FIXED-POINT THEOREMS FOR CERTAIN NONLINEAR NONEXPANSIVE MAPPINGS

BY

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1. Introduction

Let X be a Banach space, K a bounded closed and convex subset of X, and T a *nonexpansive mapping* of K into itself (thus $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in K$). In this paper we treat the problem of determining what conditions on T will assure existence of fixed point for T when K is weakly compact.

Rather extensive study of fixed-point theory for nonlinear nonexpansive mappings was initiated with the proof that if K is a bounded closed convex subset of a uniformly convex Banach space X (or more generally, if X is reflexive and K possesses "normal structure"), then a nonexpansive mapping $T: K \to K$ always has a fixed point (see Browder [3], Göhde [6], Kirk [7]). In order to remove the geometric assumptions (either uniform convexity of X or normal structure of K) so that fixed point theorems for nonexpansive mappings might be obtained which hold in *all* reflexive spaces, additional assumptions must be made about the mapping T. This motivated the introduction of the concept of "diminishing orbital diameters" in Belluce-Kirk [1].

Let $A \subset X$ and F a mapping of A into itself. For $x \in A$, let

$$O(F^n x) = \{F^n x, F^{n+1} x, F^{n+2} x, \cdots\}, \quad n = 0, 1, 2, \cdots$$

(where $F^0 x = x$), and for $B \subset A$, let

$$\delta B = \sup \{ \| x - y \| : x, y \in B\}$$

denote the *diameter* of B. A point $z \in B$ is called a *nondiametral* point of B if $\sup \{ \| z - y \| : y \in B\} < \delta B$.

Definition [1]. The mapping $F: A \to A$ is said to have *diminishing orbital diameter* (d.o.d.) at $x \in A$ if either $\delta O(x) = 0$ or

$$\lim_{n \to \infty} \delta O(F^n x) < \delta O(x).$$

F has *diminishing orbital diameters* on A if F has d.o.d. at each point of A.

Nonexpansive mappings which have diminishing orbital diameters include mappings $f: A \to A$ which satisfy:

(I) *For each $x \in A$ there is a number $\alpha(x) < 1$ such that*

$$\| Fx - Fy \| \leq \alpha(x) \| x - y \|$$

for each $y \in A$.

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This assumption, which is a weakening of the classical Banach Contraction Mapping condition, arises naturally in the theory of Frechét differentiable functions. For example, suppose A is a bounded convex open set and suppose $F: A \to A$ has a Frechét derivative $F'(x)$ at each point $x \in A$. Then if for each $x \in A$ the bounded linear operator $F'(x)$ has norm less than 1, it follows that Condition (I) is satisfied in A. This observation is contained in Kirk [9] where fixed point theorems for mappings satisfying (I) are shown to yield existence of solutions of a certain nonlinear functional equation.

The strength of the condition of diminishing orbital diameters is illustrated in the following general fixed point theorem which holds for such mappings.

THEOREM [8]. *Let K be a nonempty weakly compact subset of a Banach space X, and let $T: K \to K$ be nonexpansive and have d.o.d. on K. Then T has a fixed point in K.*

The original version of this theorem appeared as a corollary in Belluce-Kirk [1] for K convex, and convexity of K was removed later by an alteration of the original argument. For weakly compact convex K it is possible to weaken the assumptions on T in the above theorem. It has been shown [8] that for such K one need only assume that the mapping T^N has d.o.d. on K for some *fixed* positive integer N, and an effort to generalize this result further gave rise to a rather natural question [8]; namely, whether a nonexpansive mapping $T: K \to K$ always has a fixed point if it has the property that for each $x \in K$ there exists an integer n, *depending on* x such that T^n has d.o.d. at x. (A pointwise periodic isometry, for example, trivially has this latter property.) The principal result of this paper, Theorem 2.1 is the affirmative answer to this question.

For any nonexpansive mapping $T: K \to K$ the sequence

$$\{ \| T^n x - T^{n+1} x \| \}_{n=1}^{\infty}$$

is nonincreasing. T is said to be *asymptotically regular* (a.r.) on K if

$$\lim_{n \to \infty} \| T^n x - T^{n+1} x \| = 0$$

for each $x \in K$. This class of mappings is discussed in Browder-Petryshyn [4] where the condition of asymptotic regularity, along with other assumptions on T, yields theorems concerning convergence of sequences of successive approximations to fixed points. A connection between asymptotic regularity and results of this paper arises from the fact that if T is a nonexpansive a.r. mapping which has the property that for each $x \in K$ there exists an integer n such that T^n has d.o.d. at x, then T itself has d.o.d. on K. Thus for mappings of this type the weakening of d.o.d. to the assumption of T^n having d.o.d. at each $x \in K$ (n depending on x) yields no greater generality. However, it is true that for such mappings, existence of fixed points can be established under assumptions weaker than d.o.d. These facts are treated in Section 4.

