ON <8>-COBORDISM

BY

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Since Lashof [7] pointed out that one could obtain a cobordism theory with respect to any system Y of maps $f_k : Y_k \to BO_k$, $\iota_k : Y_k \to Y_{k+1}$, satisfying the obviously compatibility condition with respect to the canonical map $BO_k \to BO_{k+1}$, many people have considered various forms of these theories. See e.g. [3] and [8]. See also Stong [14] for a very general treatment of the subject.

In this note we consider the cobordism theory associated to the 7-connected covering $BO\langle 8\rangle$ of BO. The algebraic structure is complicated and results are given in low dimensions. The cohomology of the Thom space $MO\langle 8\rangle$ is examined, and its homotopy is partially calculated. The introduction of the relative theory $\Omega^{\langle 8\rangle,\mathrm{Spin}}$ permits the determination of some differentials, and the 2 primary part of $\pi_*(MO\langle 8\rangle)$ is given in dimensions ≤ 16. In the final section the corresponding 3-torsion is calculated, and it is shown that there is no p-torsion, for primes $p > 3$. Straightforward but complicated computations are not reproduced.

The results are partially contained in the author's thesis (MIT, 1966).

1. Definitions

The term manifold will be used for compact C^∞ manifolds, with or without boundary. BO_k is the classifying space for k-dimensional vector bundles, and γ_k the universal $\mathbb{R}^k$ bundle over BO_k. Let $BO\langle 8\rangle_k$ denote the 7-connected covering of BO_k and $p_k : BO\langle 8\rangle_k \to BO_k$ the canonical projection, i.e., p_k is a fibration, $BO\langle 8\rangle_k$ is 7 connected, and the homomorphism

$$p_{k\#} : \pi_i(BO\langle 8\rangle_k) \to \pi_i(BO_k)$$

is an isomorphism if $i \geq 8$. An n-dimensional manifold M embedded in $\mathbb{R}^{n+k}$ is said to have an $\langle 8\rangle_k$ structure if the map $\nu : M \to BO_k$ classifying the normal bundle has a lifting $\tilde{\nu} : M \to BO\langle 8\rangle_k$.

The "inclusion" $i_k : BO_k \to BO_{k+1}$ lifts to map $\iota : BO\langle 8\rangle_k \to BO\langle 8\rangle_{k+1}$ which is unique up to homotopy (and which may be chosen specifically). Since the diagram

$$\begin{array}{ccccc} & & BO\langle 8\rangle_k & \xrightarrow{\iota_k} & BO\langle 8\rangle_{k+1} \\ & \overset{\tilde{\nu}}{\nearrow} & \downarrow & & \downarrow p_{k+1} \\ M & \xrightarrow{\nu} & BO_k & \xrightarrow{i_k} & BO_{k+1} \end{array}$$

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is commutative, on $\langle 8 \rangle$ structure on M induces an $\langle 8 \rangle_{k+1}$ structure. Thus we obtain a sequence of $\langle 8 \rangle_r$ structures, $r = k, k + 1, \cdots$. Two such sequences are identified if they are the same for some r. A manifold has an $\langle 8 \rangle$ structure if it has an $\langle 8 \rangle_k$ structure for some k. Letting $BO\langle 8 \rangle = \lim_k BO\langle 8 \rangle_k$, we can reformulate this as follows.

DEFINITION. A manifold M is said to have an $\langle 8 \rangle$-structure if the classifying map $\nu : \ \rightarrow BO$ of its stable normal bundle admits a lifting $\nu : M \rightarrow BO\langle 8 \rangle$. Note that this is equivalent to requiring that the first two Stiefel-Whitney classes vanish (implying that M is a Spin manifold), and that $\nu_1^*(W_4) = 0$, where $W_4 \in H^4(\text{BSpin}, Z)$ is a generator, ν_1 being any lifting of ν to BSpin. An $\langle 8 \rangle$-structure on M is a homotopy class of such liftings A manifold with an $\langle 8 \rangle$-structure will be called an $\langle 8 \rangle$-manifold. A cobordism relation ($\langle 8 \rangle$-cobordism) is defined as usual: Two n-dimensional $\langle 8 \rangle$-manifolds are cobordant if there is an $(n + 1)$-dimensional $\langle 8 \rangle$-manifold W such that

