^{-1} : k \in h \text{ and } k \leq h\})\hat{\alpha} = \bigvee \{k\alpha^{-1}\hat{\alpha} : k \in G \text{ and } k \leq h\} = h$$

so that $\hat{\alpha}$ is onto.

Remark. Example 11 of §8 shows that $\hat{\alpha} \in P^+(\hat{G})$ can be onto without $\alpha \in P^+(G)$ being onto.

As usual let G be an archimedean l-group, and $\hat{G}$ its completion. Then it follows from 3.10 that $P_2 \equiv P(\hat{G})$ is a complete f-ring with identity, and it follows from 3.11 that there is a natural embedding of $P_1 = P(G)$ into P_2. Example 5, §8 shows that P_2 need not be the completion of the archimedean f-ring P_1. However we are able to show

3.13 THEOREM. *If G is a divisible archimedean l-group with a strong order unit and a basis, then $P_2 = P(\hat{G})$ is the completion of $P_1 = P_1(G)$.*

Proof. We may assume that $P_1 \subseteq P_2$. Thus since P_2 is complete, it is sufficient to show that for $\beta \in P_2^+ \backslash \{0\}$, there are $\alpha, \gamma \in P_1$ such that $0 < \alpha < \beta < \gamma$ (See [6]).

Now since G is divisible and has a basis, we may assume that

$$\sum \{R_\lambda : \lambda \in P\} \subseteq G \subseteq \prod \{R_\lambda : \lambda \in P\}$$

where $\{R_\lambda : \lambda \in P\}$ is a family of divisible subgroups of the reals. Then [4] $\hat{G}$ is the l-ideal of $\prod \{T_\lambda : \lambda \in \Gamma\}$ ($T_\lambda = R$ for all $\lambda \in \Gamma$) generated by G and

$$\sum \{T_\lambda : \lambda \in \Gamma\} \subseteq \hat{G} \subseteq \prod \{T_\lambda : \lambda \in \Gamma\}.$$

By Theorem 2.8, $\beta \in P_2^+ \backslash \{0\}$ has the form $(\cdots, \beta_\lambda, \cdots)$ where $\beta_\lambda \in R^+$. Suppose that $\beta_\lambda > 0$ and choose n so that $1/n < \beta_\lambda$ and let

$$\alpha = (0, \cdots, 0, 1/n, 0, \cdots, 0)$$

where the $1/n$ is in the λ-*th* place. Then $\alpha \in P_1$ since $G \supseteq \sum \{R_\lambda : \lambda \in \Gamma\}$ and each R_λ is divisible. Moreover, $0 < \alpha < \beta$. Finally, we can assume without loss of generality that $(1, 1, \cdots)$ is a strong order unit for G and hence for $\hat{G}$. But then $(1, 1, \cdots)\beta = (\cdots, \beta_\lambda, \cdots)$ and the set $\{\beta_\lambda : \lambda \in \Gamma\}$ is bounded, by n say. Thus if we set $x\gamma = nx$ for $x \in G$, we have a $\gamma \in P_1$ such that $\beta < \gamma$.

Remarks. 3.13 fails without the assumption of divisibility (See Example 5 of §8). However, we do not know if any of the other assumptions can be omitted.

Note that the intersection of all the laterally complete l-subgroups of $D(X)$ that contain G is the lateral completion of G (see [5]). In particular, each p-endomorphism α of G has a unique extension to a p-endomorphism of the lateral completion of G.

4. A characterization of $P^+(G)$

In this section we prove a single theorem. It shows, for an archimedean l-group G, that the largest f-ring of $B(G)$ containing the identity is $P(G)$. Precisely

4.1 THEOREM. *Let G be an archimedean l-group, and let F be a subring of the ring $B(G)$ of ordered-bounded endomorphisms of G containing the identity* 1. *Moreover, suppose that F is an f-ring where the order is given by the cone $F^+ = B(G)^+ \cap F$. Then $F^+ \subseteq P^+(G)$ and hence* (See 3.3)*$P(G)$ is the largest sub-po ring of $B(G)$ which is an f-ring and contains* 1.