2. Preliminaries

In this section we state the principal theorem of this paper, discuss one of its corollaries, and state a lemma which will be helpful in the proof of this theorem.

THEOREM 2.1. *Let K be a nonempty weakly compact convex subset of a Banach space, X, and let $T : K \to K$ be nonexpansive. Suppose that for each $x \in K$ there exists an integer n, depending on x, such that the mapping T^n has d.o.d. at x. Then T has a fixed point in K.*

With K as in Theorem 2.1 let $T : K \to K$ be nonexpansive and satisfy the condition:

(II) *For each $x \varepsilon K$ there exists a number $\alpha(x) < 1$ and an integer $N(x)$ such that $\| T^N x - T^N y \| \leq \alpha(x) \| x - y \|$ for each $y \in K$.*

Under this assumption it is readily seen that T satisfies the hypothesis of Theorem 2.1: Suppose T satisfies (II) and let $x \in K$. Then for some $\alpha(x) < 1$ and $N = N(x)$,

$$\begin{aligned}\alpha(x)\delta\{x, T^N x, T^{2N} x, \cdots\} &= \alpha(x) \sup_i \| x - T^{iN} x \| \\ &\geq \sup_i \| T^N x - T^{iN} x \| \\ &= \delta\{T^N x, T^{2N} x, \cdots\},\end{aligned}$$

and thus the mapping T^N has d.o.d. at x. This is analogous to the remark in the introduction that the condition (I) implies T has d.o.d. This latter observation was made in [1] where it is also noted that if T satisfies (I) *and if T has a fixed point x_0*, then for each $y \in K$, $\lim_{n\to\infty} T^n y = x_0$. The same conclusion holds if T is nonexpansive and satisfies (II). To see this, suppose T is such a mapping, and let $Tx_0 = x_0$. Then for some number $\alpha(x) < 1$ and some positive integer $N = N(x_0)$ we have, for each $y \in K$,

$$\| T^N y - T^N x_0 \| = \| T^N y - x_0 \| \leq \alpha(x_0) \| y - x_0 \| .$$

Induction yields

$$\| T^{kN} y - x_0 \| \leq [\alpha(x_0)]^k \| y - x_0 \| \qquad (k = 1, 2, \cdots),$$

whence $\lim_{k\to\infty} T^{kN} y = x_0$, and since T is nonexpansive and x_0 fixed, this in turn implies $\lim_{n\to\infty} T^n y = x_0$. The following, then, is a corollary of Theorem 2.1.

COROLLARY. *Let K be a nonempty weakly compact convex subset of a Banach space and suppose $T : K \to K$ is nonexpansive and satisfies (II). Then T has a unique fixed point in K and for each $y \in K$ the sequence of iterates $\{T^n y\}$ converges to this fixed point.*

We will use the following lemma in the proof of Theorem 2.1.

LEMMA. *Let $\mathfrak{A} = \{A_0, \cdots, A_{n-1}\}$ be a family of nonempty subsets of a set*

A. *Suppose f is a mapping of A into itself such that $f(A_i) \subset A_{i+1}$, $0 \le i \le n-1$, (where $A_n = A_0$). Then there is a family of nonempty subsets $\{A_0', \cdots, A_{n-1}'\}$ of A such that $f(A_i') \subset A_{i+1}'$, $0 \le i \le n-2$, $f(A_{n-1}') \subset A_0'$, and such that for all i, j,*

(i) $A_i' \cap A_j' = \emptyset$ *or* $A_i' = A_j'$,
(ii) $A_i' \subset A_i$,
(iii) *if* $A_i' \cap A_j \neq \emptyset$ *then* $A_i' \subset A_j$.

