THE π-LAYER OF A FINITE GROUP

BY

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1. Introduction

A fundamental property of a π-separable group H, π a set of primes, is the fact that H is *π-constrained*; that is, $O_\pi(H/O_{\pi'}(H))$ contains its own centralizer in $H/O_{\pi'}(H)$ (Theorem 6.3.2 of [3][3]). This concept plays a basic role in almost all of the general classification problems solved to date (see, for example, [1], [2], [4], [5], [6], [7], [8]). On the other hand, in the classification of groups with abelian Sylow 2-subgroups [9], large portions of this analysis involve non-constrained subgroups of the simple group under consideration. It is clear that the latter situation will be typical in more general classification problems.

It is therefore natural to ask whether there exists in an arbitrary finite group H a suitably chosen subgroup of $H/O_{\pi'}(H)$ containing its own centralizer which can be used effectively in place of $O_\pi(H/O_{\pi'}(H))$. In Theorem 1, we prove the existence of such a subgroup. Then in Theorem 2, we study the inverse image of this subgroup in H. This leads to the concept of the *π-layer* and the *π-components* of the group H. In Section 3 we derive a number of elementary properties of the π-layer and the π-components of a group; and in Section 4, we prove some general lemmas which are useful in analyzing the π-components of a given group.

It will be convenient to adopt what we may call the "bar" convention: Namely, if H is a group and X is an element, subset, or subgroup of H and if $\bar{H}$ is a homomorphic image of H, then $\bar{X}$ will always denote the image of X in $\bar{H}$.

2. The π-layer of a group

We say that a group G is *quasisimple* if it possesses a perfect normal subgroup H with the following properties: (i) $H/Z(H)$ is a nonabelian simple group, and (ii) $C_G(H) \subseteq Z(H)$. These conditions imply that H is the unique such subgroup of G and hence that H is characteristic in G. Furthermore, since H is perfect, no proper subgroup of H covers $H/Z(H)$. If $G = H$, we say that G is *perfect quasisimple*.

A central product L of perfect quasisimple groups L_i, $1 \le i \le r$, will be said to be *semisimple*. (This definition is an extension of the usual notion of semisimple.) The factors L_i of L are actually uniquely determined. Indeed, our conditions imply that $Z(L) = \prod_{i=1}^{r} Z(L_i)$, whence $\bar{L} = L/Z(L)$

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[3] In general, we follow the terminology of [3].

is the direct product of its subgroups $\bar{L}_i$, each of which is simple, $1 \leq i \leq r$. Because the $\bar{L}_i$ are nonabelian, $i = 1, 2, \ldots, r$, they form the set of minimal normal subgroups of $\bar{L}$. Hence they are the uniquely determined components of $\bar{L}$. Since any normal group of L_i is normal in L and since no proper subgroup of L_i covers $L_i/Z(L_i)$, it follows that L_i is the unique minimal normal subgroup of L whose image is $\bar{L}_i$ and consequently the L_i are uniquely determined, $1 \leq i \leq r$. They will be called the *components* of the semisimple group L. For completeness, we also call L semisimple if $L = 1$.

If L is semisimple, any product of components of L will be called a *semisimple factor* of L. Clearly any such product is a perfect semisimple normal subgroup of L.

We remark parenthetically that for some applications it is convenient for technical reasons to include the central product of two copies of $SL(2, q)$, q odd, $q > 3$, in the definition of quasisimple group and hence also in that of component of a semisimple group.

We note two useful general properties of semisimple groups. First, if G is a group such that $\bar{G} = G/Z(G)$ is the direct product of nonabelian simple groups, then the derived group G' of G is semisimple and covers $\bar{G}$. Indeed, if L_i, $1 \leq i \leq r$, denote minimal normal subgroups of G which map onto the distinct components of $\bar{G}$, then each L_i is perfect and their product L covers $\bar{G}$. This implies that $L \subseteq G'$ and that $G' \subseteq LZ(G)$. Since G' is perfect, these conditions force $G' = L$. Furthermore, we have $[L_i, L_j] \subseteq Z(G)$ for all $i \neq j$, whence $[L_i, L_j, L_i] = 1$. Since L_i is perfect, the three-subgroup lemma yields that L_i centralizes L_j for all $i \neq j$. Clearly also $L_i \cap Z(G) = Z(L_i)$ and so each L_i is a perfect quasisimple group. Thus G' is the central product of perfect quasisimple groups and so is semisimple.