Proof. Consider G as an l-group of almost finite extended real-valued continuous functions on the Stone space X of the Boolean algebra of polars as in §3. Then it is sufficient to show that the support of $g\rho$ is contained in the support of g for each $g \in G^+$ and each $\rho \in F^+$.

To this end suppose that there is an $z \in$ support $(g\rho)$ which is not in support (g). Then there is a neighborhood V_1 of z such that $g(V_1) = \{0\}$. However, since $z \in$ support $(g\rho)$, there is a $y \in V_1^0$ such that $(g\rho)(y) > 0$. Since $g\rho$ is continuous, there is a neighborhood V_2 of y contained in V_1 such that $(g\rho)(\omega) > 0$ for each $\omega \in V_1$. It follows that there is $z_1 \in X$ and a neighborhood V_3 of z_1 such that $g(V_3) = \{0\}$ and $(g\rho)(\omega) > 0$ for all $\omega \in V_3$. Now $g\rho$ is finite on an open dense subset of X and hence there is a $y \in V_3$ such that $0 < (g\rho)(y) + \infty$. Again, it follows that there is a $z_2 \in X$ and a neighborhood V_4 of z_2 such that $g(V_4) = \{0\}$, and $0 < (g\rho)(\omega) < +\infty$ for all $\omega \in V_4$. Also we have that $(g\rho^2)(y) < +\infty$ for some $y \in V_4$. Consequently there is an $x \in X$ such that $g(x) = 0$, $0 < (g\rho)(x) < +\infty$, and $0 \leq (g\rho^2)(x) < +\infty$.

Pick an integer $n > 0$ such that $ng\rho(x) > g\rho^2(x)$ and let $\delta = \rho \wedge n1$. Then $0 = (\rho - \delta) \wedge (n1 - \delta)$ so that $(\rho - \delta)(n1 - \delta) = 0$ since $\rho - \delta$ and $n1 - \delta$ are disjoint elements of the f-ring F. We obtain a contradiction to our assumption that support $(g\rho)$ is not contained in support (g) by showing that $g(\rho - \delta)(n1 - \delta) \neq 0$. To this end note that

$$g(\rho - \delta)(x) = (g\rho(x) - (g\delta)(x) = (g\rho)(x)$$

since $0 \leq \delta \leq n1$ and $g(x) = 0$. Thus $g(\rho - \delta) = h$ where $0 < h < g\rho$ and $h(x) = g\rho(x)$. Moreover, $hn1(x) = ng\rho(x)$ and $h\delta(x) \leq h\rho(x) \leq g\rho^2(x) < ng\rho(x)$. Thus

$$g(\rho - \delta)(n1 - \delta)(x)$$
$$= h(n1 - \delta)(x) = hn1(x) - h\delta(x) = ng\rho(x) - h\delta(x) > 0,$$

and we're done.

Remark. By a similiar proof, one can show the following result. Theorem. Let G be an archimedean l-group and let F be a subring of $B(G)$ which contains 1, is an l-ring in the partial order $F^+ = B(G)^+ \cap F$, and satisfies: $\alpha, \beta \in F$ and $g \in G^+$ imply $g(\alpha \wedge \beta) = g\alpha \wedge g\beta$. Then $F^+ \subseteq P^+(G)$.

