STABLE MANIFOLDS OF SEMI-HYPERBOLIC FIXED POINTS

BY
J. W. ROBBIN

In this paper we show that the argument given in [2] (see also [3, p. 234]) proving the existence of the stable and unstable manifolds of a hyperbolic fixed point of a smooth map can be sharpened so as to (1) work in the infinite-dimensional case and (2) yield smoothness with respect to parameters. The former improvement is essential for the application of the stable manifold theory to hyperbolic invariant sets as in [5]; the latter is essential for applications of transversality theory as in [1]. The proof goes through for stable manifolds of semi-hyperbolic fixed points, answering affirmatively a question raised in [4] (where the unstable manifold of a semi-hyperbolic fixed point is constructed; I presume that by the time [5] was written Pugh and Hirsch knew the answer also).

For our proof we must assume not only that the given map is smooth but also that its derivatives are (locally) uniformly continuous. We obtain the corresponding smoothness for the stable manifold. This assumption is, of course, vacuous in the finite-dimensional case.

Throughout, our notation is that of [1]. The main theorems of the paper are 4.1, 6.3, and 7.1.

1. Notation and terminology

If $f : X_1 \to X_2$ is a map from a metric space X_1 with metric d_1 to a metric space X_2 with metric d_2, then $L(f)$ denotes the *minimum Lipschitz constant* for f. In other words, $L(f)$ is the infimum of all real numbers K such that $d_2(f(x), f(y)) \leq K\, d_1(x, y)$ for all $x, y \in X_1$. (We set $L(f) = \infty$ if no such K exists.)

If X is a topological space and $f : X \to X$ is a map, a point $x_0 \in X$ is an *attractive fixed point* iff $f(x_0) = x_0$ and for all $x \in X, f^n(x) \to x_0$ as $n \to \infty$. If X is Hausdorff and x_0 is an attractive fixed point of f, then x_0 is the only fixed point of f. The *contraction principle* guarantees the existence of an attractive fixed point when X is a complete metric space and $L(f) < 1$.

If f is a function with values in a Banach space, $\| f \|_0$ denotes the *sup norm* of f; i.e. $\| f \|_0$ is the supremum of the real numbers $\| f(x) \|$ as x ranges over the domain of definition of f.

If $\mathbf{E}$ is a Banach space and r is a positive real number, then $\mathbf{B}^r\mathbf{E}$ denotes the open ball of radius r about the origin; i.e. $\mathbf{B}^r\mathbf{E}$ is the set of all $x \in E$ such that $\| x \| < r$.

Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces. The product space $\mathbf{E} \times \mathbf{F}$ is always given the product norm; i.e.

$$\| (x, y) \| = \max (\| x \|, \| y \|)$$

Received April 11, 1969.

for $x \in \mathbf{E}$ and $y \in \mathbf{F}$. Thus $\mathbf{B}^r(\mathbf{E} \times \mathbf{F}) = \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F}$. $L(\mathbf{E}, \mathbf{F})$ denotes the Banach space of bounded linear maps from $\mathbf{E}$ to $\mathbf{F}$ and for $k = 2, 3, \cdots L_s^k(\mathbf{E}, \mathbf{F})$ denotes the Banach space of all bounded, symmetric k-linear maps from $\mathbf{E}^k$ to $\mathbf{F}$. The Banach space $J^k(\mathbf{E}, \mathbf{F})$ is defined by

$$J^k(\mathbf{E}, \mathbf{F}) = L(\mathbf{E}, \mathbf{F}) \times L_s^2(\mathbf{E}, \mathbf{F}) \times \cdots \times L_s^k(\mathbf{E}, \mathbf{F}).$$

If $U \subseteq \mathbf{E}$ is open and $f : U \to \mathbf{F}$ is a C^k $(k \geq 1)$ map (see [1]), then $j^k f : U \to J^k(\mathbf{E}, \mathbf{F})$ is defined by

$$j^k f(x) = (Df(x), D^2 f(x), \cdots, D^k f(x))$$

for $x \in U$. A map f is *uniformly* C^k iff f is C^k and $j^k f$ is uniformly continuous. If U is bounded and convex, f is uniformly C^k if f is C^k and $D^k f$ is uniformly continuous. This is by the mean value theorem and the fact that a uniformly continuous function defined on a bounded set is bounded. If $\bar{U}$ is compact (so that $\mathbf{E}$ is finite dimensional) and f extends to a C^k function defined on $\bar{U}$, then f is uniformly C^k.

Let $\mathbf{G}$ be a Banach space, $Z \subseteq \mathbf{G}$ be an open set, and $f : Z \to \mathbf{G}$ be a map. The *local stable manifold* of f is denoted by $W^s(f)$ and is defined to be the set of all points $z \in Z$ such that $f^n(z) \in Z$ for all $n = 0, 1, 2, \cdots$. Note that $f(W^s(f)) \subseteq W^s(f)$ by definition.

2. Stable manifolds for Lipschitz maps

Throughout §§2 and 3, $\mathbf{E}$ and $\mathbf{F}$ denote Banach spaces, $\mathbf{G}$ denotes the product space $\mathbf{G} = \mathbf{E} \times \mathbf{F}$, and r is a positive real number. Recall that $\mathbf{B}^r\mathbf{G} = \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F}$.

2.1. THEOREM. *Let $f : \mathbf{B}^r\mathbf{G} \to \mathbf{G}$ be of the form $f = (\varphi, \psi)$ where*

$$\varphi : \mathbf{B}^r\mathbf{G} \to \mathbf{E} \quad \textit{and} \quad \psi : \mathbf{B}^r\mathbf{G} \to \mathbf{F}.$$

Suppose ψ has the form

$$\psi(x, y) = By + S(x, y)$$

for $(x, y) \in \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F} = \mathbf{B}^r\mathbf{G}$ where $B \in L(\mathbf{F}, \mathbf{F})$ and $S : \mathbf{B}^r\mathbf{G} \to \mathbf{F}$. Assume B is invertible and let e be a real number with $0 < e \leq 1$. Assume further that

(1) $\varphi(B^r\mathbf{G}) \subseteq B^r\mathbf{E}$,
(2) $\| B^{-1} \| (r + \| S \|_0) < r$,
(3) $\| B^{-1} \| (eL(\varphi) + L(S)) < e$,
(4) $\| B^{-1} \| (eL(\varphi) + 1 + L(S)) < 1$.

Then there is a unique function $g : \mathbf{B}^r\mathbf{E} \to \mathbf{B}^r\mathbf{F}$ such that $L(g) \leq e$ and

$$f(\text{graph}(g)) \subseteq \text{graph}(g).$$

Moreover if

(5) $L(\varphi) < 1$,

then $f \mid \text{graph}(g)$ has an attractive fixed point.

Proof. Let $\mathcal{G}$ be the metric space of all maps $g : \mathbf{B}^r\mathbf{E} \to \mathbf{B}^r\mathbf{F}$ such that $L(g) \leq e$. The metric on $\mathcal{G}$ is $d(g_1, g_2) = \| g_1 - g_2 \|_0$. $\mathcal{G}$ is not complete (because $\mathbf{B}^r\mathbf{F}$ is open) but this is not important. For $g \in \mathcal{G}$ define $\Gamma(g) : \mathbf{B}^r\mathbf{E} \to \mathbf{F}$ by

$$\Gamma(g) = B^{-1}(g \circ \varphi \circ (1, g) - S \circ (1, g)).$$

(Here 1 denotes the identity map of E.) By (1), $\Gamma(g)$ is well defined. Note that f (graph $(g)) \subseteq$ graph (g) if and only if $\Gamma(g) = g$.