Second, we claim that any group G possesses a unique maximal normal semisimple subgroup. Indeed, if L and M are semisimple normal subgroups of G, we need only show that $K = LM$ is semisimple. Clearly $Z(L)$ is normal in G, whence

$$[Z(L), M] \subseteq Z(L) \cap M \subseteq Z(M).$$

The three-subgroup lemma together with the perfectness of M now yields that M centralizes $Z(L)$. Thus $Z(L) \subseteq Z(K)$. Similarly $Z(M) \subseteq Z(K)$. Setting $\bar{K} = K/Z(L)Z(M)$, we have $\bar{K} = \bar{L}\bar{M}$ with $\bar{L}$, $\bar{M}$ normal in $\bar{K}$ and with each the direct product of non-abelian simple groups. Hence also $\bar{K}$ is the direct product of nonabelian simple groups. We conclude therefore from the preceding paragraph that K' is a semisimple subgroup of K which covers $\bar{K}$. In particular, $K = K'Z(L)Z(M) = K'Z(K)$. But $K = LM$ is perfect as L, M, are perfect and each is normal in K. Thus $K = K'$ and so K is semisimple, as required. In particular, our argument shows that every normal semisimple subgroup of G is a semisimple factor of the unique maximal normal semisimple subgroup of G.

We now prove

THEOREM 1. *Let H be a group in which $O_{\pi'}(H) = 1$, π a set of primes, let*

L^* be the maximal normal semisimple subgroup of H, set $L = O^{\pi}(L^*)$, and $K = LO_{\pi}(H)$. Then

(i) $C_H(K) \subseteq O_{\pi}(H)$,

(ii) *L is semisimple, $L = O^{\pi 1}(L)$, $[L, O_{\pi}(H)] = 1$, and $L \cap O_{\pi}(H) = Z(L)$.*

Proof. Set $D = O_{\pi}(H)$. We first verify that L satisfies the various parts of (ii). Since $Z(L^*)$ is a normal abelian subgroup of H and $O_{\pi'}(H) = 1$, we have $Z(L^*) \subseteq D$. In particular, $Z(L_1) \subseteq D$ for any component L_1 of L^*. Hence if L_1 is not a π-group, the simplicity of $L_1/Z(L_1)$ and the perfectness of L_1 imply that $O^{\pi}(L_1) = L_1$ and $O_{\pi}(L_1) = Z(L_1)$. It follows at once now from the definition of L that L is a semisimple factor of L^*, that $L^* = LO_{\pi}(L^*)$, that $O_{\pi}(L) = Z(L)$, and that L centralizes $O_{\pi}(L^*)$. Also

$$[L, D] \subseteq L^* \cap D \subseteq O_{\pi}(L^*),$$

whence $[L, D, L] \subseteq [O_{\pi}(L^*), L] = 1$. Since L is perfect, the three-subgroup lemma now yields that $[L, D] = 1$. We note also that $O_{\pi}(L^*)$, being characteristic in L^*, is normal in H and hence is contained in D. But then $L^* = LO_{\pi}(L^*) \subseteq LD = K$.

Finally suppose $O^{\pi'}(L_1) \subset L_1$ for some component L_1 of L. Then $O^{\pi'}(L_1) \subseteq Z(L_1)$ as $L_1/Z(L_1)$ is simple, whence $L_1/Z(L_1)$ is a π'-group. But then by the Schur-Zassenhaus theorem, $Z(L_1)$ possesses a complement M_1 in L_1 and so $L_1 = M_1 \times Z(L_1)$. Since L_1 is perfect, this forces $Z(L_1) = 1$ and L_1 to be a π'-group. Thus $L_1 \subseteq O_{\pi'}(L^*)$ and so $O_{\pi'}(L^*) \neq 1$. On the other hand, $O_{\pi'}(L^*)$ is characteristic in L^* and so is normal in H, whence $O_{\pi'}(L^*) \subseteq O_{\pi'}(H) = 1$, a contradiction. Thus $O^{\pi'}(L_1) = L_1$ for each component L_1 of L and therefore $O^{\pi'}(L) = L$. Hence all parts of (ii) hold.