This lemma is doubtless well known and its proof is routine. By letting letting $\mathfrak{T}$ denote the collection of all nonempty subsets of A which are obtained by taking finite intersections of sets of $\mathfrak{A}$, and taking $\mathfrak{T}'$ to be those elements of $\mathfrak{T}$ which are *minimal* with respect to set inclusion, one can easily show that f maps sets in $\mathfrak{T}'$ into sets in $\mathfrak{T}'$. Thus if $A_0' \in \mathfrak{T}$ and $A_0' \subset A_0$, then $A_0' = \bigcap_{i=1}^{k} A_{n_i}$. Select $A_1' \in \mathfrak{T}'$ such that $A_1' \subset \bigcap_{i=1}^{k} A_{n_i+1}$. Since $A_1' \in \mathfrak{T}'$, $A_1' = \bigcap_{i=1}^{p} A_{m_i}$; select $A_2' \in \mathfrak{T}'$ so that $A_2' \subset \bigcap_{i=1}^{p} A_{m_i+1}$. In this manner a sequence $\{A_0', \cdots, A_{n-1}'\}$ may be generated with all the desired properties.

3. Proof of Theorem 2.1

The general line of argument follows the approach of [1] in using the assumptions on T to obtain nondiametral points in some minimal weakly compact T-invariant subset H of K if $\delta H > 0$, and then using this to obtain a weakly compact T-invariant proper subset of H yielding a contradiction.

Since K is weakly compact and convex we may use Zorn's Lemma to obtain a minimal nonempty weakly compact convex subset K_1 of K which is mapped into itself by T. Now let H be a subset of K_1 which is minimal with respect to being nonempty, weakly compact, and invariant under T. Choose $x_0 \in H$. By hypothesis, there exists a positive integer N such that the mapping T^N has d.o.d. at x_0. *Let $F = T^N$ and assume $Fx_0 \neq x_0$.* Then a number r may be chosen so that

$$(*) \qquad \lim_{n\to\infty} \delta O(F^n x_0) < r < \delta\{x_0, Fx_0, F^2x_0, \cdots\}.$$

Define N subsets of K_1 as follows:

$$S_j = \{z \in K_1 : \text{there exists } i_0 \text{ such that } \| z - T^{iN+j}x_0 \| \le r \text{ if } i \ge i_0\}, \qquad 0 \le j \le N-1.$$

Because of $(*)$, S_0 is not empty (i.e., S_0 contains $F^n x_0$ if n is sufficiently large).

Furthermore, it is easy to see that

(i) $T(S_j) \subset S_{j+1}$, $\quad 0 \le j \le N-2$,
(ii) $T(S_{N-1}) \subset S_0$,
(iii) S_j is convex, $\quad 0 \le j \le N-1$.

By (iii), the sets $\{\bar{S}_j\}$ are weakly compact, and we consider the N weakly compact sets

$$H_j = \bar{S}_j \cap H, \qquad 0 \le j \le N - 1.$$

The fact that $T(H) \subset H$, along with (i), (ii), enables us to apply the procedure of the lemma to the family $\{H_j\}_{j=0}^{N-1}$ to obtain a family

$$\{H_0', H_1', \cdots, H_{N-1}'\}$$

of nonempty *weakly compact* sets such that for all i, j, $0 \le i, j \le N - 1$,

(i)$'$ $T(H_i') \subset H_{i+1}'$ (with $H_N' = H_0'$),
(ii)$'$ $H_i' \subset H_i$,
(iii)$'$ $H_i' \cap H_j' = \emptyset$ or $H_i' = H_j'$,
(iv)$'$ if $H_i' \cap H_j \ne \emptyset$ then $H_i' \subset H_j$.

Since $H' = \bigcup_{i=0}^{N-1} H_i'$ is a weakly compact subset of H which is invariant under T, minimality of H implies $H' = H$. Thus for some index k, $x_0 \in H_k'$. By (i)$'$, $T^N(H_k') \subset H_k'$ and thus

$$\{x_0, Fx_0, F^2x_0, \cdots\} \subset H_k'.$$

Now, since $F^ix_0 \in S_0$ for i sufficiently large,

$$H_k' \cap H_0 = H_k' \cap (\bar{S}_0 \cap H) \ne \emptyset.$$

Thus by (iv)$'$, $H_k' \subset H_0 \subset \bar{S}_0$. Let $x \in H_k'$ and $\varepsilon > 0$. Then there exist $x' \in S_0$ such that $\| x - x' \| \le \varepsilon$ and such that for $i \ge i_0$, $\| x' - F^ix_0 \| \le r$. Then $\| x - F^ix_0 \| \le r + \varepsilon$ if $i \ge i_0$. By weak compactness of H_k' there exists a point $t \in H_k'$ such that

$$t \in \bigcap_{i=0}^{\infty} \overline{\text{conv}}\, \{F^ix_0, F^{i+1}x_0, \cdots\}.$$