As $\| g \circ \varphi \circ (1, g) \|_0 \leq \| g \|_0$ and $\| S \circ (1, g) \|_0 \leq \| S \|_0$ it follows that

$$(\#) \qquad \| \Gamma(g) \|_0 \leq \| B^{-1} \| \; (\| g \|_0 + \| S \|_0)$$

for $g \in \mathcal{G}$. But $\| g \|_0 \leq r$, hence by (2), $\| \Gamma(g) \|_0 < r$. Hence

$$\Gamma(g) : \mathbf{B}^r\mathbf{E} \to \mathbf{B}^r\mathbf{F}.$$

By (3),

$$\begin{aligned} L(\Gamma(g)) &\leq L(B^{-1})(L(g)L(\varphi)L(1, g) + L(S)L(1, g)) \\ &\leq \| B^{-1} \| \; (eL(\varphi) + L(S)) \\ &\leq e \end{aligned}$$

for $g \in \mathcal{G}$. We have shown that $\Gamma : \mathcal{G} \to \mathcal{G}$.

For $g_1, g_2 \in \mathcal{G}$

$$\begin{aligned} &\| g_1 \circ \varphi \circ (1, g_1) - g_2 \circ \varphi \circ (1, g_2) \|_0 \\ &\leq \| g_1 \circ \varphi \circ (1, g_1) - g_1 \circ \varphi \circ (1, g_2) \|_0 + \| g_1 \circ \varphi \circ (1, g_2) - g_2 \circ \varphi \circ (1, g_2) \|_0 \\ &\leq L(g_1)L(\varphi) \| g_1 - g_2 \|_0 + \| g_1 - g_2 \|_0 \\ &\leq (eL(\varphi) + 1) \| g_1 - g_2 \|_0 \end{aligned}$$

and

$$\| S \circ (1, g_1) - S \circ (1, g_2) \|_0 \leq L(S) \| g_1 - g_2 \|_0 .$$

Hence

$$\| \Gamma(g_1) - \Gamma(g_2) \|_0 \leq \| B^{-1} \| \; (eL(\varphi) + 1 + L(S)) \| g_1 - g_2 \|_0 .$$

Thus by (4), $\Gamma : \mathcal{G} \to \mathcal{G}$ is a contraction map. Note that $\Gamma(\mathcal{G}) \subseteq \mathcal{G}'$ where $\mathcal{G}'$ is the set of all $g \in \mathcal{G}$ with $\| g \|_0 \leq \| B^{-1} \| \; (r + \| S \|_0) < r$ (by (#) above). As $\mathcal{G}'$ is a complete metric space, it follows from the contraction principle that Γ has a unique attractive fixed point g. Clearly f (graph$(g)) \subseteq$ graph(g) by the definition of Γ.

Now assume (5). Then $f \mid$ graph (g) is a contraction map in the metric d on graph(g) defined by

$$d(z_1, z_2) = \| x_1 - x_2 \|$$

for $z_1 = (x_1, g(x_1))$, $z_2 = (x_2, g(x_2)) \in$ graph(g). This metric determines the topology on graph(g) which it inherits as a subset of $\mathbf{B}^r\mathbf{G}$. This completes the proof.

2.2 Corollary. *Suppose f is as in 2.1 and satisfies (1)–(4) and also*
(6) $\| B^{-1} \| L(\varphi) < 1$.
Let g be as in the conclusion of 2.1 and define a real number K by

$$K = \| B^{-1} \|^{-1} - \frac{\| B^{-1} \| L(S) L(\varphi)}{1 - \| B^{-1} \| L(\varphi)} - L(S).$$

Then

$$K \| y - g(x) \| \leq \| \psi(x, y) - g(\varphi(x, y)) \|.$$

Hence if
(7) $K > 1$;
then graph $(g) = W^s(f)$. If (5) holds, f has a unique fixed point which is an attractive fixed point of $f \mid W^s(f)$.

Proof. (6) insures that K is well defined. Note that (5), together with (4), implies (6) but we will need 2.2 under the weaker hypothesis of (6).
As $\Gamma(g) = g$,

$$Bg(x) = g(\varphi(x, g(x))) - S(x, g(x))$$

for $x \in \mathbf{B}^r\mathbf{E}$. Now for $(x, y) \in \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F}$

$$\begin{aligned}
&\| \psi(x, y) - g(\varphi(x, y)) \| \\
&= \| B(y - g(x) + g(x)) + S(x, y) - g(\varphi(x, y)) \| \\
&= \| B(y - g(x)) + g \circ \varphi(x, g(x)) - S(x, g(x)) + S(x, y) - g \circ \varphi(x, y) \| \\
&= \| B(y - g(x)) - [g \circ \varphi(x, y) - g \circ \varphi(x, g(x))] - [S(x, g(x)) - S(x, y)] \| \\
&\geq \| B(y - g(x)) \| - L(g) L(\varphi) \| y - g(x) \| - L(S) \| y - g(x) \| \\
&\geq (\| B^{-1} \|^{-1} - L(g) L(\varphi) - L(S)) \| y - g(x) \|.
\end{aligned}$$

As $g = B^{-1}(g \circ \varphi \circ (1, g) - S \circ (1, g))$ and as $L(1, g) \leq 1$, it follows that $L(g) \leq \| B^{-1} \| (L(g) L(\varphi) + L(S))$ or

$$L(g) \leq \frac{\| B^{-1} \| L(S)}{1 - \| B^{-1} \| L(\varphi)}.$$

Combining these two inequalities gives

$$\| \psi(x, t) - g(\varphi(x, y)) \| \geq K \| y - g(x) \|.$$

Now suppose $K > 1$. Let $z = (x_0, y_0) \in \mathbf{B}^r\mathbf{G} = \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F}$. If $z \in \text{graph}(g)$, then $f^n(z) \in \text{graph}(g)$ for $n = 0, 1, 2, \cdots$ (as $f(\text{graph}(g)) \subseteq \text{graph}(g)$). Assume $z \notin \text{graph}(g)$; that is, $y_0 \neq g(x_0)$. We must show that $z \notin W^s(f)$; i.e. it is not the case that $f^n(z) \in \mathbf{B}^r\mathbf{G}$ for all n. Suppose the contrary. Then $f^n(z)$ is defined for all n and we may define $(x_n, y_n) = f^n(z)$. In view of the first part of the theorem,

$$\| y_{n+1} - g(x_{n+1}) \| \geq K \| y_n - g(x_n) \|.$$

Hence by induction

$$\| y_n - g(x_n) \| \geq K^n \| y_0 - g(x_0) \|.$$

This says that the distance from y_n to $g(x_n)$ is growing exponentially, contradicting the fact that $y_n, g(x_n) \in \mathbf{B}^r\mathbf{F}$.

If (5) holds, $f \mid W^s(f)$ has an attractive fixed point by 2.1. But any fixed point of f is in $W^s(f)$ by definition. Hence this is the only fixed point of f.