Now set $E = C_H(K)$. Then E is characteristic in H and so $O_{\pi'}(E) = 1$. Let E^* be the maximal semisimple normal subgroup of E. Then E^* is characteristic in E and so is normal in H, whence $E^* \subseteq L^*$. But we have already shown that $L^* \subseteq K$ and therefore $E^* = [E^*, E^*] \subseteq [E, K] = 1$.

Observe next that $O_{\pi}(E) \subseteq D$ and hence

$$E = C_H(K) \subseteq C_H(D) \subseteq C_H(O_{\pi}(E)).$$

Thus $O_{\pi}(E) \subseteq Z(E)$. Applying the Schur-Zassenhaus theorem now to $O_{\pi, \pi'}(E)$, it follows that

$$O_{\pi, \pi'}(E) = O_{\pi}(E) \times O_{\pi'}(E).$$

Since $O_{\pi'}(E) = 1$, we conclude that $O_{\pi, \pi'}(E) = O_{\pi}(E)$.

We shall now argue that $E = O_{\pi}(E)$, which will suffice to prove (i), as then $E = C_H(K) = O_{\pi}(E) \subseteq D = O_{\pi}(H)$. Assume false and set $\bar{E} = E/O_{\pi}(E)$. Our conditions imply that $O_{\pi}(\bar{E}) = O_{\pi'}(\bar{E}) = 1$ and that $\bar{E} \neq 1$. Let $\bar{N}$ be a minimal normal subgroup of $\bar{E}$. Since any prime p is either in π or π', $\bar{N}$ is not an elementary abelian group of prime power order and so $\bar{N}$, being characteristically simple, is necessarily the direct product of isomorphic nonabelian simple groups. If N denotes the inverse image of $\bar{N}$

in E, then $Z(N) = Z(E)$ and $N/Z(N) = \bar{N}$. It follows that N' is a semisimple group which covers $\bar{N}$. In particular, $N' \neq 1$. On the other hand, N' is characteristic in N and N is normal in E, whence N' is normal in E. But then $N' \subseteq E^*$, contrary to the fact that $E^* = 1$. This establishes (i) and completes the proof.

On the basis of the preceding theorem, we introduce the following terminology:

DEFINITION 2. If H is a group in which $O_{\pi'}(H) = 1$ for some set of primes π, we denote the uniquely determined subgroups K and L of Theorem 1 by $O_\pi^*(H)$ and $L_\pi(H)$ respectively. We call $L_\pi(H)$ the *π-layer* of H and if $L_\pi(H) \neq 1$, we call its uniquely determined components the *π-components* of H.

In this notation, we have

$$O_\pi^*(H) = L_\pi(H)O_\pi(H),$$

where $L_\pi(H)$ is a semisimple group centralizing $O_\pi(H)$ and, moreover, $O_\pi^*(H)$ contains its own centralizer in H.

By analogy with the notation $O_{\pi',\pi}(H)$, we define the following additional terms:

DEFINITION 3. For any group H and any set of primes π, we let $O_{\pi',\pi}^*(H)$ and $L_{\pi'\pi}^*(H)$ denote the inverse images in H of $O_\pi^*(H/O_{\pi'}(H))$ and $L_\pi(H/O_{\pi'}(H))$ respectively.

The group $L_{\pi',\pi}^*(H)$ is normal in $O_{\pi',\pi}^*(H)$ and covers the π-layer of $H/O_{\pi'}(H)$. Our next result will show that there is a unique subgroup of $O_{\pi',\pi}^*(H)$ which is minimal subject to these conditions; it is this latter subgroup which we shall call the π-layer of H.

THEOREM 2. *Let H be a group in which the π-layer of $\bar{H} = H/O_{\pi'}(H)$ is nontrivial, π a set of primes. Let $\bar{L}_i$, $1 \leq i \leq r$, be the π-components of $\bar{H}$, let L_i be a minimal normal subgroup of $O_{\pi',\pi}^*(H)$ which covers $\bar{L}_i$, $1 \leq i \leq r$, and let L be the minimal normal subgroup of $O_{\pi',\pi}^*(H)$ which covers the π-layer of $\bar{H}$. Then the groups L_i and L are uniquely determined, $1 \leq i \leq r$, and we have*

(i) $L = \prod_{i=1}^r L_i$ *and* $L_{\pi',\pi}^*(H) = LO_{\pi'}(H)$.
(ii) L *is characteristic in* H.
(iii) $O^{\pi'}(L_i) = O^\pi(L_i) = [L_i, L_i] = L_i$, $1 \leq i \leq r$.
(iv) $O_{\pi'}(L_i) = L_i \cap O_{\pi'}(H)$ *and* $O_{\pi',\pi}(L_i) = L_i \cap O_{\pi',\pi}(H)$, $1 \leq i \leq r$.

