THE SCHUR MULTIPLIER OF A SEMI-DIRECT PRODUCT[1]

BY

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1. Let G be a p-group which is the semi-direct product of an *abelian* normal subgroup A and a complementary subgroup K. It will be shown that *if p is odd* then

$$H_2(G, Z) \cong H_2(K, Z) \oplus H_1(K, A) \oplus H_2(A, Z)_K \qquad \text{(Theorem 2.1)}.$$

Moreover, a method is described for *systematically* expressing $H_2(G, Z)$ in terms of generators and relations. (The results of the first part grew largely out of conversations which the author has had from time to time with Norman Blackburn, and he is as much responsible for the ideas involved if not the final form presented here as is the author.)

To illustrate the method, the Schur Multipliers of the p-Sylow subgroups of the general linear groups, the symplectic groups, and the even dimensional orthogonal groups over finite fields of characteristic p are computed. (The case $p = 2$, and for the symplectic case $p = 2, 3$ are omitted.) Each of these groups is a unipotent subgroup of a Chevalley group and is readily expressed in terms of generators and relations. *Presumably* this allows a somewhat more direct method of calculating the Schur Multiplier than is presented here. In view of this and to avoid tedium, I have omitted most of the details of the calculation. However, the step of computing $H_1(K, A)$ which is perhaps of independent interest and in which interesting homological phenomena occur (Proposition 5.1) is given a more extended treatment.

Notation. All tensor products are over Z.

If G is a group, A a left G-module, we have the group of n-chains

$$C_n(G, A) = \sum Z\langle g_1, g_2, \cdots, g_n\rangle \otimes A.$$

We denote by $B_n(G, A)$ and $Z_n(G, A)$ the groups of boundaries and cycles respectively.

If $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$.

If $g \in G$ and $a \in A$, we write $[g, a] = g(a) - a$.

Finally, we alternate between additive and multiplicative notation freely where convenient.

2. Let K be a group, A a K-module. We are interested in $H_2(G, Z)$ for G the semidirect product $K \cdot A$. The spectral sequence argument for the group extension $1 \to A \to G \to K \to 1$ yields

$$H_2(G, Z) \cong H_2(K, Z) \oplus K_2(G, Z),$$

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where $K_2(G, Z) = \operatorname{Ker}\{H_2(G, Z) \to H_2(K, Z)\}$, and also a natural exact sequence

$$(1) \qquad H_2(K, A) \xrightarrow{d_{21}^2} H_2(A, Z)_K \to K_2(G, Z) \to H_1(K, A) \to 0.$$

(Z may be replaced by any trivial A-module M if A is replaced in the above sequence by $H_1(A, M)$.)

THEOREM 2.1. *With the notation as above, suppose A is a finitely generated abelian group; then $2d_{21}^2 = 0$. If A is finite abelian of odd order, then the sequence*

$$(2) \qquad 0 \to H_2(A, Z)_K \to K_2(G, Z) \to H_1(K, A) \to 0$$

is exact and splits.

Proof. The first statement has been proved by Charlap and Vasquez; we include it only for completeness, but it is not used in what follows. To prove the second statement we show that if A is finite abelian of odd order, then

$$H_2(A, Z)_K \xrightarrow{h} K_2(G, Z)$$

has a left inverse; it follows immediately that the former arrow h is a monomorphism and the sequence (2) splits. In particular, it follows independently that $d_{21}^2 = 0$.

Define $\iota \in H^2(A, H_2(A))$ by the cocycle formula $i(a, b) = a \cap b$, for $a, b \in A$. (By "$\cap$" we mean the Pontryagin product in the ring $H_*(A, Z)$; also, we have identified A with $H_1(A, Z)$.) The universal coefficient theorem yields the natural "evaluation" morphism

$$\alpha : H^2(A, H_2(A)) \to \operatorname{Hom}(H_2(A), H_2(A))$$

and one checks easily that $\alpha(\iota) = 2 \text{ id}$. Namely, since A is finite, $H_2(A, Z)$ is generated by all $a \cap b$, and $a \cap b$ is represented by the cycle $(\langle a, b\rangle - \langle b, a\rangle) \otimes 1$.