5. The additive subgroup of an archimedean f-ring with identity

The object of this section is to prove

(5.1 THEOREM. *Let* $(G, +)$ *be an archimedean l-group with weak order unit* e. *Then* (a) *there is at most one multiplication on* G *so that* $(G, +, \cdot)$ *is an f-ring with identity* e, *and* (b) *such a multiplication exists if and only if*

$$\{e\alpha : \alpha \in P^+(G)\} = G^+.$$

Before proving 5.1 we lay some groundwork. Let G be an archimedean l-group with a weak order unit e, and let $eP^+(G) = \{e\alpha : \alpha \in P^+(G)\}$. For $\alpha, \beta \in P^+(G)$, let $e\alpha + e\beta = e(\alpha + \beta)$, $e\alpha \wedge e\beta = e(\alpha \wedge \beta)$ and $e\alpha \vee e\beta = e(\alpha \vee \beta)$. Then $eP^+(G)$ is a subsemigroup and a sublattice of G^+. Now for $e\alpha, e\beta \in P^+(G)$, define $(e\alpha)(e\beta) = e\alpha\beta$. Then since e is a weak order unit, it follows from 3.2 that if $e\mu = e\nu$ where $\mu, \nu \in P^+(G)$, then $\mu = \nu$. Hence the multiplication $(e\alpha)(e\beta) = e\alpha\beta$ is well defined. It follows that $eP^+(G)$ is a semiring. Let G^e denote the subgroup of G generated by $eP^+(G)$. Then

$$G^e = \{e\alpha - e\beta : \alpha, \beta \in P(G)\} = \{e\mu - e\nu : \mu, \nu \in P(G) \text{ and } e\mu \wedge e\nu = 0\}.$$

For if $\alpha, \beta \in P(G)$, then $e\alpha = e\alpha \wedge e\beta + s$ and $e\beta = e\alpha \wedge e\beta + t$ where $s \wedge t = 0$. Thus

$$s = e\alpha - e(\alpha \wedge \beta) = e(\alpha - (\alpha \wedge \beta)) = e\mu \quad \text{where } \mu = \alpha - (\alpha \wedge \beta) \in P(G)$$

and

$$t = e\beta - e(\alpha \wedge \beta) = e(\beta - (\alpha \wedge \beta)) = e\nu \quad \text{where } \nu = \beta - (\alpha \wedge \beta) \in P(G).$$

Thus $e\alpha - e\beta = e\mu - e\nu$. In particular, $e\mu = (e\alpha - e\beta)^+$ and $e\nu = (e\alpha - e\beta)^-$. Thus $(e\alpha - e\beta)^+, (e\alpha - e\beta)^- \in eP^+(G)$, so that G^e is an l-subgroup of G with positive cone $eP^+(G)$. Now by linearity, one can extend the multiplication on $eP^+(G)$ to G^e, and it then follows easily that the map $\alpha \to e\alpha$ from $P(G)$ onto G^e is an l-isomorphism. Thus $P(G)$ is l-isomorphic to G^e. Thus if $G^+ = eP^+(G)$, we have that $(G, +)$ can be made into an f-ring with e as identity. This proves one way of part (b) of 5.1. To see the converse of part b), let $g \in G^+$ and assume that $(G, +)$ is an f-ring with e as identity. Then since the map $x \to xg$ is a p-endomorphism and $eg = g$, we have that $eP^+(G) = G^+$. This completes the proof of part (b) of 5.1.

To see part (a) let $\circ$ and $\cdot$ be two multiplications on G making G^+ into an archimedean f-ring with identity e. Then for $g \in G^+ \backslash \{0\}$, the map $x \to x \circ g$ and $x \to x \cdot e$ are p-endomorphisms of G. But a p-endomorphism is determined by its action on a weak order unit and since $e \circ g = g = e \cdot g$, we have that $x \circ g = x \cdot g$ for all $x \in G$ and $g \in G^+$. It follows that $x \circ y = x \cdot y$ for all $x, y \in G$ and we're done.

This theorem is more or less "well known" (see [8]). The novelty is the discription in terms of $P(G)$.

6. Contractors on archimedean l-groups

Previously we have considered those endomorphisms of an l-group G which leave invarient all minimal prime subgroups of G (see 2.1). We now turn our attention to those endomorphisms of G which leave invarient all prime subgroups of G.