Proof. As each $\bar{L}_i$ and $L_\pi(\bar{H})$ is normal in $O_\pi^*(\bar{H})$, it follows that any such minimal normal subgroups L_i and L of $O_{\pi',\pi}^*(H)$ map onto $\bar{L}_i$ and $L_\pi(\bar{H})$ respectively, $1 \leq i \leq r$. To establish the uniqueness of the L_i, suppose M_i is another minimal normal subgroup of $O_{\pi',\pi}^*(H)$ covering $\bar{L}_i$. Since $\bar{M}_i = \bar{L}_i$ and $\bar{L}_i$ is perfect, it follows that $[M_i, L_i]$ also has $\bar{L}_i$ as its image. But

$[M_i, L_i] \subseteq M_i \cap L_i$ and $[M_i, L_i]$ is normal in $O^*_{\pi',\pi}(H)$. The minimality of M_i and L_i now forces $[M_i, L_i] = M_i$ and $[M_i, L_i] = L_i$, whence $M_i = L_i$ and the uniqueness of the L_i is proved, $1 \leq i \leq r$. The uniqueness of L is similarly proved.

Since $L_\pi(\bar{H}) = \prod_{i=1}^r \bar{L}_i$, L covers each $\bar{L}_i$ and hence $L \supseteq L_i$, $1 \leq i \leq r$. Thus $L \supseteq M = \prod_{i=1}^r L_i$. On the other hand, clearly M is a normal subgroup of $O^*_{\pi',\pi}(H)$ which covers $L_\pi(\bar{H})$, so $L \subseteq M$ by the minimality of L. We conclude that $L = M$. Since $L^*_{\pi',\pi}(H)$ is the inverse image of $L_\pi(\bar{H})$ in H, we also have that $L^*_{\pi',\pi}(H) = LO_{\pi'}(H)$, so both parts of (i) hold.

By Theorem 1, $L_\pi(\bar{H}) = O^\pi(\bar{L}^*)$, where $\bar{L}^*$ denotes the unique maximal normal semisimple subgroup of $\bar{H}$, and consequently $L_\pi(\bar{H})$ is characteristic in $\bar{H}$. Since $O_{\pi'}(H)$ and $O^*_{\pi',\pi}(H)$ are each characteristic in H, it follows that L^α is a normal subgroup of $O^*_{\pi',\pi}(H)$ which covers $L_\pi(\bar{H})$ for any α in $\mathrm{Aut}(H)$. Since L^α and L have the same order, the minimality of L forces $L^\alpha = L$. We conclude that L is characteristic in H, proving (ii).

By Theorem 1, we have $O^{\pi'}(\bar{L}_i) = O^\pi(\bar{L}_i) = [\bar{L}_i, \bar{L}_i] = \bar{L}_i$ and consequently $O^{\pi'}(L_i)$, $O^\pi(L_i)$, and $[L_i, L_i]$ each maps onto $\bar{L}_i$. Since each of these three groups is normal in $O^*_{\pi',\pi}(H)$, being characteristic in L_i, the minimality of L_i now forces $O^{\pi'}(L_i) = O^\pi(L_i) = [L_i, L_i] = L_i$, $1 \leq i \leq r$, so (iii) also holds.