3. Smoothness of the stable manifold

In this section we retain the notation of §2. Our aim is to show that the local stable manifold is as smooth as the map.

3.1 THEOREM. *Let* $f : \mathbf{B}^r\mathbf{G} = \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F} \to \mathbf{G} = \mathbf{E} \times \mathbf{F}$ *have the form* $f = (\varphi, \psi)$ *where* $\varphi : \mathbf{B}^r\mathbf{G} \to \mathbf{E}$ *and* $\psi : \mathbf{B}^r\mathbf{G} \to \mathbf{F}$. *Suppose* φ *and* ψ *have the forms*

$$\varphi(x, y) = Ax + R(x, y), \qquad \psi(x, y) = By + S(x, y)$$

for $x \in \mathbf{B}^r\mathbf{E}$ *and* $y \in \mathbf{B}^r\mathbf{F}$ *where* $A \in L(\mathbf{E}, \mathbf{E})$, $B \in L(\mathbf{F}, \mathbf{F})$, $R : \mathbf{B}^r\mathbf{G} \to \mathbf{E}$, *and* $S : \mathbf{B}^r\mathbf{G} \to \mathbf{F}$. *Let* δ *be a positive real number. Assume that* f *is uniformly* C^k $(k \geq 1)$, B *is invertible, and*

(8) $\varphi(\mathbf{B}^r\mathbf{G}) \subseteq \mathbf{B}^r\mathbf{E}$,

(9) $\| B^{-1} \| (r + \| S \|_0) < r$,

(10) $\| B^{-1} \| < 1$, $\| A \| \leq 1$,

(11) $\| R \|_0, \| S \|_0, \| j^k R \|_0, \| j^k S \|_0 \leq \delta$.

Then there is a real number $\delta_0 > 0$ *depending only on* $\| B^{-1} \|$ *and* $\| A \|$ *(and not on* r*) such that if* $\delta \leq \delta_0$, *then* $W^s(f)$ *is the graph of a uniformly* C^k *function* $g : B^rE \to B^rF$. *If, in addition,*

(12) $\| A \| < 1$,

then (if $\delta \leq \delta_0$*),* f *has a unique fixed point and this fixed point is an attractive fixed point of* $f \mid W^s(f)$.

Proof. We will show that (8)–(11) imply (1)–(4) and (6)–(7) (for suitable e), and that (8)–(12) imply (5). Then we show that the g which results from 2.1 is uniformly C^k. This (by §2) will complete the proof.

Hence assume (8)–(11). As (1), (2), and (8), (9) are identical, (1) and (2) hold. Next note that $L(R) \leq \| j^k R \| < \delta$ and $L(S) \leq \| j^k S \|_0 < \delta$ by the mean value theorem and (11). Hence

$$L(\varphi) \leq L(A) + L(R) \leq \| A \| + \delta \leq 1 + \delta$$

by (10) and (11). But $\| B^{-1} \| < 1$ by (11) and hence

$$\| B^{-1} \| L(\varphi) \leq \| B^{-1} \| (1 + \delta) < 1$$

for δ sufficiently small. This proves (6).

Now $L(\varphi) \leq 1 + \delta$, $L(S) \leq \delta$, and $\| B^{-1} \| < 1$. Furthermore, (4) clearly holds when $e = L(S) = 0$. Hence we may choose e with $0 < e \leq 1$ so that

(4) still holds (for sufficiently small δ). Because $\| B^{-1} \| L(\varphi) < 1$, (3) holds when $L(S) = 0$ and hence as $L(S) \leq \| j^k S \|_0 < \delta$, (3) continues to hold for small δ. Finally (7) holds when $L(S) = 0$ and hence still holds when δ is sufficiently small.

If we assume (12) in addition, then by

$$L(\varphi) \leq \| A \| + \| j^k R \|_0 \leq \| A \| + \delta$$

we have (5) if δ is small.

Now 2.1 and 2.2 assures the existence of a (Lipschitz) map $g : \mathbf{B}^r\mathbf{E} \to \mathbf{B}^r\mathbf{F}$ satisfying the conclusions of the theorem. All that remains is to show that g is uniformly C^k. For this purpose we employ the Fiber contraction principle [4] which we state without proof.

FIBER CONTRACTION PRINCIPLE. *Let $\mathcal{G}$ be a topological space, $\mathcal{H}$ a complete metric space, and $\Phi : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H}$ a map of form*

$$\Phi(g, h) = (\Gamma(g), \Delta_g(h))$$

for $g \in \mathcal{G}$ and $h \in \mathcal{H}$ (where $\Gamma : \mathcal{G} \to \mathcal{G}$ and $\Delta_g : \mathcal{H} \to \mathcal{H}$ for each $g \in \mathcal{G}$). Let ρ be a real number with $0 \leq \rho < 1$. Assume

(13) *$\Delta_g(h)$ is continuous in g for each fixed $h \in \mathcal{H}$,*

(14) *$\Gamma : \mathcal{G} \to \mathcal{G}$ has an attractive fixed point,*

(15) *$L(\Delta_g) \leq \rho$ for each $g \in \mathcal{G}$.*

Then Φ has an attractive fixed point.

Now let $\mathcal{G}$ be the metric space defined in the proof of 2.1 and $\Gamma : \mathcal{G} \to \mathcal{G}$ the contraction map also defined there. Choose $g \in \mathcal{G}$ such that g is C^k but otherwise arbitrary and let $\tilde{g} = \Gamma(g)$; i.e.

$$\tilde{g} = B^{-1}(g \circ \varphi \circ (1, g) - S \circ (1, g)).$$

The higher derivatives of $\tilde{g}$ evaluated at a point $x \in \mathbf{B}^r\mathbf{E}$ are "polynomials" of the derivatives of g at x, of g at $x' = \varphi(x, g(x))$, of φ at $(x, g(x))$, and of S at $(x, g(x))$. (See the composite mapping formula of [1].) In other words, for $x \in \mathbf{B}^r\mathbf{E}$

$$j^k\tilde{g}(x) = \mathcal{E}(j^k g(x'), j^k g(x), j^k \varphi(z), j^k S(z))$$

where $z = (x, g(x))$ and $x' = \varphi(z)$ and

$$\mathcal{E} : J^k(\mathbf{E}, \mathbf{F}) \times J^k(\mathbf{E}, \mathbf{F}) \times J^k(\mathbf{G}, \mathbf{F}) \times J^k(\mathbf{G}, \mathbf{F}) \to J^k(\mathbf{E}, \mathbf{F})$$

is a "polynomial" and is hence C^∞ and maps bounded sets to bounded sets.