1. the boundary of W is the disjoint union of M_1 and M_2, and
2. for each $i = 1, 2$, the following diagram is commutative:

$$\begin{array}{ccc} W & \xrightarrow{\tilde{\nu}} & BO\langle 8 \rangle \\ \uparrow & \overset{\tilde{\nu}_{M_i}}{\nearrow} & \downarrow p \\ M_i & \xrightarrow[\nu_{M_i}]{} & BO. \end{array}$$

Let $\Omega_n^{\langle 8 \rangle}$ be the set of equivalence classes of n-dimensional $\langle 8 \rangle$-manifolds with respect to the above relation, and $\Omega_*^{\langle 8 \rangle}$ the associated graded group. Then we have the following result.

THEOREM 1.1. *$\Omega_*^{\langle 8 \rangle}$ is a graded ring with operations induced by disjoint union and cartesian product. Moreover*

$$\Omega_n^{\langle 8 \rangle} = \pi_n(MO\langle 8 \rangle) = \lim_k \pi_{n+k}(MO\langle 8 \rangle).$$

Proof. The proof that $\Omega_*^{\langle 8 \rangle}$ is a graded group isomorphic to $\pi_*(MO\langle 8 \rangle)$ is a special case of a theorem of Lashof [7]. To complete the proof, note that the map $BO \times BO \rightarrow BO$ induced by Whitney sum has a unique (up to homotopy) lifting $BO\langle 8 \rangle \times BO\langle 8 \rangle \rightarrow BO\langle 8 \rangle$. This gives a unique $\langle 8 \rangle$ structure on the product of two $\langle 8 \rangle$ manifolds.

A simple surgery argument [7] shows that every $\langle 8 \rangle$-manifold of dimension ≥ 16 is $\langle 8 \rangle$-cobordant to a 7-connected manifold. Hence $\Omega_n^{\langle 8 \rangle}$, $n \geq 16$ can be considered as the cobordism group of n-dimensional 7-connected manifolds.

2. The E_2 term

In this section the E_2 term of the Adams spectral sequence for the 2-primary part of $\pi_*(MO\langle 8 \rangle)$ is computed. Here, and in §3, all homotopy and cohomology will be with coefficients in Z_2, and π_i will denote the 2-primary part of the ith homotopy group. A will be the mod 2 Steenrod algebra, and A^+ the ideal of terms of positive degree in A.

In order to compute the E_2 term, we need the cohomology of $MO\langle 8\rangle$ a module over the Steenrod algebra. The following lemma is the first step.

LEMMA 2.1. *$H^*(BO\langle 8\rangle)$ is a polynomial algebra with generators*

$$w_i \in H^i(BO\langle 8\rangle)$$

such that $i - 1$ has at least three ones in its dyadic expansion. Moreover each w_i is the image of the ith Stiefel-Whitney class in $H^i(BO)$ under the map p_* induced by the projection.

Proof. This is Theorem A of Stong [13].

Since $H^*(MO\langle 8\rangle)$ is isomorphic as a graded group to $H^*(BO\langle 8\rangle)$, we have only to compute the A-module structure in $H^*(BO\langle 8\rangle)$. But this is a quotient of $H^*(BO)$, and one can compute the action of A on any element of $H^*(MO\langle 8\rangle)$ with the Cartan and Wu formulas, and the fact that $Sq^i(U) = w_i U$, U being the Thom class in $H^0(MO)$ or $H^0(MO\langle 8\rangle)$. As an example we have the following lemma:

LEMMA 2.2. *Let A_2 be the Hopf-subalgebra of A generated by the elements Sq^0, Sq^1, and Sq^2. Let A_2^+ be the ideal in A_2 of terms of positive degree. The map*

$$e : A /\!/ AA_2^+ \to H^*(BO\langle 8\rangle)$$

given by $e(x) = xU$ is a monomorphism.