Finally we clearly have

$$O_{\pi'}(H) \cap L_i \subseteq O_{\pi'}(L_i) \quad \text{and} \quad O_{\pi',\pi}(H) \cap L_i \subseteq O_{\pi',\pi}(L_i),$$

so to prove (iv) it suffices to establish the reverse inclusions. Set $K = O^*_{\pi',\pi}(H)$. Since L_i is normal in K and $O_{\pi'}(L_i)$, $O_{\pi'\pi}(L_i)$ are characteristic in L_i, we have that $O_{\pi'}(L_i)$ and $O_{\pi',\pi}(L_i)$ are normal in K, whence

$$O_{\pi'}(L_i) \subseteq O_{\pi'}(K) \quad \text{and} \quad O_{\pi',\pi}(L_i) \subseteq O_{\pi',\pi}(K).$$

But K is normal in H and so by the same argument

$$O_{\pi'}(K) \subseteq O_{\pi'}(H) \quad \text{and} \quad O_{\pi',\pi}(K) \subseteq O_{\pi',\pi}(H).$$

We conclude at once that

$$O_{\pi'}(L_i) \subseteq L_i \cap O_{\pi'}(H) \quad \text{and} \quad O_{\pi',\pi}(L_i) \subseteq L_i \cap O_{\pi',\pi}(H), 1 \leq i \leq r,$$

thus completing the proof of (iv).

DEFINITION 4. We call the subgroup L of Theorem 2 the *π-layer* of H and denote it by $L_{\pi',\pi}(H)$. Moreover, we call its subgroups L_i the *π-components* of H, $1 \leq i \leq r$. For completeness, we set $L_{\pi',\pi}(H) = 1$ if the π-layer of $H/O_{\pi'}(H)$ is trivial.

We note that these definitions are consistent with those of Definition 2; that is to say, if $O_{\pi'}(H) = 1$, then the terms π-component and π-layer of Definition 4 have the identical meanings as those given in Definition 2.

In the applications of these results to the local group-theoretic analysis in general classification problems, it is very often necessary to pass back and forth between the π-components of a group H and their images in $\bar{H} = H/O_{\pi'}(H)$. To facilitate this procedure, it is useful to introduce the following terminology:

DEFINITION 5. Let L be a π-component of a group H, π a set of primes, and let $\bar{L}$ be the image of L in $\bar{H} = H/O_{\pi'}(H)$. Then we call $\bar{L}$ the *associated perfect quasi-simple component* of L and we call L the *associated covering component* of $\bar{L}$.

3. Some properties of π-components

In this section we establish several additional results concerning the π-components and π-layer of a group.

As an immediate consequence of the various preceding definitions, we have the following criterion for π-constraint:

THEOREM 3. *A group H is π-constrained for a set of primes π if and only if*

$$O^*_{\pi',\pi}(H) = O_{\pi',\pi}(H) \quad \textit{and} \quad L_{\pi',\pi}(H) = 1.$$

We next describe the action of H on its π-components:

THEOREM 4. *For any group H and any set of primes π, H induces under conjugation a group of permutations of the set of π-components of H.*

Proof. We can clearly assume without loss that $L_{\pi',\pi}(H) \neq 1$. Let L_i, $1 \leq i \leq r$, be the π-components of H and set $\bar{H} = H/O_{\pi'}(H)$. Then the subgroups $\bar{L}_i$, $1 \leq i \leq r$, are the unique minimal normal nonsolvable subgroups of $L_\pi(\bar{H})$ and so each element of $\bar{H}$ acts as a permutation of the $\bar{L}_i$, $1 \leq i \leq r$. This implies that for any h in H and any i, L_i^h maps onto $\bar{L}_j$ for some j, $1 \leq i, j \leq r$. But L_i is normal in $O^*_{\pi',\pi}(H)$ and hence so is L_i^h. Hence by the uniqueness and minimality of L_j, it follows that $L_j \subseteq L_i^h$. The same reasoning applied to $L_j^{h^{-1}}$ shows that $L_i \subseteq L_j^{h^{-1}}$, whence $L_i^h \subseteq L_j$. Thus $L_i^h = L_j$ and we conclude that h induces a permutation of the π-components L_i of H, $1 \leq i \leq r$. Thus the theorem follows.

The following result is a direct corollary of Theorems 1 and 2:

THEOREM 5. *For any group H and any set of primes π, the image in $H/O_{\pi',\pi}(H)$ of each π-component of H is a nonabelian simple group and the image of $L_{\pi'\pi}(H)$ is the direct product of the images of the distinct π-components of H.*

Our next result is a direct extension of a well-known property of π-separable groups.