When R and S are identically zero, $\tilde{g}(x) = B^{-1}g(Ax)$ and hence

$$j^k\tilde{g}(x) = (B^{-1}Dg(Ax)A, B^{-1}D^2g(Ax) \otimes^2 A, \cdots, B^{-1}D^kg(Ax) \otimes^k A)$$

where for $C \in L_s^i(\mathbf{E}, \mathbf{F})$ and $i = 2, \cdots, k$, $C \otimes^i A \in L_s^i(E, F)$ is defined by

$$C \otimes^i A(e_1, \cdots, e_i) = C(Ae_1) \cdots (Ae_i)$$

for $e_1, \cdots, e_i \in \mathbf{E}$. Thus when R and S are identically zero

$$\mathcal{E}(p, q, j^k\varphi(z), j^kS(z)) = (B^{-1}p_1 A, B^{-1}p_2 \otimes^2 A, \cdots, B^{-1}p_k \otimes^k A)$$

for

$$q, p = (p_1, p_2, \cdots, p_k) \in J^k(\mathbf{E}, \mathbf{F}) = L(\mathbf{E}, \mathbf{F}) \times L_s^2(\mathbf{E}, \mathbf{F}) \times \cdots \times L_s^k(\mathbf{E}, \mathbf{F})$$

For fixed g and x,

$$\mathcal{E}(p, q) = \mathcal{E}(p, q, j^k\varphi(z), j^kS(z))$$

is linear in (p, q) when R and S are identically zero and is hence its own derivative. This derivative has norm $\leq \| B^{-1} \| < 1$. Hence if ρ is a real number with $\| B^{-1} \| < \rho < 1$ we have that

$$(*) \qquad \| \mathcal{E}(p, q) - \mathcal{E}(p', q') \| \leq \rho \max (\| p - p' \|, \| q - q' \|)$$

for $p, p', q, q' \in J^k(\mathbf{E}, \mathbf{F})$ and as $\mathcal{E}(0, 0) = 0$ we have also that

$$(**) \qquad \| \mathcal{E}(p, q) \| \leq \rho \max (\| p \|, \| q \|).$$

Now by (11) (since $\mathcal{E}(p, q)$ varies continuously with $\varphi(z)$ and $S(z)$ and $D\mathcal{E}(p, q)$ varies continuously with $j^k\varphi(z)$ and $j^kS(z)$), we may assume $(*)$ and $(**)$ continue to hold even when R and S are not identically zero (provided that δ is sufficiently small).

Let $\mathcal{H}$ be the space of all uniformly continuous maps $h : \mathbf{B}^r\mathbf{E} \to J^k(\mathbf{E}, \mathbf{F})$ such that $\| h \|_0 \leq 1$. $\mathcal{H}$ is a complete metric space (in the sup norm). For $g \in \mathcal{G}$ and $h \in \mathcal{H}$ define $\Delta_g(h) : \mathbf{B}^r\mathbf{E} \to J^k(\mathbf{E}, \mathbf{F})$ by

$$\Delta_g(h)(x) = \mathcal{E}(h(x'), h(x), j^k\varphi(z), j^kS(z))$$

for $x \in \mathbf{B}^r\mathbf{E}$ (where $z = (x, g(x))$ and $x' = \varphi(z)$). By $(**)$ above $\| \Delta_g(h) \|_0 \leq \rho < 1$. As $h, j^k\varphi, j^kS, \varphi$ and g are all uniformly continuous, and as $\mathcal{E}$ is uniformly continuous on bounded sets, if follows that $\Delta_g(h)$ is uniformly continuous. Thus $\Delta_g(h) \in \mathcal{H}$. Thus $\Delta_g : \mathcal{H} \to \mathcal{H}$.

Fix $h \in \mathcal{H}$. As $h, j^k\varphi, j^kS$, and φ are uniformly continuous and as $\mathcal{E}$ is uniformly continuous on bounded sets, it follows that $\Delta_g(h)$ is continuous in g. This verifies (13). (14) was verified in the proof of 2.1 and (15) follows immediately from $(*)$. Thus by the fiber contraction principle the map $\Phi : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H}$ defined by $\Phi(g, h) = (\Gamma(g), \Delta_g(h))$ for $g \in \mathcal{G}$ and $h \in \mathcal{H}$ has an attractive fixed point. Let (g, h) denote this fixed point. As $\Gamma(g) = g$, g is the function of the conclusion of our theorem.

Now let $g_0 : \mathbf{B}^r\mathbf{E} \to B^rF$ be identically zero. Then $g_0 \in \mathcal{G}$, and if $h_0 = j^kg_0$, then $h_0 \in \mathcal{H}$. Let $(g_n, h_n) = \Phi^n(g_0, h_0)$ for $n = 0, 1, 2, \cdots$. Then $g_n \to g$ and $h_n \to h$ uniformly as $n \to \infty$. Clearly each g_n is uniformly C^k and by the definition of Δ and induction of n we have that $j^kg_n = h_n$. Thus g_n and its derivatives up to order k converge uniformly; hence g is C^k and $j^kg = h$. Hence g is uniformly C^k as was to be shown.

4. Dependence on a parameter

In this section we show that if f depends smoothly on a parameter, then the dependence of the stable manifold $W^s(f)$ on that parameter is just as smooth. To make this precise we make the following definitions.

DEFINITION. Let $\mathfrak{A}$, U, and V be open sets in (possibly different) Banachable spaces and let $C^k(U, V)$ denote the set of all C^k maps from U to V. A function $\rho : \mathfrak{A} \to C^k(U, V)$ is called a *C^k representation* (resp. a *uniformly C^k representation*) iff the evaluation map $ev_\rho : \mathfrak{A} \times U \to V$ defined by

$$ev_\rho(a, x) = \rho(a)(x)$$

for $a \in \mathfrak{A}$ and $x \in U$ is C^k (resp. uniformly C^k).

DEFINITION. Let $\mathbf{G}$ be a Banachable space. A linear operator $C \in L(\mathbf{G}, \mathbf{G})$ is *semi-hyperbolic* iff the spectrum of C contains no complex number of modulus one. Let $Z \subseteq \mathbf{G}$ be open and $f : Z \to \mathbf{G}$ be a C^1 map. A point $z \in Z$ is a *semi-hyperbolic fixed point* of f iff $f(z) = z$ and $Df(z) \in L(G, G)$ is semi-hyperbolic.

4.1 PARAMETERIZED LOCAL STABLE MANIFOLD THEOREM. *Let Z be an open neighborhood of* 0 *in a Banachable space* $\mathbf{G}$, *and let $\rho : \mathfrak{A} \to C^k(Z, \mathbf{G})$ be a uniformly C^k representation ($k \geq 1$). Let $a_0 \in \mathfrak{A}$ be such that* 0 *is a semi-hyperbolic fixed point of $\rho(a_0) : Z \to \mathcal{G}$. Then there is a closed splitting*

$$\mathbf{G} = \mathbf{E} \oplus \mathbf{F} \cong \mathbf{E} \times \mathbf{F}$$

of $\mathbf{G}$, *neighborhoods $\mathfrak{B}$, U, and V of a_0*, 0, *and* 0 *in $\mathfrak{A}$*, $\mathbf{E}$, *and* $\mathbf{F}$ *respectively, and a uniformly C^k representation $\pi : \mathfrak{B} \to C^k(U, V)$ such that $U \times V \subseteq Z$ and for $b \in \mathfrak{B}$,*

$$\text{graph}(\pi(b)) = W^s(\rho(b) \mid U \times V).$$

Furthermore, for each $b \in \mathfrak{B}$, $\rho(b) \mid U \times V$ has a unique fixed point and this fixed point is an attractive fixed point of $\rho(b) \mid$ graph $(\pi(B))$.