Since $\| x - F^ix_0 \| \le r + \varepsilon$, $i \ge i_0$, this implies $\| x - t \| \le r + \varepsilon$. But $\varepsilon > 0$ is arbitrary, so $\| x - t \| \le r$. Letting $\bar{U}(p, \rho)$ denote the spherical ball centered at p with radius ρ, we have proved that $t \in \bar{U}(x, r)$ for arbitrary $x \in H_k'$; thus

$$t \in \bigcap_{x \in H_k'} \bar{U}(x, r).$$

This implies $H_k' \subset \bar{U}(t, r)$. Let $G_k = H_k' \cap Z_k$ where

$$Z_k = \{z \in K_1 : H_k' \subset \bar{U}(z, r)\}.$$

Clearly Z_k is closed and convex; hence G_k is weakly compact. Furthermore, letting wc denote "weak closure," minimality of H implies $wcT(H_i') = H_{i+1}'$. (Otherwise the union of the family $\{H_j'\}_{i=0}^{N-1}$ with H_i' replaced by $wcT(H_i')$ whenever $H_j' = H_i'$ would be a T-invariant weakly compact *proper* subset of H.) It follows, then, that if $x \in G_k$ then $T(H_k') \subset \bar{U}(Tx, r)$ which in turn implies

$$H'_{k+1} = wcT(H'_k) \subset \bar{U}(Tx, r)$$

and hence, $Z_{k+1} \cap H'_{k+1} \neq \emptyset$ where

$$Z_{k+1} = \{z \in K_1 : H'_{k+1} \subset \bar{U}(z, r)\}.$$

Continuing this procedure it can be shown that $Z_i \cap H'_i \neq \emptyset$ where

$$Z_i = \{z \in K_1 : H'_i \subset \bar{U}(z, r)\}, \qquad 0 \leq i \leq N - 1.$$

Let $G_i = Z_i \cap H'_i$, $0 \leq i \leq N - 1$. Then if $G_i \cap G_j \neq \emptyset$ it follows that $H'_i \cap H'_j \neq \emptyset$ from which $H'_i = H'_j$ and thus $G_i = G_j$. Also for $0 \leq i \leq N$ (with $G_N = G_0$) we have $T(G_i) \subset G_{i+1}$. To see this, suppose $x \in G_i$. Then, as before, $H'_i \subset \bar{U}(x, r)$ from which

$$H'_{i+1} = wcT(H'_i) \subset \bar{U}(Tx, r).$$

Furthermore, G_k is a proper subset of H'_k because

$$\delta H'_k \geq \delta\{x_0, Fx_0, F^2x_0, \cdots\} > r \geq \delta G_k.$$

The set $G = \bigcup_{i=0}^{N-1} G_i$ is therefore a nonempty ($t \in G_k$) weakly compact *proper* subset of $H = \bigcup H'_i$ which is invariant under T, contradicting the minimality of H. Hence the assumption $T^N x_0 \neq x_0$ leads to a contradiction.

On the other hand, if $T^N x_0 = x_0$, then let

$$M = \{x_0, Tx_0, \cdots, T^{N-1}x_0\}.$$

If $\delta M > 0$ (i.e., if $Tx_0 \neq x_0$), then the closed convex hull $\overline{\text{conv}}\, M$ of M is compact so there is a point $z \in \overline{\text{conv}}\, M$ which is a nondiametral point of $\overline{\text{conv}}\, M$. (See [2] or [5, Lemma 1].) Let $r < \delta M$ be a number such that $M \subset \bar{U}(z, r)$. It is easily shown in this case (cf. [7]) that the set

$$C = \{z \in K_1 : M \subset \bar{U}(z, r)\}$$

is a nonempty closed convex proper subset of K_1 which is invariant under T contradicting the minimality of K_1. This completes the proof.

4. Asymptotically regular maps

In this section we consider mappings $T : K \to K$ which are *asymptotically regular* [4], that is, for which

$$\lim_{n\to\infty} \| T^n x - T^{n+1} x \| = 0, \qquad x \in K.$$

In Theorem 2.1 it is assumed that for each $x \in K$ there is an integer $n = n(x)$ such that T^n has d.o.d. at x; thus if $\delta\{x, T^n x, T^{2n} x, \cdots\} = \delta > 0$, then for some integer i,

$$\delta\{T^{in}x, T^{(i+1)n}x, \cdots\} < \delta.$$

For mappings which are asymptotically regular the sequence $\{T^{in}x\}_{j=1}^{\infty}$ above may be replaced by *any* subsequence of $O(x) = \{T^n x\}_{n=0}^{\infty}$.