THEOREM 6. *If D is a π'-subgroup of the group H, π a set of primes, and if $[D, O^*_{\pi',\pi}(H)]$ is a π'-group, then $D \subseteq O_{\pi'}(H)$.*

Proof. Setting $\bar{H} = H/O_{\pi'}(H)$, $\bar{K} = O_\pi^*(\bar{H})$, and $\bar{X} = [\bar{D}, \bar{K}]$, our conditions imply that $\bar{X}$ is a π'-group, as $\bar{K}$ is the image of $O_{\pi',\pi}^*(H)$ in $\bar{H}$, and that $\bar{X}$ is normal in $\bar{K}$. But by Theorem 1, $O_{\pi'}(\bar{K}) = 1$, whence $\bar{X} = 1$ and so $\bar{D}$ centralizes $\bar{K}$. Again by Theorem 1, $C_{\bar{H}}(\bar{K})$ is a π-group, thus forcing $\bar{D} = 1$. We conclude that $D \subseteq O_{\pi'}(H)$.

We also have the following variation of Theorem 6.

THEOREM 7. *If D is a subgroup of the group H and if $[L_{\pi',\pi}(H), D]$ is π-solvable, π a set of primes, then $[L_{\pi',\pi}(H), D] \subseteq O_{\pi'}(H)$.*

Proof. Set $L = L_{\pi',\pi}(H)$ and $\bar{H} = H/O_{\pi'}(H)$, so that by definition $\bar{L} = L_\pi(\bar{H})$. Our conditions imply that $[\bar{L}, \bar{D}]$ is a π-solvable normal subgroup of $\bar{L}$. But by Theorem 1, either $\bar{L} = 1$ or $\bar{L}/Z(\bar{L})$ is the direct product of nonabelian simple groups none of which is a π-group or a π'-group. In either case it follows that $[\bar{L}, \bar{D}] \subseteq Z(\bar{L})$, whence $\bar{L}$ centralizes $\bar{D}$ by the three-subgroup lemma. Thus $[L, D] \subseteq O_{\pi'}(H)$, as asserted.

Our next theorem gives conditions for a product of π-components of a group to be semisimple.

THEOREM 8. *For any group H and any set of primes π, the following conditions hold:*

(i) *If L is a π-component of H, then L is perfect quasisimple if and only if L centralizes $O_{\pi'}(H)$.*

(ii) *Any product of perfect quasisimple π-components of H is semisimple.*

(iii) *The π-layer of H is semisimple if and only if it centralizes $O_{\pi'}(H)$.*

Proof. Let L be a perfect quasisimple π-component of H, in which case $L/Z(L)$ is simple. If $D = O_{\pi'}(L)$, then D is a proper normal subgroup of L inasmuch as $O^{\pi'}(L) = L$ and consequently $D \subseteq Z(L)$. On the other hand, as L and $O_{\pi'}(H)$ are normal in $O_{\pi',\pi}^*(H)$,

$$[L, O_{\pi'}(H)] \subseteq L \cap O_{\pi'}(H) = D$$

by part (iv) of Theorem 2. Since $D \subseteq Z(L)$, it follows that L centralizes $[L, O_{\pi'}(H)]$, whence L centralizes $O_{\pi'}(H)$ by the three-subgroup lemma.

Conversely suppose a π-component L of H centralizes $O_{\pi'}(H)$, in which case L centralizes $L \cap O_{\pi'}(H) = D$. Setting $\bar{H} = H/O_{\pi'}(H)$, we know that $\bar{L}$ is perfect quasisimple. Hence if E denotes the inverse image of $Z(\bar{L})$ in L, we have $D \subseteq E$, E/D is isomorphic to $Z(\bar{L})$, and L/E is a nonabelian simple group. Furthermore, $Z(\bar{L})$ is a π-group by Theorem 1. Since $D \subseteq Z(L)$, this implies that $E = D \times F$, where F is a π-group which maps onto $Z(\bar{L})$ and is normal in L. Since $\bar{L}$ centralizes $\bar{F} = Z(\bar{L})$, it follows that L centralizes F, whence L centralizes E. Thus $E \subseteq Z(L)$ and, as L is perfect, we conclude that L is perfect quasisimple, completing the proof of (i).