6.1 Definition. *A contractor α on an l-group G is a group endomorphism α of G such that $G^+\alpha \subseteq G^+$ and for each $g \in G^+$ there is an integer $n = n(g)$ such that $g\alpha \leq ng$.*[3]

6.2 Proposition. *Let G be an l-group and let α be an endomorphism of G such that $G^+\alpha \subseteq G^+$. Then the following are equivalent:*

(i) *α is a contractor on G;*

(ii) *α leaves invarient each convex l-subgroup of G; and*

(iii) *α leaves invarient each prime subgroup of G.*

Proof. That (i) is equivalent to (ii) is clear. That (ii) is equivalent to (iii) follows from the fact [4] that each convex l-subgroup is the interesection of prime convex l-subgroups of G.

Note that the set $C^+(G)$ of contractors on G is closed under multiplication and if G is abelian. it is closed under addition. Thus for abelian l-groups we have that

$$C(G) = \{\alpha - \beta : \alpha, \beta \in C^+(G)\}$$

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[3] Langford's contractors [10] are our contractors with $n(g) = 1$ for all $g \in G$. Langford asks if contractors on archimedean l-groups commute. Since each contractor is a p-endomorphism the answer is yes.

is a po subring of $P(G)$ with positive cone

$$C(G) \cap P^{+}(G) = C^{+}(G).$$

For if $\alpha - \beta \in C(G) \cap P^{+}(G)$ and g belongs to a convex l-subgroup M of G, then $g\alpha$, $g\beta \in M$ and so $g(\alpha - B) = g\alpha - g\beta \in M$. Therefore $\alpha - \beta \in C^{+}(G)$.

Now suppose that G is an archimedean l-group and assume as in §3 that $G \subseteq D(X)$ where X is Stone space of the Boolean algebra of polars on G. Moreover, suppose that G has a weak order unit which we can assume is the constant function 1 in $D(X)$. Now let $\alpha \in C(G)$. Then since α is a p-endomorphism of G, α is multiplication by a function $\bar{\alpha}$ of $D(X)$ (Theorem 3.2). Moreover, there is an integer n such that $0 \leq e\bar{\alpha} \leq ne$ and hence for each $x \in X$ we have that $0 \leq \bar{\alpha}(x) \leq n$. Thus $\bar{\alpha} \in C(X)$, the real-valued continuous functions on X. Consequently, $C^{+}(G)$ is isomorphic to a subring of $C(X)$.

Example 8 of §8 shows that if G does not have a weak order unit, then $C(G)$ need not be contained in $C(X)$. However, we are able to prove

6.3 PROPOSITION. *Let G be an archimedean l-group. Then there is a topological space Y such that $C(G)$ is isomorphic to a subring of $C(Y)$.*

Proof. Let $\{e_\lambda : \lambda \in \Gamma\}$ be a maximal disjoint of subset of G and assume that $G \subseteq D(X)$ where X is the Stone space of the Boolean algebra of polars of G as in §3. Let $\{X_\lambda : \lambda \in \Gamma\}$ be a family of compact open subsets of X such that e_λ is the characteristic function of X_λ.

Let $Y = \bigcup \{X_\lambda : \lambda \in \Gamma\}$ and let $\alpha \in C(G)$. Assume that α is represented by the function $\bar{\alpha} \in D(X)$. (Theorem 3.2) Now let $x \in Y$ and let $\lambda \in \Gamma$ be such that $x \in X_\lambda$. Then there is an integer n such that $0 \leq e_\lambda \bar{\alpha} \leq ne_\lambda$ so that $0 \leq \bar{\alpha}(x) \leq n$. Thus $\bar{\alpha}(x)$ is real for each $x \in X_\lambda$, and hence $\bar{\alpha}$ is real on Y. It follows that $C(G)$ is isomorphic to a subring of $C(Y)$.