Proof. Let $f = \rho(a_0) : Z \to \mathbf{G}$. By the hypothesis that 0 is a semi-hyperbolic fixed point of f, the unit circle separates the spectrum of $Df(0) \in L(\mathbf{G}, \mathbf{G})$. Hence by the spectral theorem we may assume that $\mathbf{G} = \mathbf{E} \times \mathbf{F}$ and $Df(0) = (A, B)$ where $A \in L(\mathbf{E}, \mathbf{E})$ has spectrum inside the unit circle and $B \in L(\mathbf{F}, \mathbf{F})$ has spectrum outside the unit circle. Thus B is invertible and by [7] we may choose norms on $\mathbf{E}$ and $\mathbf{F}$ so that $\| A \| < 1$ and $\| B^{-1} \| < 1$. We suppose without loss of generality that a_0 is the origin of the ambient space of $\mathfrak{A}$. We choose a norm for the ambient space of $\mathfrak{A}$ and then choose a positive real number q such that $\mathbf{B}^q\mathbf{E} \times \mathbf{B}^q\mathbf{F} \subseteq Z$ and $\mathbf{B}^q\mathfrak{A} \subseteq \mathfrak{A}$ ($\mathbf{B}^q\mathfrak{A}$ denotes the open ball of radius q centered at the origin in the ambient space of $\mathfrak{A}$). We give $\mathbf{G} = \mathbf{E} \times \mathbf{F}$ the product norm so that $\mathbf{B}^q\mathbf{G} = \mathbf{B}^q\mathbf{E} \times \mathbf{B}^q\mathbf{F}$.

For $a \in \mathbf{B}^q\mathfrak{A}$ let $f_a : \mathbf{B}^q\mathbf{G} \to \mathbf{G}$ be defined by $f_a = \rho(a) \mid \mathbf{B}^q\mathbf{G}$. By Taylor's formula, $f_a = (\varphi_a, \psi_a)$ where $\varphi_a : \mathbf{B}^q\mathbf{G} \to \mathbf{E}$ and $\psi_a : \mathbf{B}^q\mathbf{G} \to \mathbf{F}$ have the forms

$$\varphi_a(x, y) = Ax + R_a(x, y), \qquad \psi_a(x, y) = By + S_a(x, y)$$

where $R_a : \mathbf{B}^q\mathbf{G} \to \mathbf{E}$ and $S_a : \mathbf{B}^q\mathbf{G} \to \mathbf{F}$ for each $a \in \mathbf{B}^q\mathfrak{A}$ and where $R_a(x, y)$, $S_a(x\ y)$, $DR_a(x, y)$, and $DS_a(x, y)$ all vanish when $a = 0$ and $(x, y) = (0, 0)$. Define

$$R : \mathbf{B}^q\mathfrak{A} \times \mathbf{B}^q\mathbf{G} \to \mathbf{E} \quad \text{and} \quad S : \mathbf{B}^q\mathfrak{A} \times \mathbf{B}^q\mathbf{G} \to \mathbf{F}$$

by $R(a, x, y) = R_a(x, y)$ and $S(a, x, y) = S_a(x, y)$ for $(a, x, y) \in \mathbf{B}^q\mathfrak{A} \times \mathbf{B}^q\mathbf{G}$.

Let r and t be real numbers with $0 < t \leq 1$ and $r \leq t^{-1}q$. Define

$$R_t : \mathbf{B}^r\mathfrak{A} \times \mathbf{B}^r\mathcal{G} \to \mathbf{E} \quad \text{and} \quad S_t : \mathbf{B}^r\ \mathfrak{A} \times \mathbf{B}^r\mathbf{G} \to \mathbf{F}$$

by

$$R_t(a, x, y) = t^{-1}R(t^2a, tx, ty) \quad \text{and} \quad S_t(a, x, y) = t^{-1}S(t^2a, tx, ty)$$

for $(a, x, y) \in \mathbf{B}^r\mathfrak{A} \times \mathbf{B}^r\mathbf{E} \times \mathbf{B}^r\mathbf{F}$. For $a \in \mathbf{B}^r\mathfrak{A}$ we define

$$R_{at} : \mathbf{B}^r\mathbf{G} \to \mathbf{E} \quad \text{and} \quad S_{at} : \mathbf{B}^r\mathbf{G} \to \mathbf{F}$$

by

$$R_{at}(x, y) = R_t(a, x, y) \quad \text{and} \quad S_{at}(x, y) = S_t(a, x, y)$$

and define $f_{at} : \mathbf{B}^r\mathbf{G} \to \mathbf{G}$ by setting $f_{at} = (\varphi_{at}, \psi_{at})$ where

$$\varphi_{at} : \mathbf{B}^r\mathbf{G} \to \mathbf{E} \quad \text{and} \quad \psi_{at} : \mathbf{B}^r\mathbf{G} \to \mathbf{F}$$

are defined by

$$\varphi_{at}(x, y) = Ax + R_{at}(x, y), \qquad \psi_{at}(x, y) = By + S_{at}(x, y)$$

for $(x, y) \in \mathbf{B}^r\mathbf{G}$.

Let $\tilde{\mathbf{E}}$ be the product of the ambient space of $\mathfrak{A}$ with $\mathbf{E}$ so that

$$\mathbf{B}^r\tilde{\mathbf{E}} = \mathbf{B}^r\mathfrak{A} \times \mathbf{B}^r\mathbf{E}$$

and define $\tilde{f}_t : \mathbf{B}^r\tilde{\mathbf{E}} \times \mathbf{B}^r\mathbf{F} \to \tilde{\mathbf{E}} \times \mathbf{F}$ by $\tilde{f}_t = (\tilde{\varphi}_t, \tilde{\psi}_t)$ where

$$\tilde{\varphi}_t(a, x, y) = \tilde{A}(a, x) + \tilde{R}_t(a, x, y), \qquad \tilde{\psi}_t(a, x, y) = By + S_t(a, x, y)$$

for $(a, x, y) \in \mathbf{B}^r(\tilde{\mathbf{E}} \times \mathbf{F})$. Here $\tilde{A} \in L(\tilde{\mathbf{E}}, \tilde{\mathbf{E}})$ is defined by $\tilde{A}(a, x) = (a, Ax)$ and $\tilde{R}_t : \mathbf{B}^r(\tilde{\mathbf{E}} \times \mathbf{F}) \to \tilde{\mathbf{E}}$ is defined by $\tilde{R}_t(a, x, y) = (0, R_t(a, x, y))$. Note that $\| \tilde{A} \| = 1$ (provided the ambient space of $\mathfrak{A}$ is not the zero-dimensional Banach space).

We will show that t and r may be so chosen that f_{at} satisfies hypotheses (8)–(12) of 3.1 for each $a \in \mathbf{B}^r\mathfrak{A}$ and $\tilde{f}_t$ satisfies (8)–(11) of 3.1. To do this choose δ_0 depending on $\tilde{A}$, A and B as in 3.1 (sufficiently small for the application of the first half of 3.1 to maps such as $\tilde{f}_t$ whose linear part is $(\tilde{A}, B)$ and also sufficiently small for the application of all of 3.1 to maps such as f_{at} with "linear part" (A, B)). To verify the hypotheses of 3.1 it suffices to show

(16) $\| R_{at} \|_0 < r(1 - \| A \|)$,
(17) $\| S_{at} \|_0 \leq r\,d$ where $d < (1 - \| B^{-1} \|)\| B^{-1} \|^{-1}$,
(18) $\| R_{at} \|_0,\ \| S_{at} \|_0 \leq \delta_0$,
(19) $\| j^k R_t \|_0,\ \| j^k S_t \|_0 \leq \delta_0$.