THEOREM 4.1. *Let K be a nonempty closed convex weakly compact subset of a Banach space and $T : K \to K$ nonexpansive and asymptotically regular. Suppose that for each $x \in K$, some subsequence of $O(x)$ has diameter less than $\delta O(x)$ when $\delta O(x) > 0$. Then T has a fixed point in K.*

Proof. Assume that K is a minimal nonempty closed convex weakly compact T-invariant set, and suppose $\delta K > 0$. Let $x \in K$. Then for some subsequence $\{p_i\}_{i=1}^{\infty}$ of integers,

$$\delta\{T^{p_1}x, T^{p_2}x, \cdots\} = r < \delta O(x) \leq \delta K.$$

Letting $y = T^{p_1}x$ and $n_i = p_i - p_1$, $i = 1, 2, \cdots$, we have

$$\delta\{y, T^{n_1}y, T^{n_2}y, \cdots\} = r.$$

Choose r_1 so that $r < r_1 < \delta K$ and select an arbitrary finite sequence $\{T^{m_1}y, T^{m_2}y, \cdots, T^{m_k}y\}$ of $\{T^n y\}$. By nonexpansiveness of T,

$$\| T^{m_j}y - T^{n_i+m_j}y \| \leq r, \quad 1 \leq i < \infty, 1 \leq j \leq k.$$

Because T is asymptotically regular there exists an integer i_0 such that if $i \geq i_0$,

$$\delta\{T^{n_i+m_1}y, T^{n_i+m_2}y, \cdots, T^{n_i+m_k}y\} < r_1 - r.$$

Thus $\{T^{n_i+m_1}y, \cdots, T^{n_i+m_k}y\} \subset \bar{U}(T^{m_j}y, r_1)$, $j = 1, \cdots, k$, $i \geq i_0$, and we see that

$$K \cap [\textstyle\bigcap_{j=1}^{k} \bar{U}(T^{m_j}y, r_1)] \neq \emptyset.$$

Consequently, the family of weakly closed sets $\{\bar{U}(T^n y, r_1)\}_{n=1}^{\infty}$ has the finite intersection property in the weakly compact set K, so there exists an element $z \in \bigcap_{n=1}^{\infty} \bar{U}(T^n y, r_1)$ and the set

$$R = \{w \in K : w \in \textstyle\bigcap_{n=t}^{\infty} \bar{U}(T^n y, r_1) \text{ for some integer } t\}$$

is not empty. The closure $\bar{R}$ of R is convex and invariant under T, so by minimality of K, $\bar{R} = K$. It follows that every element $w \in K$ has the property that for arbitrary $\varepsilon > 0$,

$$\textstyle\bigcap_{n=1}^{\infty} \overline{\text{conv}}\, \{T^n y, T^{n+1}y, \cdots\} \subset \bar{U}(w, r_1 + \varepsilon).$$

Weak compactness of K implies that the above intersection is not empty, so we have

$$\textstyle\bigcap_{w \in K} \bar{U}(w, r_1) \neq \emptyset$$

and thus there exists $t \in K$ such that $K \subset \bar{U}(t, r_1)$. Since $r_1 < \delta K$ we conclude that K possesses nondiametral points, and the argument may be continued in the standard way showing that this leads to a contradiction, cf. [1, 7], whence $\delta K = 0$ and the proof is complete.

REFERENCES

1. L. P. BELLUCE AND W. A. KIRK, *Fixed point theorems for certain classes of nonexpansive mappings*, Proc. Amer. Soc., vol. 20 (1969), pp. 141–146.

2. M. S. Brodskii and D. P. Milman, *On the center of a convex set*, Dokl. Akad. Nauk SSSR, vol. 59 (1948), pp. 837–840.
3. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., vol. 54 (1965), pp. 1041–1044.
4. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., vol. 72(1966), pp. 571–575.
5. R. DeMarr, *Common fixed points for commuting contraction mappings*, Pacific J. Math., vol. 13 (1963), pp.1139–1141.
6. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., vol. 30 (1965), pp. 251–258.
7. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, vol. 72 (1965), pp. 1004–1006.
8. ———, *Fixed point theorems for nonexpansive mappings*, Proc. Sympos. Pure Math., vol. 18, Amer. Math. Soc., Providence, R. I., 1970, pp. 162–168.
9. ———, *Mappings of generalized contractive type*, J. Math. Anal. Appl., vol. 32 (1970), pp. 567–572.

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