Now we show that a contractor α on an archimedean l-group G extends to a contractor $\hat{\alpha}$ on $\hat{G}$, the completion of G. Note that since α is a p-endomorphism, α extends to a p-endomorphism $\hat{\alpha}$ of $\hat{G}$ (see 3.11). Thus we need only show that $\hat{\alpha}$ is a contractor on $\hat{G}$. First we note that if a, b, c, and d are real numbers such that $0 \leq a \leq b$ and $bc \leq db$, then $ac \leq da$. Hence from Theorem 3.2 it follows that if α is a contractor on an archimedean l-group G and if $a, b \in G^{+}$ are such that $0 \leq a \leq b$ and $b\alpha \leq nb$ where n is a positive integer, then $a\alpha \leq na$. Now recall that if $h \in \hat{G}$, then $h = \bigvee \{k : k \in G \text{ and } k \leq h\}$ and $h\hat{\alpha}$ is defined by $h\hat{\alpha} = \bigvee \{k\alpha : k \in G \text{ and } k \leq h\}$. Moreover let $g \in G$ be such that $g \geq h$, and let n be an integer such that $g\alpha \leq ng$. Then for $k \leq h$ we have that $k\alpha \leq nk$ and hence

$$\begin{aligned} h\hat{\alpha} &= \bigvee \{k\alpha : k \in G \text{ and } k \leq h\} \leq \bigvee \{nk : k \in G \text{ and } k \leq h\} \\ &= n \bigvee \{k : k \in G \text{ and } k \leq h\} = nh. \end{aligned}$$

Hence $\hat{\alpha}$ is a contractor on $\hat{G}$. Thus we have proven

6.4 PROPOSITION. *Let G be an archimedean l-group, let α be a contractor on*

G, and let $\hat{G}$ denote the completion of $\hat{G}$. Then if we define $\hat{\alpha} : \hat{G} \to \hat{G}$ by

$$h\hat{\alpha} = \bigvee \{k\alpha : G \text{ and } k \leqq h\}$$

for $h \in \hat{G}$, then $\hat{\alpha}$ is a contractor on $\hat{G}$.

Remark. It can be shown that if $\{a_\lambda : \gamma \in \Gamma\}$ is a disjoint subset of an l-group G, $a = \bigvee a_\gamma$ and α is a contractor, then $a\alpha = \bigvee (a_\gamma \alpha)$.

7. A problem of Birkhoff

In this section E will always denote a complete vector lattice. By a *bounded contractor* on E we shall mean a linear transformation $\alpha : E \to E$ such that $E^+\alpha \subseteq E^+$ and $x\alpha \leq \lambda x$ for some $\lambda \in R^+$ and all $x \in E^+$.

Recall [2] that if E is a complete vector lattice, then the algebra $B(E)$ of order-bounded linear transformations on E is a complete l-algebra; and if $\{T_\lambda : \lambda \in \Gamma\}$ is a subset of $B(E)$ bounded above, then the supremum of $\{T_\lambda : \lambda \in \Gamma\}$ is the linear transformation $T : E \to E$ defined by

$$xT = \bigvee \{xT_\lambda : \lambda \in \Gamma\}$$

for $x \in E^+$.

Now let α be a bounded contractor on E. Then the family

$$\{\textstyle\sum_{n=0}^{k} \alpha^n/n\,! : k = 1, 2, \cdots\}$$

is bounded above by $e^{\lambda}1_E$ where 1_E denotes the identity map of E and $\lambda \in R^+$ is such that $x\alpha \leq \lambda x$ for all $x \in E^+$. Thus the supremum of the family

$$\{\textstyle\sum_{n=0}^{k} \alpha^n/n\,! : k = 1, 2, \cdots\}$$

exists in $B(E)$. We denote this supremum by e^{α}. Birkhoff's problem (Problem 154 of [2]) is now as follows. If α is a bounded contractor on a complete vector lattice E, show that the family $\{e^{t\alpha} : t \in R^+\}$ is a semigroup of operators.