((16)–(19) must hold for some t and r and all $a \in \mathbf{B}^r\mathcal{A}$.) To see this note that (16) implies that $\| A \|r + \| R_{at} \|_0 < r$. As

$$\| \varphi_{at} \|_0 \leq \| A \mid B^r E \|_0 + \| R_{at} \|_0 \leq \| A \|r + \| R_{at} \|_0,$$

(16) implies $\| \phi_{at} \|_0 < r$ which in turn implies that

$$\varphi_{at}(\mathbf{B}^r\mathbf{G}) \subseteq \mathbf{B}^r\mathbf{E} \quad \text{and} \quad \tilde{\varphi}_t(\mathbf{B}^r\tilde{\mathbf{E}} \times \mathbf{B}^r\mathbf{F}) \subseteq \mathbf{B}^r\tilde{\mathbf{E}}$$

verifying (8). Condition (17) implies that $\| B^{-1} \| (r + \| S_{at} \|_0) < r$ which is (9) for the map f_{at} and as $\| S_t \|_0$ is the supremum of $\| S_{at} \|_0$ as a varies over $\mathbf{B}^r\mathcal{A}$, this implies (9) for the map $\tilde{f}_t$ as well. (18) and (19) clearly imply (11). (10) and (12) have already been verified.

First we verify (19). We note that for integers l, m, and n with $l + m + n \leq k$,

$$(*) \qquad D_1^l D_2^m D_3^n R_t(a, x, y) = t^h D_1^l D_2^m D_3^n R(t^2 a, tx, ty)$$

for $a \in \mathbf{B}^r\mathcal{A}$ and $(x, y) \in \mathbf{B}^r\mathbf{G}$ and where $h = 2l + m + n - 1$. We first choose r so that $j^k R$ is bounded on $\mathbf{B}^r\mathcal{A} \times \mathbf{B}^r\mathbf{G}$, then choose $t \in (0, 1]$ so small that $t\| j^k R \|_0 \leq \delta_0$. Then by $(*)$ above,

$$(**) \qquad \| D_1^l D_2^m D_3^n R_t \|_0 \leq \delta_0$$

if $l > 0$ or $m > 1$ or $n > 1$. As $D_2 R_t(0, 0, 0) = 0$ and $D_3 R_t(0, 0, 0) = 0$ $(**)$ can be made to hold when $l = m = 0$ and $n = 1$ or $l = n = 0$ and $m = 1$ by making r smaller. Thus $\| j^k R_t \|_0 \leq \delta_0$; a similar argument shows that $\| j^k S_t \|_0 \leq \delta_0$ (for suitable r and t).

Now for $a \in \mathbf{B}^r\mathcal{A}$ and $(x, y) \in \mathbf{B}^r\mathbf{G}$ we have

$$\| R_{at}(x, y) \| \leq \| R_{at}(x, y) - R_{at}(0, 0) \| + \| R_{at}(0, 0) \|,$$
$$\| S_{at}(x, y) \| \leq \| S_{at}(x, y) - S_{at}(0, 0) \| + \| S_{at}(0, 0) \|$$

so that by the mean value theorem and (19)

$$(***) \qquad \begin{aligned} \| R_{at} \|_0 &\leq r\| DR_{at} \|_0 + \| R_{at}(0, 0) \|, \\ \| S_{at} \|_0 &\leq r\| DS_{at} \|_0 + \| S_{at}(0, 0) \|. \end{aligned}$$

Now as above we may make r smaller so that

$$\| DR_{at} \|_0 < (1 - \| A \|), \quad \| DS_{at} \|_0 < d, \quad r\| DR_{at} \|_0, \; r\| DS_{at} \|_0 < \delta_0.$$

By replacing the norm on the ambient space of $\mathcal{A}$ by a scalar multiple of itself we make $\mathbf{B}^r\mathcal{A}$ smaller without disturbing the inequalities already proved. As $\mathbf{B}^r\mathcal{A}$ gets smaller, so do $\| R_{at}(0, 0) \|$ and $\| S_{at}(0, 0) \|$ (for $\| R_{0t}(0, 0) \| = \| S_{0t}(0, 0) \| = 0$). Thus for $\mathbf{B}^r\mathcal{A}$ small enough, the inequalities $(***)$ imply (16), (17), and (18).

Thus we may apply 3.1 and conclude that there are uniformly C^k functions $\tilde{g}_t$ and g_{at} (for each $a \in \mathbf{B}^r\mathcal{A}$) with

$$\operatorname{graph}(\tilde{g}_t) = W^s(\tilde{f}_t) \quad \text{and} \quad \operatorname{graph}(g_{at}) = W^s(f_{at}).$$

As $z \in W^s(f_{at})$ if and only if $(a, z) \in W^s(\tilde{f}_t)$ it follows that $\tilde{g}_t(a, x) = g_{at}(x)$ for $a \in B^r\mathfrak{A}$ and $x \in B^rE$. In other words the map $\mathbf{B}^r\mathfrak{A} \to C^k(\mathbf{B}^r\mathbf{E}, \mathbf{B}^r\mathbf{F})$ which sends a to g_{at} is a uniformly C^k representation.

Now $\rho(t^2a)$ and f_{at} are related by the change of co-ordinates $(x, y) \to (tx, ty)$. To complete the proof we take $\mathfrak{B} = \mathbf{B}^p\mathfrak{A}$ (where $p = t^{-2}r$), $U = \mathbf{B}^s\mathbf{E}$, $V = \mathbf{B}^s\mathbf{F}$ (where $s = t^{-1}r$) and for $b \in \mathfrak{B}$ we define $\pi(b) : U \to V$ by

$$\pi(b)(x) = t^{-1}g_{at}(tx)$$

for $x \in U$ (where $a = t^2b$). As $ev_\pi(b, x) = t^{-1}\tilde{g}_t(t^2b, tx)$ it follows that π is a uniformly C^k representation. Clearly $\text{graph}(\pi(b)) = W^s(\rho(b) \mid U \times V)$. The proof is complete.

Taking $\mathfrak{A}$ to be a point we obtain the "unparametrized" local stable manifold theorem as a corollary to 4.2.

4.2 Local Stable Manifold Theorem. *Let Z be an open neighborhood of the origin in a Banachable space* $\mathbf{G}$ *and $f : Z \to \mathbf{G}$ be a uniformly C^k map $(k \geq 1)$ having the origin as a hyperbolic fixed point. Then there exists a closed splitting* $\mathbf{G} = \mathbf{E} \oplus \mathbf{F} \simeq \mathbf{E} \times \mathbf{F}$ *of* $\mathbf{G}$, *open neighborhoods U and V of the origin in* $\mathbf{E}$ *and* $\mathbf{F}$ *respectively, and a uniformly C^k map $g : U \to V$ such that*

$$\text{graph}(g) = W^s(f \mid U \times V).$$

Furthermore, 0 is the only fixed point of f in $U \times V$ and it is an attractive fixed point of $f \mid \text{graph}(g)$.