Proof. Since A_2 is a Hopf-subalgebra of A, $A/\!/AA_2^+$ is a coalgebra. e is a map of coalgebras, and hence it suffices [12, Prop 3.9] to prove that e is a monomorphism on the primitive elements of $A/\!/AA_2^+$. To compute the primitive elements, recall that they correspond under duality to the indecomposables in the dual. A^* is isomorphic to the polynomial ring $Z_2[\xi_1, \xi_2, \cdots]$ with generators ξ_i of degree $2^i - 1$, and $(A/AA_2^+)^*$ is isomorphic to the subring $Z_2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \cdots]$. Hence there are primitives only in degrees 8, 12, 14, and $2^i - 1$ for $i \geq 4$, and there is only one primitive in each of those degrees. The first three are the classes of Sq^8, Sq^{12}, and Sq^{14}–the only nonzero elements in those degrees, and the others are the projections of the primitives in A, usually denoted by Q_i, $i \geq 4$. To complete the proof note that

$$Sq^j U = w_j U \neq 0, \qquad j = 8, 12, 14$$

and

$$Q_i U = w_{2^i-1} U + (\text{decomposables}) U.$$

But $w_{2^i-1} \neq 0$ in $H^{2^i-1}(BO\langle 8\rangle)$ if $i \geq 4$.

COROLLARY 2.3. *As a module over the Steenrod algebra in dimensions <20,*

$H^*(MO\langle 8\rangle)$ is isomorphic to the direct sum of two copies of $A/\!/AA_2^+$, one starting in dimension 0, the other in dimension 16.

Proof. In this range there are only six elements in $H^*(MO\langle 8\rangle)$; call them the images under the Thom isomorphism of W_i, for $i = 8, 12, 14, 15, 16$, and the image of w_8^2. All but the class corresponding to w_8^2 are in the image of e, and $Sq^i(w_8)^2 = 0$ for $i < 8$.

In dimensions 50, one can compute the structure of $H^*(MO\langle 8\rangle)$ by brute force. The results are given below. In these dimensions $H^*(MO\langle 8\rangle)$ is the direct sum of cyclic modules of six different types. The modules, together with the dimensions in which A generators for them appear, are as follows (repeated integers indicating more than one summand of a given type with generators in the same dimension):

$A/\!/AA_2^+$	0, 16, 32, 32, 48, 48, 48
$A/\!/A(Sq^1, Sq^5, Sq^6, Sq^{13})$	20, 36, 36
$A/\!/A(Sq^1, Sq^9)$	40
$A/\!/A(Sq^1, Sq^5)$	44
$A/\!/A(Sq^2, Sq^2Sq^1)$	46, 46
$A/\!/A(Sq^1, Sq^2)$	48

We now compute the E_2 term of the Adams spectral sequence up to dimension 17, i.e., $\mathrm{Ext}_A(A/\!/AA_2^+, Z_2)$ for $t - s < 17$. But $\mathrm{Ext}_A(A/\!/AA_2^+, Z_2) \approx \mathrm{Ext}_{A_2}(Z_2, Z_2)$ (Liulevicius [9, Theorem 1.5]). Now $\mathrm{Ext}_{A_2}(Z_2, Z_2)$ was completely determined by the author in his thesis using the results of Peter May [10], and independently, using other methods, by N. Shimada and A. Iwai [6]. We refer the reader to either of these sources for complete results and details of the methods of computation.

THEOREM 2.4. *As an algebra, $\mathrm{Ext}_{A_2}^{s,t}(Z_2, Z_2)$, in the range $t - s < 17$, has 9 generators: h_0, h_1, h_2, c_0, ω, τ, d_0 and $\varkappa$. The gradings can be read off of Figure 1. There are the following relations: $h_i h_{i-1}$, $h_1^3 + h_0^2 h_2$, h_2^3; $h_0 h_2^2 c_0^2$, $h_\theta c_0$, $h_2 c_0$, $h_1^2 c_0$, $h_1 \tau$, $h_2 \tau + h_0 \varkappa$, $h_1 \varkappa$, $h_0^2 \varkappa + h_1 d_0$, $h_2 d_0 + h_0 c_0$, $h_0^2 d_0 + h_2^2 \omega$. Moreover, each element denoted by a latin letter is the image of the corresponding element of $\mathrm{Ext}_{A_2}(Z_2, Z_2)$, and ω corresponds to the periodicity* [10]. *Figure 1 gives $\mathrm{Ext}_A(H^*(MO\langle 8\rangle), Z_2)$. The elements x, $h_0^g x$, and $h_1 x$ come from the second summand of $H^*(MO\langle 8\rangle)$.*