The proof that $\{e^{t\alpha} : t \in R^+\}$ is a semigroup of operators is broken up into a series of steps.

1. As usual let X denote the Stone space of the Boolean algebra of polars of E, and consider E as a sub-vector lattice of $D(X)$ as in §3. Moreover, for $\alpha \in P(E)$, let $\bar{\alpha}$ denote the function in $D(X)$ associated with α as in Theorem 3.2. Thus for $f \in E$, $f\alpha = f\bar{\alpha}$ where the product on the right is the ring multiplication in $D(X)$.

2. Let $f \in D(X)$ and let $R(f) = \{x \in X : f(x) \text{ is real}\}$. Then for each $x \in R(f)$, $[\sum_{n=0}^{k} f^n/n\,!)(x) \leq e^{f(x)}$ and since the map $x \to e^{f(x)}$ is continuous on $R(f)$ (and hence extends to an element of $D(X)$ which we denote by $\exp(f)$), we have that the set $\{\sum_{n=0}^{k} f^n/n\,! : k = 1, 2, \cdots\}$ is bounded above by an element of $D(X)$. Thus the supremum of the set $\{\sum_{n=0}^{k} f^n/n\,! : k = 1, 2, \cdots\}$ exists; we denote it by e^f. Note that $e^f \leq \exp(f)$. Moreover, suppose that $g \in D(X)$ is $\geq \sum_{n=0}^{k} f^n/n\,!$ for each k. Then

$$g(x) \geq \textstyle\sum_{n=0}^{k} f^n(x)/n\,!$$

for each k and each $x \in R(f) \cap R(g)$. It follows that $g(x) \geq e^{f(x)}$ for each $x \in R(g) \cap R(f)$ and hence $g \geq \exp(f)$; and hence $e^f = \exp(f)$. Thus for each $x \in R(f)$, we have that

$$[\bigvee \{\textstyle\sum_{n=0}^{k} f^n/n! : k = 1, 2, \cdots\}](x) = e^{f(x)}.$$

Thus it is clear that for $f, g \in D(X)$, we have that $e^{(f+g)} = e^f e^g$.

3. If α is a bounded contractor on E, then e^α is also a bounded contractor by the remarks in the first paragraph of this section. Now for $g \in E^+$ we have that

$$ge^\alpha = g\bigvee \{\textstyle\sum_{n=0}^{k} \alpha^n/n! : k = 1, 2, \cdots\} = \bigvee \{\textstyle\sum_{n=0}^{k} g\alpha^n/n! : k = 1, 2, \cdots\}$$
$$= \bigvee \{\textstyle\sum_{n=0}^{k} g\bar{\alpha}^n/n! : k = 1, 2, \cdots\} = \bigvee \{\textstyle\sum_{n=0}^{k} \bar{\alpha}^n/n!)g : k = 1, 2, \cdots\}.$$

But by 3.8,

$$\bigvee \{(\textstyle\sum_{n=0}^{k} \bar{\alpha}^n/n!)g : k = 1, 2, \cdots\}$$
$$= [\bigvee \{\textstyle\sum_{n=0}^{k} \bar{\alpha}^n/n! : h = 1, 2, \cdots\}]g = e^{\alpha}g.$$

Thus

$$ge^\alpha = e^{\bar{\alpha}}g$$

and hence we have that

$$\overline{e^\alpha} = e^{\bar{\alpha}}.$$

4. For α and β bounded contractors on E, we have that $\alpha + \beta$ is a bounded contractor and

$$e^{(\alpha+\beta)} = e^{\overline{\alpha+\beta}} = e^{\bar{\alpha}+\bar{\beta}} = e^{\bar{\alpha}}e^{\bar{\beta}} = \overline{e^\alpha}\,\overline{e^\beta} = \overline{e^\alpha e^\beta}$$

so that $e^{\alpha+\beta} = e^\alpha e^\beta$.