If $\mathbf{G}$ is finite dimensional, we may weaken the hypothesis of 4.2: we need only assume that f is C^k (for then its restriction to a smaller neighborhood of 0 is uniformly C^k).

We remark that if the map f of 4.2 is C^∞, then so is g. Indeed by 4.2 it is true that for each integer $p = 1, 2, \cdots, g$ is C^p at points sufficiently near 0. But $f \mid \text{graph}(g)$ is a contraction and $\text{graph}(g)$ is invariant under f; hence g si everywhere C^p. As p is arbitrary, g is C^∞.

5 . Examples and application

Let k be a non-negative integer, α be a real number with $0 < \alpha < 1$, Z be an open set in a Banach space, and $\mathbf{G}$ be a Banach space. Then $B^k(Z, \mathbf{G})$ denotes the Banach space of all C^k maps $f : Z \to \mathbf{G}$ such that $\| f \|_0, \| j^kf \|_0 < \infty$ and $B^k_u(Z, \mathbf{G})$ denotes the closed subspace of $B^k(Z, \mathbf{G})$ consisting of those maps which are uniformly C^k (the norm on $B^k(Z, \mathbf{G})$ is $\| f \|_0 + \| j^kf \|_0$). $B^{k+\alpha}(Z, \mathbf{G})$ is the Banach space of all $f \in B^k(Z, \mathbf{G})$ such that D^kf satisfies a Hölder condition of order α; the norm on $B^{k+\alpha}(Z, \mathbf{G})$ is $\| f \|_0 + \| j^kf \| + H_\alpha(D^kf)$ where $H_\alpha(D^kf)$ is the minimum Hölder constant of order α of D^kf. If $\bar{Z}$ is compact with smooth boundary, then $C^k(\bar{Z}, \mathbf{G})$ and $C^{k+\alpha}(\bar{Z}, \mathbf{G})$ are Banach spaces (with the appropriate norms).

5.1. If $\mathfrak{A} \subseteq B^k(Z, \mathbf{G})$ is open, then the inclusion $\mathfrak{A} \to C^k(Z, \mathbf{G})$ is a C^k representation (see [1]) but unfortunately even if $\mathfrak{A} \subseteq B^k_u(Z, \mathbf{G})$ is open and bounded and even when $\bar{Z}$ is compact, the inclusion $\mathfrak{A} \to C^k(Z, \mathbf{G})$ is not a uniformly C^k representation.

5.2. If $\mathfrak{A} \subseteq B^{k+\alpha}(Z, \mathbf{G})$ is open and bounded, then the inclusion $\mathfrak{A} \to C^k(Z, \mathbf{G})$ is a uniformly C^k representation. In fact it is a $C^{k+\alpha}$ representation (where this concept is defined in the obvious way). Hence 4.1 applies and we obtain a result which may be expressed by saying that near a semi-hyperbolic fixed point of a $B^{k+\alpha}$ map $f : Z \to G$ the stable manifold is a C^k function of f. Inspection of the proof of 4.1 shows that each stable manifold is in fact the graph of a $B^{k+\alpha}$ function and the representation π is a $C^{k+\alpha}$ representation.

5.3. We emphasize 4.1 does not imply that the map $\pi : \mathfrak{B} \to C^k(U, V)$ is C^k; however (at least when $\mathbf{G}$ is finite dimensional) it is continuous (provided that $\bar{U} \times \bar{V}$ is compact and $\bar{U} \times \bar{V} \subseteq Z$).

5.4. Even though the representation of 5.1 is not uniformly C^k we may still conclude a rather strong theorem in this direction. Namely, suppose $k \geq 2$ and $\mathfrak{A} \subseteq B^k(Z, \mathbf{G})$ is open and $\rho : \mathfrak{A} \to C^{k-1}(Z, \mathbf{G})$ is the inclusion. Then (in the context of 4.1) we may assume (by making $\mathfrak{A}$ smaller so that it is bounded) that ρ is a uniformly C^{k-1} representation. Hence by 4.1, π is a uniformly C^{k-1} representation. By 4.2, if G is finite dimensional, each map $\pi(b) : U \to V$ is C^k (and not just C^{k-1}).

5.5. More generally, if $\rho : \mathfrak{A} \to C^k(Z, \mathbf{G})$ is any C^k representation (not necessarily uniformly C^k but otherwise as in 4.1) and if $k \geq 2$ and if each $\rho(a) : Z \to \mathbf{G}$ (for $a \in \mathfrak{A}$) is uniformly C^k, then π is a C^{k-1} representation and each $\pi(b)$ (for $b \in \mathfrak{B}$) is a C^k map.

5.6. Let $\bar{Z}$ be compact with smooth boundary. If $\mathfrak{A} \subseteq C^k(\bar{Z}, \mathbf{G})$ is open and bounded, the inclusion $\mathfrak{A} \to C^k(Z, \mathbf{G})$ is not a uniformly C^k representation, but the remarks of 5.5 apply. If $\mathfrak{A} \subseteq C^{k+\alpha}(\bar{Z}, \mathbf{G})$ is open and bounded, then the inclusion $\mathfrak{A} \to C^k(Z, \mathbf{G})$ is a uniformly C^k representation (in fact, it is a $C^{k+\alpha}$ representation) and the remarks of 5.2 apply.

6. Unstable manifolds

In [4] the stable manifold of a map f with a hyperbolic fixed point is obtained as the unstable manifold of f^{-1}. We reverse the process here obtaining an analog of 4.1 for unstable manifolds. We first make the pertinent definitions. Throughout this section Z is an open subset of a Banach space $\mathbf{G}$.

Definition. Let $f : Z \to \mathbf{G}$. The *unstable manifold* of f is denoted by $W^u(f)$ and defined by

$$W^u(f) = \bigcap_{n=1}^{\infty} f^n(Z).$$

The following lemma follows immediately from the definition.

6.1 Lemma. *If $f : Z \to f(Z) \subseteq \mathbf{G}$ is a bijection, then $W^u(f) = W^s(f^{-1})$.*

DEFINITION. A linear operator $C \in L(\mathbf{G}, \mathbf{G})$ is *hyperbolic* iff it is invertible and semi-hyperbolic. A point $z \in Z$ is a *hyperbolic fixed point* of a C^1 map $f : Z \to \mathbf{G}$ iff $f(z) = z$ and $Df(z) \in L(\mathbf{G}, \mathbf{G})$ is hyperbolic.

The following lemma is the key step in deriving 6.3 below from 4.1.