3. Differentials

An inspection of Figure 1 yields that the only elements which may not be permanent cycles are τ, $\varkappa$, e_0, x and their multiples with h_0. We will show that x is the only cycle among these four. This follows from the following theorems.

s \ $t-s$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
9																	↑	$h_1\omega^2$
8																	ω^2	$h_1\omega c_0$
7									↑								ωc_0	$h_0^3 e_0$
6									$h_0^2\omega$		$h_1^2\omega$	$h_0h_2\omega$	↑		$h_0^2 d_0$			$h_0^2 e_0$
5									$h_0\omega$	$h_1\omega$		$h_2\omega$	$h_0^2\tau$		$h_0 d_0$	$h_0^2\kappa$		$h_0 e_0$
4	↑								ω	$h_1 c_0$			$h_0\tau$		d_0	$h_0\kappa$		e_0
3	h_0^3			h_1^3					c_0				τ			κ	↑	
2	h_0^2		h_1^2	h_0h_2			h_2^2										$h_0^2 x$	
1	h_0	h_1	h	h_2													$h_0 x$	$h_1 x$
0	1																x	

FIGURE 1
$\mathrm{Ext}_A^{s,t}(H^*(MO\langle 8\rangle, Z_2), Z_2)$ for $t - s \leq 17$

THEOREM 3.1. $\Omega_{14}^{\langle 8\rangle} = Z_2$.

THEOREM 3.2. $\Omega_{15}^{\langle 8\rangle} = Z_2$.

These will be proved at the end of this section.

THEOREM 3.3. *In the Adams spectral sequence for* $\Omega_*^{\langle 8\rangle}$ *we have*

(i) $d_2(\varkappa) = h_0 d_0$, $d_2(\tau) = h_2\omega$,
(ii) $d_r(x) = 0$ *for all* r,
(iii) $d_3(e_0) = \omega c_0$.

Furthermore, these determine all differentials in the range $t - s \leq 17$.

Proof. To prove (i), we use Theorem 3.1. Since d_0 is a permanent cycle so are $h_0 d_0$ and $h_0^2 d_0$. But $\Omega_{14}^{\langle 8\rangle} = Z_2$, hence two of these three elements must be boundaries. Hence $d_2(\varkappa) = h_0 d_0$. Now $d_2(h_0\varkappa) = h_0^2 d_0$, but $h_0\varkappa = h_2\tau$. and hence $d_2(h_2\tau) = h_0^2 d_0 = h_2^2\omega$. Hence $d_2(\tau) = h_2\omega$ and this proves (i). For dimensional reasons all other d_2's are 0. Consider E_3. By 3.2, $\Omega_{15}^{\langle 8\rangle}$ is Z_2, and there is only one element in E_3 in dimension 15, namely the class of $h_0^2\varkappa$, this must remain to E_∞. Hence there is no candidate for $d_r(x)$, and it must be 0 for all r.

To see (iii), note that the canonical map $S^n \to MO\langle 8\rangle_n$ induces a map of spectra, and hence a map of spectral sequences from the Adams spectral sequence for the homotopy of the sphere spectrum S to that of $MO\langle 8\rangle$. Furthermore both ωc_0 and e_0 are in the image of this map. If we denote the pre-images by the same letters, then according to May [10] we have $d_3(e_0) = \omega c_0$ in the spectral sequence for the sphere. Naturality gives the desired result, and the proof of the theorem is complete.

From E_2 (Figure 1) and Theorem 3.2 one obtains the E_∞ term. Recall that multiplication in E_∞ with h_0 corresponds to multiplication by 2 in the homotopy. This fact gives all the extensions except in dimensions 8 and 9. In dimension 9, we have $2[h_1 c_0] = 2([h_1][c_0]) = (2[h_1])[c_0] = 0$. To see that the extension in dimension 8 is trivial, note that c_0 (and $[c_0]$) are in the image of the map induced from the canonical map $S \to MO\langle 8\rangle$, and hence $[c_0]$ has order 2.