Write $M = H_2(A, Z)$ and let the central extension

$$(3) \qquad 1 \to M \to E \to A \to 1$$

represent ι. Then one knows [3, Sections 2, 4] that the cotransgression $\delta : H_2(A, Z) \to M$ is $\alpha(\iota)$, that is, 2 id.

Let K act on A and by the induced action on M. Observe that the cocycle i defined above is K-invariant. Hence, K acts also on E consistently with its actions on A and M. Let G be the semidirect product $K \cdot A$ and U the semidirect product $K \cdot E$. If $\phi : U \to G$ is defined by $ke \mapsto ka$ where $e \mapsto a$, then we may identify $\operatorname{Ker} \phi$ with M, and we have the commutative diagram with exact rows

$$(4) \qquad \begin{array}{ccccccccc} 1 & \to & M & \to & E & \to & A & \to & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 1 & \to & M & \to & U & \to & G & \to & 1. \end{array}$$

Because of the naturality of cotransgression, (4) yields the commutative diagram

$$\begin{tikzcd} H_2(A) \arrow[r, "\delta"] \arrow[d] & M \arrow[d] \\ H_2(G) \arrow[r, "\delta'"] & M_G \end{tikzcd} \tag{5}$$

where δ' is the cotransgression for the lower extension in (4). Since A acts trivially on M, we have $M_G = M_K$. Thus (5) induces the commutative diagram

$$\begin{tikzcd} H_2(A)_K \arrow[dd] \arrow[dr, "\delta_K"] & \\ & M_K \\ H_2(G) \arrow[ur, "\delta'"'] & \end{tikzcd} \tag{6}$$

Since $H_2(A) = M$ is of odd order, and since $\delta = 2\cdot\text{id}$, the desired result follows. (Remember that $H_2(A)_K \to K_2(G)$ is a factor of $H_2(A)_K \to H_2(G)$.)

3. Let K be a group, A an abelian group. Put $L = K * A$, the free product of K and A. If K acts on A,

$$k_1 a k_2 \mapsto k_1(a) k_1 k_2$$

defines an epimorphism $K * A \to K \cdot A = G$. Let S be the kernel of this epimorphism. One checks easily that S is the subgroup of L generated by all $kak^{-1}k(a)^{-1}$ where $k \in K$ and $a \in A$. (Notice that since ka makes sense in L and in G, there is a possibility of confusion. Generally, the context will make clear what is meant.) To see this, the following identities in $K * A$ are useful: Write $(k, a) = kak^{-1}k(a)^{-1}$; then

$$\begin{aligned} h(k, a)h^{-1} &= (hk, a)(h, k(a))^{-1} \quad \text{for } h \in K, \\ b(k, a)b^{-1} &= (k, k^{-1}(b))^{-1}(k, ak^{-1}(b)) \quad \text{for } b \in A. \end{aligned} \tag{7}$$

Remark 1. In fact, one can prove that S is free on the generators (k, a) with $k \neq 1$ and $a \neq 1$.

The fundamental homology sequence for the sequence

$$1 \to S \to L \to G \to 1$$

yields

$$H_2(L) \to H_2(G) \to S/[L, S] \to L/L' \to G/G' \to 0. \tag{8}$$

We know $G/G' \cong K/K' \oplus A/[K, A]$. Also, by a result of Rinehart and Barr, [1, Section 4], for $i \geq 1$,

$$H_i(K * A, Z) \cong H_i(K, Z) \oplus H_i(A, Z),$$

the isomorphism being induced by the inclusions of K and A in $K * A$.

Thus (8) becomes

$$\begin{tikzcd} H_2(K) \oplus H_2(A) \arrow[r, "a"] & H_2(G) \arrow[r, "b"] & S/[L, S] \arrow[r, "c"] & K/K' \oplus A \arrow[d, "d"] \\ & & & K/K' \oplus A/[K, A] \arrow[d] \\ & & & 0 \end{tikzcd} \tag{9}$$

where the homomorphisms are the obvious ones. In particular, a is induced by the inclusions of K and A in G, and it may be factored through

$$a' : H_2(K) \oplus H_2(A)_K \to H_2(G)$$

under which the first factor goes isomorphically onto a direct summand and the second factor goes into $K_2(G, Z)$.