6.2 LEMMA. *Let $\rho : \mathcal{A} \to C^k(Z, \mathbf{G})$ be a uniformly C^k representation with $k \geq 1$. Let $a_0 \in \mathcal{A}$, $z_0 \in Z$, $f = \rho(a_0) : Z \to \mathbf{G}$ and suppose that $Df(z_0)$ is a linear isomorphism. Then there is a uniformly C^k representation $\tilde{\rho} : \mathcal{B} \to C^k(W, \mathbf{G})$ where $\mathcal{B}$ is a neighborhood of a_0 in $\mathcal{A}$ and W is a neighborhood of $f(z_0)$ in $\mathbf{G}$ such that for each $b \in \mathcal{B}$ $\tilde{\rho}(b)$: $W \to \mathbf{G}$ is a diffeomorphism onto $\tilde{\rho}(b)(W)$ and $\tilde{\rho}(b)^{-1} = \rho(b) \mid \tilde{\rho}(b)(W)$.*

Proof. Define $g : \mathcal{A} \times Z \to \mathcal{A} \times \mathbf{G}$ by $g(a, z) = (a, \rho(a)(z))$ for $a \in \mathcal{A}$ and $z \in Z$. $Dg(a_0, z_0)$ is invertible and so by the uniformly C^k inverse function theorem there exists neighborhoods $\mathcal{B}$ of a_0 and W of $f(z_0)$ and a uniformly C^k map $g^{-1} : \mathcal{B} \times W \to \mathcal{A} \times Z$ which is right and left inverse to $g \mid g^{-1}(\mathcal{B} \times W)$. g^{-1} has the form $g^{-1}(b, w) = (b, h(b, w))$ for $b \in \mathcal{B}$ and $w \in W$ and we may define, for $b \in \mathcal{B}$, $\tilde{\rho}(b) : W \to \mathbf{G}$ by $\tilde{\rho}(b)(w) = h(b, w)$ for $w \in W$. $\tilde{\rho}$ clearly has the desired properties.

6.3 PARAMETRIZED LOCAL UNSTABLE MANIFOLD THEOREM. *Let*

$$\rho : \mathcal{A} \to C^k(Z, \mathbf{G})$$

be a uniformly C^k representation ($k \geq 1$) such that $0 \in Z$ is a hyperbolic fixed point of $\rho(a_0)$f or some $a_0 \in \mathcal{A}$. Then there is a closed splitting $\mathbf{G} = \mathbf{E} \oplus \mathbf{F} \simeq \mathbf{E} \times \mathbf{F}$ of $\mathbf{G}$, neighborhoods $\mathcal{B}$, U, and V of a_0, 0, and 0 in $\mathcal{A}$, E, and F respectively and a uniformly C^k representation $\pi : \mathcal{B} \to C^r(U, V)$ such that $U \times V \subseteq Z$ andfor $b \in \mathcal{B}$,

$$\operatorname{graph}(\pi(b)) = W^u(\rho(b) \mid U \times V).$$

Furthermore for $b \in \mathcal{B}$, $\rho(b) \mid U \times V$ is a diffeomorphism and has a unique fixed point which is an attractive fixed point of $\rho(b)^{-1} \mid \operatorname{graph}(\pi(b))$.

6.3 follows immediately from 4.1, 6.1, and 6.3. There is an obvious corollary of 6.3 analogous to 4.2 the statement of which we leave to the reader.

7. Globalization

The unparametrized, finite-dimensional version of theorem 7.1 below is due to Smale [6]. The proof of 7.1 is a straightforward generalization of Smale's proof and is therefore omitted.

The notions of *uniformly C^k Banach manifold* and *uniformly C^k map* are defined in the obvious way. If $\mathcal{A}$, X, and Y are uniformly C^k manifolds, a map $\rho : \mathcal{A} \to C^k(X, Y)$ is a *uniformly C^k representation* iff the evaluation map $ev_\rho : \mathcal{A} \times X \to Y$ is uniformly C^k. Note that C^{k+1} Banach manifolds, maps, and representations are uniformly C^k (mean value theorem).

Let X be a C^1 manifold, $f : X \to X$ a C^1 diffeomorphism, and $x \in X$ a fixed point of f. The *stable manifold of f through x* is denoted by $W^s(f, x)$ and defined to be the set of all points $y \in X$ such that $f^n(y) \to x$ as $n \to \infty$. The *unstable manifold of f through x* is denoted by $W^u(f, x)$ and defined by $W^u(f, x) = W^s(f^{-1}, x)$. x is a *hyperbolic fixed point* iff $T_x f : T_x X \to T_x X$ is a hyperbolic linear operator. An *admissible parametrization of $W^s(f, x)$* is an injective immersion $p : \mathbf{E} \to X$ where $\mathbf{E}$ is a Banach space such that $p(\mathbf{E}) = W^s(f, x)$ and $p^{-1} \circ f \circ p : \mathbf{E} \to \mathbf{E}$ is a diffeomorphism and a contraction map. (The fixed point of $p^{-1} \circ f \circ p$ is necessarily $p^{-1}(x)$.) An *admissible parametrization* of $W^u(f, x)$ is an admissible parametrization of $W^s(f^{-1}, x)$. The importance of admissible parametrizations is explained below.

Recall that a Banach space is *uniformly C^k smooth* iff it admits a uniformly C^k, real valued function of bounded support which is not identically zero.

7.1 Parametrized Global Stable-unstable Manifold Theorem. *Let $\rho : \mathfrak{A} \to C^k(X, X)$ be a uniformly C^k representation ($k \geq 1$) such that for each $a \in \mathfrak{A}$, $\rho(a) : X \to X$ is a diffeomorphism. Let $a_0 \in \mathfrak{A}$ and $x_0 \in X$ and suppose x_0 is a hyperbolic fixed point of $\rho(a_0)$. Suppose further that X is modelled on a uniformly C^k smooth Banach space $\mathbf{G}$. Then there exists a neighborhood $\mathfrak{B}$ of a_0 in $\mathfrak{A}$, a closed splitting $\mathbf{G} = \mathbf{E} \oplus \mathbf{F} \cong \mathbf{E} \times \mathbf{F}$ of $\mathbf{G}$, and uniformly C^k representations*

$$\pi : \mathfrak{B} \to C^k(\mathbf{E}, X) \quad \text{and} \quad \pi' : \mathfrak{B} \to C^k(\mathbf{F}, X)$$

such that for each $b \in \mathfrak{B}$, $\pi(b)$ [resp. $\pi'(b)$] is an admissible parametrization of $W^s(f, x_0)$ [resp. $W^u(f, x_0)$].

Admissible parametrizations are important because of the following lemma, the proof of which we leave for the reader.

Lemma. *Let $x_0 \in X$ be a hyperbolic fixed point of a uniformly C^1 diffeomorphism $f : X \to X$ and let p_1 and p_2 be admissible parametrizations of $W^s(f, x_0)$ [or $W^u(f, x_0)$]. Let $W \subseteq X$ be a C^1 submanifold. Then p_1 is transversal to W if and only if p_2 is. Hence if y_0 is also a hyperbolic fixed point of f (not necessarily distinct from x_0) and q_1 and q_2 are admissible parametrizations of $W^u(f, y_0)$, then p_1 and q_1 are transversal if and only if p_2 and q_2 are.*

This lemma enables us to define transversal intersection properties for stable-unstable manifolds which are independent of the choice of admissible parametrizations. The need for this lemma can be seen by considering a "figure eight" in the plane which crosses itself transversally. Such a figure eight can occur as the stable manifold of a hyperbolic fixed point of a diffeomorphism of the plane. Let W be one of the tangent lines at the crossing. Then there are two injective immersions from the real line onto the figure eight, one of which is transversal to W while the other is not.

Added in proof. After this paper was written, M. C. Irwin published a very elegant proof of 4.2. in *On the stable manifold theorem*, Bull. Amer. Math.

Soc. (2), vol. 76 (1970), pp. 196–198. Presumably Irwin's technique also yields the parametrized version of the stable manifold theorem (4.1).

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The University of Wisconsin
Madison, Wisconsin