We thus have proven

7.1 THEOREM. *Let E be a complete vector lattice and let α be a linear transformation such that $E^+\alpha \subseteq E^+$ and $x\alpha \leq \lambda x$ for some $\lambda \in R^+$ and all $x \in E^+$.* Then the family $\{e^{t\alpha} : t \in R^+\}$ is a semigroup of operators where

$$e^{t\alpha} = \bigvee \{\textstyle\sum_{n=0}^{k} t^n \alpha^n/n! : k = 1, 2, \cdots\}.$$

8. Examples

In examples (1), (2), (3) and (4), let $G = R \oplus R$ be the cardinal sum of two copies of the reals R. Then each o-endomorphism of G is linear,

$$B(G)^+ = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \geq 0\}$$

and $B(G)$ is the full ring of linear transformations on G. Moreover,

$$P^+(G) = C(G) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) : a, d \in R^+\}.$$

1. Let $\tilde{S} = \{\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) : a \in R\}$. The $\tilde{S}$ is a sub-f-ring of l-endomorphisms of $B(G)$ not contained in $P(G)$.

2. $S = \{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) : x, y \in R^+\backslash\{0\}\}$ is a subsemiring of l-endomorphisms of $B(G)$ which contains I. But $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in (\tilde{S})^+\backslash S$.

3. Let $\tilde{S} = \{ \left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right) : a, x \in R\}$. Then $\tilde{S}$ is a commutative subring of $B(G)$ which contains 1_G and is generated by the l-endomorphisms $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Now $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \wedge 1_G = 0$, but $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)1_G = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ so that $\tilde{S}$ is not an f-ring. Also $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not an l-endomorphism since $(1, 0)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = (1, 1)$ and $(0, 1)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = (0, 1)$ are not disjoint. Finally $\tilde{S}$ is not real representable since $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^2 = 0$.

4. Let $S = \{ \left(\begin{smallmatrix} x & 0 \\ 0 & 2x+y \end{smallmatrix}\right) : x, y \in R^+\}$. Then S is a subsemiring of $P^+(G)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1_G \in \tilde{S}\backslash S$. In particular, $(\tilde{S})^+ \not\subseteq S$.

5. *An example where $P(G)$ is complete and proper subring of $P(\hat{G})$.* Let G be the additive group of all rational numbers with square free denominators. Then clearly $\hat{G} = R$ and so $P(\hat{G}) = R^+$, but $P^+(G) = Z^+$. For if the map $x \to kx$ belongs to $P^+(G)$, then $1 \to k$ so that $k \in G$ and also $k^2 \in G$ since $k \to k^2$. Now write $k^2 = m^2/p_1^{2e_1} \cdots p_n^{2e_n}$ where p_{e_i} are primes and m is an integer. Then it follows since $k^2 \in G$ that k must be an integer and hence $P^+(G) = Z^+$.

6. *A non-archimedean o-group G for which $P^+(G)$ is commutative.* Let $G = Z \oplus Q$ where $(n, q) \leq (0, 0)$ if $q > 0$ or $q = 0$ and $n \geq 0$. Then all o-endomorphisms commute since o-endomorphisms are of the form $(n, q) \to (kn, pq)$ where $k \in Z^+$ and $p \in Q^+$.

7. *A contractor which is an l-automorphism but whose inverse is not a contractor.* Let G be the direct product of countably many copies of the reals. For $(x_1, x_2, \cdots) \in G$, let $(x_1, x_2, \cdots)\alpha = (x_1, x_2/2, x_3/3, \cdots)$. Then

$$(1, 1, \cdots)\alpha^{-1} = (1, 2, 3, \cdots) \not\subseteq n(1, 1, 1 \cdots)$$

for any integer n.