The following tables gives the two primary components of $\Omega_n^{\langle 8\rangle}$ for $n \leq 16$.

n	0	1	2	3	4	5	6	7	8	9
$\Omega^{\langle 8\rangle}$	Z	Z_2	Z_2	Z_8	0	0	Z_2	0	$Z \oplus Z_2$	$Z_2 \oplus Z_2$

n	10	11	12	13	14	15	16
$\Omega_n^{\langle 8\rangle}$	Z_2	0	Z	0	Z_2	Z_2	$Z \oplus Z$

We return now to the proof of Theorem 3.1. To do this we introduce the relative cobordism theory $\Omega^{\langle 8\rangle,\mathrm{Spin}}$, which will be defined as follows: Let

$$\pi : MO\langle 8\rangle \to \mathrm{MSpin}$$

be the map induced by the canonical projection $p : BO\langle 8\rangle \to \mathrm{BSpin} = BO\langle 4\rangle$. Let R denote the mapping cone of π. From the Puppe Sequence for p one obtains an exact sequence

$$\cdots \to \pi_i(\mathrm{MSpin}) \to \pi_i(R) \to \pi_{i-1}(MO\langle 8\rangle \to \cdots .$$

Denote by $\Omega_i^{\langle 8\rangle,\mathrm{Spin}}$ the group $\pi_i(R)$. The proof of Theorems 3.1 and 3.2 follows from the following two lemmas.

LEMMA 3.4. $\pi_{14}(\mathrm{MSpin}) = \pi_{15}(\mathrm{MSpin}) = Z_2$.

Proof. See Anderson, Brown, and Peterson [1].

LEMMA 3.5. $\Omega_{14}^{\langle 8\rangle,\mathrm{Spin}} = \Omega_{15}^{\langle 8\rangle,\mathrm{Spin}} = 0$.

Proof of Theorem 3.1. We have the exact sequence

$$\Omega_{15}^{\langle 8\rangle,\mathrm{Spin}} \to \Omega_{14}^{\langle 8\rangle} \to \Omega_{14}^{\mathrm{Spin}} \to \Omega_{14}^{\langle 8\rangle,\mathrm{Spin}}$$

and by Lemmas 3.4 and 3.5, this reduces to

$$0 \to \Omega_{14}^{\langle 8\rangle} \to Z_2 \to 0.$$

Proof of Theorem 3.2. From 3.5, we have the exact sequence

$$\Omega_{15}^{\langle 8\rangle} \to \Omega_{15}^{\mathrm{Spin}} \to \Omega_{15}^{\langle 8\rangle,\mathrm{Spin}},$$

and hence there is an epimorphism $\Omega_{15}^{\langle 8\rangle} \to Z_2$. By Theorems 3.1 and 3.3 (i) however, $d_2(\varkappa) = h_0 d_0$ and $d_2(h_0 \varkappa) = h_0^2 d_0$. Hence there is at most one nonzero element surviving to E^∞ in dimension 15.

Proof of Lemma 3.5. The proof uses the unsatisfactory technique of very primitive calculation, which in the required dimensions is not difficult. We need the following lemma.

LEMMA 3.6. *As a module over the Steenrod Algebra, $H^*(R)$ in dimensions ≤ 15 is the direct sum of 3-cyclic modules: $A/\!/A(Sq^1, Sq^4)$, $A/\!/AA_1^+$, and $A/\!/A(Sq^3)$. The generator for the first type appears in dimension 4, for the* *second in dimension* 8, *and for the third in dimension* 10, *i.e.*,

$$H^*(R) = (A/\!\!/A(Sq^1, Sq^4))_4 + (A/\!\!/A(Sq^1, Sq^2))_8 + (A/\!\!/A(Sq^3))_{10}.$$

Proof. There is a long exact sequence

$$\cdots \to H^k(R) \to H^k(\text{MSpin}) \xrightarrow{\pi^*} H^k(MO\langle 8\rangle) \to H^{k+1}(R) \to \cdots.$$

But Lemma 2.1 implies that π^* is an epimorphism, hence we have a collection of short exact sequences

$$0 \to H^k(R) \to H^k(\text{MSpin}) \to H^k(MO\langle 8\rangle \to 0.$$

Since $H^*(R)$ is embedded in $H^*(\text{MSpin})$ as an A-submodule, one easily makes the required computation.