Next let L_i, $1 \leq i \leq m$, be distinct perfect quasisimple π-components of H. Then each $\bar{L}_i$ is a component of the π-layer $L_\pi(\bar{H})$ of $\bar{H} = H/O_{\pi'}(H)$, so $[\bar{L}_i, \bar{L}_j] = 1$ for all $i \neq j$. Thus $[L_i, L_j] \subseteq O_{\pi'}(H)$, $i \neq j, 1 \leq i, j \leq m$.

But L_i centralizes $O_{\pi'}(H)$ by (i) as L_i is perfect quasisimple and consequently L_i centralizes L_j by the three-subgroup lemma for all $i \neq j$. Hence the product of the L_i is semisimple, proving (ii).

Finally, the π-layer $L_{\pi',\pi}(H)$ is, by definition, the product of all the π-components of H (with $L_{\pi',\pi}(H) = 1$ if H has no π-components). Hence (iii) is an immediate consequence of (i) and (ii).

Our last result is a direct corollary of Theorem 1:

THEOREM 9. *If H is a group in which $O_{\pi'}(H) = 1$, π a set of primes, and if L is a normal semisimple subgroup of H such that $O^{\pi}(L) = L$, then $L \subseteq L_{\pi}(H)$ and each component of L is a π-component of H.*

Proof. If L^* denotes the unique maximal semisimple normal subgroup of H, we know that L is a semisimple factor of L^*. Moreover, by Theorem 1, $L_{\pi}(H) = O^{\pi}(L^*)$. Since $O^{\pi}(L) = L$, it follows that $L \subseteq L_{\pi}(H)$. Thus L is, in fact, a semisimple factor of $L_{\pi}(H)$ and so each of its components is a component of $L_{\pi}(H)$ and hence is a π-component of H.

4. Some general lemmas

In this section we shall prove some general results which are useful when studying the π-components of a given group.

If the π-group A acts on the π'-group B, it is well known that $[B, A, A] = [B, A]$. We can establish quite easily the following variation of this result:

LEMMA 1. *Let A be a group which acts on the π'-group B, π a set of primes. If $O^{\pi'}(A) = A$, then*

$$[B, A, A] = [B, A].$$

Proof. Let $\{P_i, 1 \leq i \leq m\}$ denote the set of all Sylow p-subgroups of A as p ranges over the primes in π. Clearly the P_i generate a normal subgroup N of A such that A/N is a π'-group. Since $O^{\pi'}(A) = A$ by assumption, A/N must be trivial and consequently

$$A = \langle P_i \mid 1 \leq i \leq m \rangle.$$

Since B is a π'-group and P_i is a p-group for some p in π, we have $[B, P_i] = [B, P_i, P_i]$, whence

$$[B, P_i] \subseteq [B, A, A]$$

for all i, $1 \leq i \leq m$.

We argue now that if P_{i_j}, $1 \leq j \leq k$, are chosen arbitrarily from the set $\{P_i \mid 1 \leq i \leq m\}$, then

$$[B, P_{i_1} P_{i_2} \cdots P_{i_k}] \subseteq [B, A, A].$$

Indeed, we have already proved this above when $k = 1$. We proceed by induction on k. Hence if we set $Q = P_{i_1} P_{i_2} \cdots P_{i_{k-1}}$, then we can assume that

$$[B, Q] \subseteq [B, A, A]$$

and we must prove that

$$[B, QP_{i_k}] \subseteq [B, A, A].$$

But it is an immediate consequence of the commutator identity of Lemma 2.4 (ii) of [3] together with the fact that $[B, Q]$ and $[B, P_{i_k}]$ are each normal in B that

$$[B, QP_{i_k}] \subseteq [B, P_{i_k}][B, Q][B, Q, P_{i_k}].$$

Since $[B, Q] \subseteq [B, A, A] \subseteq B$ and since $[B, P_{i_k}] \subseteq [B, A, A]$, it follows that each term on the right side lies in $[B, A, A]$ and hence so does $[B, QP_{i_k}]$.

Finally, every element x of A lies in $P_{i_1} P_{i_2} \cdots P_{i_k}$ for some k and some choice of the groups P_{i_j}, $1 \leq j \leq k$, and consequently

$$[B, x] \subseteq [B, A, A]$$

for all x in A. Hence $[B, A] \subseteq [B, A, A]$ and the lemma follows.