8. *A contractor in $D(X)\backslash C(X)$.* Let G be the direct sum of countably many copies of the reals and let $\rho = (1, 2, 3, \cdots)$. Then since the Stone space X of the Boolean algebra of polars of G is the Stone-Cech compactification of the rational numbers, it is clear that $\rho \in D(X)\backslash C(X)$.

9. *An archimedean l-group, a p-endomorphism α, and a prime l-ideal P such that $P\alpha \not\subseteq P$.* Let G be the direct product of countably many copies of the integers and let α be the p-endomorphism given by $(1, 2, 3, \cdots)$. Then if $G(1, 1, 1, \cdots)$ denotes the l-ideal generated by $(1, 1, 1, \cdots)$ we have that $(1, 1, 1, \cdots)\alpha \notin G(1, 1, 1, \cdots)$. Thus there is a prime l-ideal P of G containing $G(1, 1, 1, \cdots)$ but not containing $(1, 1, 1, \cdots)\alpha$. Thus $P\alpha \not\subseteq P$.

10. *An o-endomorphism of the l-group of a vector lattice which is not linear.* Let $G = R \oplus R$ where $(x, y) \geq (0, 0)$ if $x > 0$ or $x = 0$ and $y \geq 0$. Then $\left(\begin{smallmatrix} a & f \\ 0 & b \end{smallmatrix}\right)$, where $a, b \in R^+$ and f is a non-linear group endomorphism of R, is a non-linear o-endomorphism of G.

11. *An archimedean l-group G, and a p-endomorphism α of G such that $\hat{\alpha} : \hat{G} \to \hat{G}$ is onto but α is not onto.* Let $G = \sum_{i=1}^{\infty} \oplus Qt^i$ where Q denotes the rationals and t is a transcendental number. Then $G \subseteq R$ and we put the

natural induced order on G. Now $\hat{G} = R$ and the map $\alpha : G \to G$ given by $y \to ty$ is not onto but $\hat{\alpha} : \hat{G} \to \hat{G}$ is onto.

Added in proof. There is some overlap between the theory developed here, and that in A. Bigard and K. Keimel, *Sur les endomorphismes conservant les polaires d'un groupe réticulé archimédien*, Bull. Soc. Math. France, vol. 97 (1969) pp. 381–398.

Their paper was submitted after but published before ours.

REFERENCES

1. S. J. BERNAU, *Unique representation of archimedean lattice groups and normal archimedean lattice rings*, Proc. London Math. Soc., vol. XV (1965), pp. 599–631.
2. G. BIRKHOFF, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., vol. XXV, Amer. Math. Soc., Providence, R. I., 1967.
3. R. BYRD, *Complete distributivity in lattice-ordered groups*, Pacific J. Math., vol. 20 (1967), pp. 423–432.
4. P. CONRAD, *Introduction a la theorie des groupes reticules*, Secretariat Mathematique, Paris, 1967.
5. ———, *Lateral completions of lattice-ordered groups*, Proc. London Math. Soc., vol. 19 (1969), pp. 444–480.
6. P. CONRAD AND D. MCALISTER, *The completion of a lattice ordered group*, J. Australian Math. Soc., vol. 9 (1968), pp. 182–200.
7. L. FUCHS, *Partially ordered algebraic systems*, Pergamon, London, 1963.
8. L. KANTOROVITCH, B. VULICH AND A. PINSKER, *Functional analysis in partially ordered spaces*, Gostekhizdat, Moscow, 1950.
9. D. JOHNSON AND J. KIST, *Prime ideals in vector lattices*, Canadian J. Math., vol. 14 (1962), pp. 517–528.
10. E. LANGFORD, *Some results on linear operators on lattice groups*, Amer. Math. Monthly, vol. 72 (1965), pp. 841–846.
11. F. SIK, *Theory of partially ordered groups*, Czech. J. Math., vol. 81 (1956), pp. 1–25.

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