To prove Lemma 3.5, one constructs resolutions of the three modules above. This has in part already been done [9]. The case $\text{Ext}_A^{s,t}(A/\!\!/A(Sq^1, Sq^4); Z_2)$, which has not appeared in print, is not difficult (nor are the other cases) in the required dimensions, and the construction is left to the reader. One notes the following relations:

$$\text{Ext}_A^{s,t}(A/\!\!/A(Sq^1, Sq^4); Z_2) = 0 \quad \text{if} \quad t - s = 10, 11;$$

$$\text{Ext}_A^{s,t}(A/\!\!/A(Sq^1, Sq^2); Z_2) = 0 \quad \text{if} \quad t - s = 5, 6, 7;$$

$$\text{Ext}_A^{s,t}(A/\!\!/A(Sq^3); Z_2) = 0 \quad \text{if} \quad t - s = 4, 5.$$

Hence, putting this together we see that $\text{Ext}_A^{s,t}(H^*(R), Z_2) = 0$ if $t - s = 14$, 15, i.e., the E_2 term of the Adams spectral sequence vanishes. Hence $\pi_{14}(R) = \pi_{15}(R) = 0$ and the lemma is proved.

4. Odd torsion

The methods for p odd are essentially the same as for $p = 2$, except that most of the work has been done. Let p be a fixed odd prime, A_p the mod p Steenrod algebra. We apply the following theorems from [4].

LEMMA 4.1.

$$H^*(BO\langle 8\rangle, Z_p) \cong Z_p[\mathcal{K}_i \mid 2i \neq p^j + 1] \otimes [\xi_i \mid i = 1, 2, \cdots]$$

where $\mathcal{K}_i \in H^{4i}(BO\langle 8\rangle; Z_p)$ *and* $\xi_i \in H^{2(pi+1)}(BO\langle 8\rangle; Z_p)$.

LEMMA 4.2. *If* $p > 3$, $H^*(MO\langle 8\rangle; Z_p)$ *is a free module over the subalgebra* $'A_p$ *of* A_p *generated by the reduced power operations.*

Proof. The first statement is Theorem 1, the second Corollary 1 of [4].

These two lemmas, together with the results of either Milnor [11] or Brown-Peterson [2] yield the following corollary.

THEOREM 4.3. $\Omega_*^{\langle 8\rangle}$ *has no p-torsion for* $p > 3$. *Thus we have determined the structure of* $\Omega_*^{\langle 8\rangle}$ *in low dimensions up to elements of order a power of* 3.

It seems to be quite difficult to determine the 3 primary part of $\Omega_*^{\langle 8\rangle}$. The map e of Lemma 4.2 is no longer a monomorphism.

$$e(\mathcal{P}^1) = \mathcal{P}^1 \text{ (Thom class) } \in H^4(MO\langle 8\rangle; Z_p) = 0.$$

Moreover, $H^*(MO\langle 8\rangle; Z_p)$ is not the direct sum of cyclic modules. To see this, consider the element

$$\mathcal{P}_3 U \in H^{12}(MO\langle 8\rangle; Z_p) = Z_3,$$

where $U \in H^0(MO\langle 8\rangle; Z_p)$ is the Thom class. By Milnor [11] $\mathcal{P}^3 U = \pm\mathcal{K}_3 U$. Now

$$\xi_1 U \in H^8(MO\langle 8\rangle; Z_3)$$

and we claim $\mathcal{P}^1(\xi_1 U) = \pm \mathcal{K}_3 U$. This is true also in $BO\langle 8\rangle$, i.e., $\mathcal{P}^1\xi_1 = \pm\mathcal{K}_3$. One can see this geometrically, or note that the projection

$$BO\langle 12\rangle \xrightarrow{p} BO\langle 8\rangle$$

carries $\mathcal{K}_3$ to 0, where upon a cursory inspection of the Serre spectral sequence for the fibration p gives the result.