Another result similar in spirit is the following:

LEMMA 2. *Let H be a group of the form AB, where B is a normal π'-subgroup of H and $O^{\pi'}(A) = A$. If K is a subgroup of A such that $A = K(A \cap B)$ then*

$$[B, K] = [B, A].$$

Proof. Setting $X = [B, K]$, we have that X is normal in $\langle B, K\rangle$. But

$$H = AB = K(A \cap B)B = KB,$$

so X is normal in H. Moreover, KX/X centralizes BX/X and consequently also KX is normal in H. Thus $KX \cap A$ is normal in A. But $A/KX \cap A$ is isomorphic to $AX/KX = KX(A \cap B)/KX$ and so is a π'-group inasmuch as B is a π'-group. However, $O^{\pi'}(A) = A$ by assumption, whence $A/KX \cap A = 1$ and therefore $A \subseteq KX$. We conclude that $[B, A] \subseteq [B, KX]$.

We claim that $[B, KX] \subseteq [B, K]$. Indeed, if $\bar{K}\bar{B} = KB/X$, then $[\bar{B}, \bar{K}\bar{X}] = [\bar{B}, \bar{K}] = 1$ as $X = [B, K]$, and the assertion follows. Thus $[B, A] \subseteq [B, K]$. On the other hand, $[B, K] \subseteq [B, A]$ as $K \subseteq A$ and we obtain the desired conclusion $[B, K] = [B, A]$.

As an immediate consequence of the three-subgroup lemma, we also have

LEMMA 3. *If the group A acts on the perfect group B and if A centralizes $B/Z(B)$, then A centralizes B.*

Finally, we prove

LEMMA 4. *Let L be a group such that $O^{\pi'}(L) = O^{\pi}(L) = L$ and $L/O_{\pi'}(L)$ is perfect quasisimple for some set of primes π. If K is a normal subgroup of L with $K \nsubseteq O_{\pi',\pi}(L)$, then $L = K$.*

Proof. Setting $\bar{L} = L/O_{\pi'}(L)$, our conditions imply that $O^{\pi}(\bar{L}) = \bar{L}$, that $\bar{L}$ is perfect quasisimple, and that $O_{\pi'}(\bar{L}) = 1$. In particular, $Z(\bar{L})$

is a π-group and $O_\pi(\bar{L}) \neq \bar{L}$. Since $\bar{L}/Z(\bar{L})$ is simple, this forces $O_\pi(\bar{L}) = Z(\bar{L})$.

Furthermore, $\bar{K}$ is normal in $\bar{L}$ and $\bar{K} \not\subseteq O_\pi(\bar{L})$, whence $\bar{K} \not\subseteq Z(\bar{L})$. Since $\bar{L}$ is perfect quasisimple, it follows that $\bar{K} = \bar{L}$ and hence that $L = KO_{\pi'}(L)$. This in turn implies that L/K is a π'-group. Since $O^{\pi'}(L) = L$, we conclude that $L = K$.

Bibliography

1. J. L. Alperin, R. Brauer, and D. Gorenstein, *Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups*, Trans. Amer. Math., Soc., vol. 151 (1970), pp. 1–261.
2. W. Feit, and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math., vol. 13 (1963), pp. 775–1029.
3. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
4. ———, *Finite groups in which the centralizer of every involution has a normal 2-complement*, Canad. J. Math., vol. 21 (1969), pp. 335–357.
5. ———, *On the centralizers of involutions in finite groups*, J. Algebra, vol. 11 (1969), pp. 243–277.
6. D. Gorenstein and J. H. Walter, *On the maximal subgroups of finite simple groups*, J. Algebra, vol. 1 (1964), pp. 168–213.
7. ———, *The characterization of finite groups with dihedral Sylow 2-subgroups*, I, II, III, J. Algebra, vol. 2 (1965), pp. 85–151, pp. 218–270, pp. 334–393.
8. J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*, sections 1–6 appear in Bull. Amer. Math. Soc., vol. 74 (1968), pp. 383–437; sections 7–9, Pacific J. Math., vol. 33 (1970), pp. 451–539; balance to appear.
9. J. H. Walter, *The characterization of finite groups with abelian Sylow 2-subgroups*, Ann. of Math., vol. 89 (1969), pp. 405–514.

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