One can, however, make the computation in some dimensions.

To do this we introduce the bordism groups $\Omega_n^{\langle 8\rangle,\mathrm{fr}}$ of $\langle 8\rangle$-manifolds with framed boundary. These groups can be described algebraically as follows: Let $\alpha : S \to MO\langle 8\rangle$ be the inclusion of the sphere spectrum into $MO\langle 8\rangle$. Let X be the cofibre of α. Then the stable homotopy of X is just the bordism group $\Omega_*^{\langle 8\rangle,\mathrm{fr}}$. See [13] for details.

LEMMA 4.4. *For $n \le 18$, $\Omega_n^{\langle 8\rangle,\mathrm{fr}}$ has no 3-primary torsion, and*

$$\Omega_n^{\langle 8\rangle,\mathrm{fr}} \otimes Q = 0$$

unless $n \equiv 0 \bmod 4$.

Proof. Look at the exact cohomology sequence of the cofibration defining X. Then since $H^q(S) = 0$ if $q > 0$, we have $H^q(X) = H^q(MO\langle 8\rangle)$ for $q > 0$, $H^0(X) = 0$. Using Z_3 coefficients, we have for $q \le 18$, $H^*(X; Z_3) = Z_3[\xi_1, \theta_3, \theta_4]$. Now $\mathcal{P}^1\xi_1 = \pm\theta_3$ and $\mathcal{P}^1\theta_3 = \pm\theta_4$ so in these dimensions $H^*(X; Z_3)$ is a free module over $'A$. Hence $\pi_q(X)$ has no 3-torsion for $q < 18$.

The second statement follows from the facts that the statement is true for $MO\langle 8\rangle$, and that $\pi_q(S) \otimes Q$ is 0 if $q > 0$.

THEOREM 4.5. *For $i < 18$, $i \not\equiv 3 \bmod 4$, the map $\Omega_i^{\mathrm{fr}} \to \Omega_i^{\langle 8\rangle}$ is a monomorphism onto the 3-torsion.*

Proof. This follows immediately from Lemma 4.4 and the exact sequence

$$\Omega_n^{\langle 8\rangle,\mathrm{fr}} \to \Omega_{n-1}^{\mathrm{fr}} \to \Omega_{n-1}^{\langle 8\rangle} \to \Omega_{n-1}^{\langle 8\rangle,\mathrm{fr}}$$

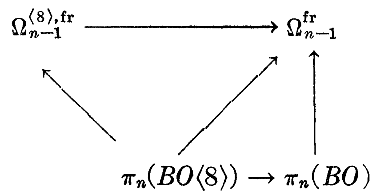
obtained by applying the stable homotopy function to the cofibration $S \to MO\ \langle 8\rangle \to X$.

THEOREM 4.6. $\Omega_3^{\langle 8\rangle} \approx Z_3$, $\Omega_{4k-1}^{\langle 8\rangle} \approx 0$ *for* $k = 2, 3, 4$.

Proof. We have $\Omega_{4k}^{\langle 8\rangle,\mathrm{fr}} \to \Omega_{4k-1}^{\mathrm{fr}} \to \Omega_{4k-1}^{\langle 8\rangle} \to 0$. Now $\Omega_4^{\langle 8\rangle,\mathrm{fr}} = 0$, so

$$\Omega_3^{\langle 8\rangle} \approx \Omega_3^{\mathrm{fr}} \approx Z_3 .$$

According to Harris [5], there is a commutative diagram



such that the map $\pi_n(BO) \to \Omega_{n-1}^{\mathrm{fr}}$ is just the J homomorphism, and the map $\pi_n(BO\langle 8\rangle) \to \pi_n(BO)$ is induced by the projection. But for $n = 8$, 12, or 16, the J homomorphism is onto the 3 component, hence the map $\Omega_{n-1}^{\langle 8\rangle,\mathrm{fr}} \to \Omega_{n-1}^{\mathrm{fr}}$ is onto the three torsion.

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