We wish to compute

$$\operatorname{Ker} \{S/[L, S] \to L/L' \cong K/K' \oplus A\}.$$

First, we may write $(k, a) = [k, a]ak(a)^{-1}$ where $[k, a] = kak^{-1}a^{-1}$ is computed in L. $[K, A]$ is a normal subgroup of L, and it is contained in Ker $(L \to L/L')$. The element

$$x = (k_1, a_1)^{e_1}(k_2, a_2)^{e_2} \cdots (k_r, a_r)^{e_r}, \quad e_i = \pm 1,$$

in S is congruent modulo $[K, A]$ to the element

$$b = (a_1 k_1(a_1)^{-1})^{e_1} \cdots (a_r k_r(a_r)^{-1})^{e_r}$$

in A. On the other hand, since $(k, a) \mapsto ak(a)^{-1}$, the element x is in Ker $(S \to L/L')$ if and only if $b = 1$ in A. Hence Ker $(S \to L/L') = S \cap [K, A]$.

Putting together this information we obtain the exact sequence

$$H_2(K) \oplus H_2(A)_K \xrightarrow{a'} H_2(G) \xrightarrow{b} S \cap [K, A]/[L, S] \longrightarrow 0. \tag{10}$$

Consider next the group $S \cap [K, A]/[L, S]$. It is evidently functorial in pairs (K, A). Blackburn [1, Section 1] shows that it is naturally isomorphic to $H_1(K, A)$. An outline of his proof is as follows: Define an isomorphism $C_1(K, A)/B_1(K, A) \cong S/[L, S]$ by letting

$$\langle k \rangle \otimes a \leftrightarrow (k, a).$$

Then $\sum \langle k_i \rangle \otimes a_i$ is a cycle if and only if, in view of the above computation, $(k_1, a_1)(k_2, a_2) \cdots (k_r, a_r)$ is in $[K, A]$.

Remark 2. In view of the discussion above one concludes that modulo the direct summand $H_2(K)$ the two sequences (1) (or (2)) and (10) are the same. To prove this one would have to identify $K_2(G, Z) \to H_1(G, A)$ as the cotransgression b. We see no transparent way of verifying this, so we content ourselves with the formulation above which is sufficient for what is done here.

The sequence (10) is a useful tool for computing H_2 particularly if it is to be described in terms of defining relations for the group. Write $K = F_K/R_K$ and $A = F_A/R_A$ where F_K and F_A are free. Let $F = F_K * F_A$; if we map F onto $G = K \cdot A$ in the obvious way, then we may write $G \cong F/R$ where $R \supseteq R_K * R_A$. The relationships among the various groups of interest are summarized in the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & R_K * R_A \arrow[r, equal] \arrow[d] & R_K * R_A \arrow[d] & & \\
1 \arrow[r] & R \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & S \arrow[r] \arrow[d] & K * A \arrow[r] \arrow[d] & G \arrow[r] & 1. \\
 & 1 & 1 & &
\end{tikzcd} \tag{11}$$

The cotransgressions

$$H_2(G) \to S/[L, S], \quad H_2(G) \to R/[F, R],$$

$$H_2(K) \to R_K/[F_K, R_K], \quad H_2(A) \to R_A/[F_A, R_A]$$

are natural homomorphisms; also, the last three yield isomorphisms $H_2(G) \cong R \cap F'/[F, R]$, etc. In particular, the isomorphism

$$H_2(A) \cong R_A \cap F'_A/[F_A, R_A] = F'_A/[F_A, R_A]$$

is described as follows. Let x and y in F_A represent a and b in A; then $a \cap b \in H_2(A)$ corresponds to the coset of $[x, y]$ on the right. Furthermore, this isomorphism is a K-isomorphism provided we let K act on the right as follows. For each k in K, choose an endomorphism of F_A covering the automorphism of A produced by letting k act on A. This endomorphism carries F'_A into itself and induces an automorphism of $F'_A/[F_A, R_A]$ depending only on k. Adding this information to the sequence (10), we obtain the following diagram with exact rows:

$$
\begin{array}{ccccccc}
H_2(K) \oplus H_2(A)_K & \xrightarrow{a'} & H_2(G) & \xrightarrow{b} & S \cap [K, A]/[L, S] & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
R_K \cap F'_K/[F_K, R_K] & & & & & & \\
\oplus\ (F'_A/[F_A, R_A])_K & \xrightarrow{a''} & R \cap F'/[F, R] & \xrightarrow{b''} & S \cap [K, A]/[L, S] & \longrightarrow & 0.
\end{array} \tag{12}
$$

Notice a'' and b'' are induced by the obvious group homomorphisms.

According to Theorem 2.1, if A is of odd order, then a' and a'' are monomorphisms onto direct summands. Also, $S \cap [K, A]/[L, S]$ is isomorphic to $H_1(K, A)$ by means of the explicit isomorphism described above. Hence, if we assume $H_2(K)$ has already been computed, the computation of $H_2(G)$ reduces to the computation of $H_1(K, A)$ and $H_2(A)_K$ each of which is readily expressed in terms of relations.

4. We wish to illustrate the method discussed above by computing the Schur Multiplier for certain interesting groups which we describe below.

Fix a finite field $k = GF(p^m)$. Denote by $G_n = G_n(p^m)$ the group of all upper triangular $n \times n$ matrices with entries in k and ones on the diagonal. G_n is the p-Sylow subgroup of the general linear group $Gl(n, k)$.

Let V_n be the ambient vector space for G_n realized as the space of column vectors with n entries in k. V_n may also be realized as the normal subgroup of G_{n+1} consisting of all matrices of the form

$$
\begin{bmatrix} 1_n & v \\ 0 & 1 \end{bmatrix}, \qquad v \text{ in } V_n. \tag{13}
$$

Moreover, we may imbed G_n in G_{n+1} as all matrices of the form

$$
\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}, \qquad g \text{ in } G_n. \tag{14}
$$

Let $g \in G_n$ and $v \in V_n$; then gvg^{-1} computed in G_{n+1} corresponds to gv in V_n. Clearly, G_n intersects V_n trivially in G_{n+1} so that G_{n+1} is the semi-direct product $G_n \cdot V_n$.

Suppose now p is odd. Let A_n be the additive group of all skew-symmetric $n \times n$ matrices with entries in k. We identify A_n with the exterior power $\wedge_k^2 V_n$ when convenient. Imbed A_n in G_{2n} as the normal subgroup of all matrices of the form

$$
\begin{bmatrix} 1_n & a \\ 0 & 1_n \end{bmatrix}, \qquad a \text{ in } A_n. \tag{15}
$$

Let $g^* = (g^{-1})^T$. Imbed G_n in G_{2n} as the subgroup of all matrices of the form

$$\begin{bmatrix} g & 0 \\ 0 & g^* \end{bmatrix}. \tag{16}$$

Then, for g in G_n, a in A_n, the automorphism $a \mapsto gag^{-1}$ computed in G_{2n} coincides with $a \mapsto gag^T$ in A_n, that is, with $v \wedge w \mapsto gv \wedge gw$. Clearly, G_n intersects A_n trivially in G_{2n} so that we may form the semi-direct product $H_n = G_n \cdot A_n$. H_n is the p-Sylow subgroup of the even dimensional orthogonal group $O(2n, k)$.

Let S_n be the additive group of all $n \times n$ symmetric matrices with entries in k. When convenient we identify S_n with the symmetric power $\mathbb{S}_k^2 V_n$. As above in (15) we imbed S_n in G_{2n}, and we consider the semi-direct product $J_n = G_n \cdot S_n$. (Notice the symmetric product vw should replace the wedge product $v \wedge w$ in the formula above.) Then J_n is the p-Sylow subgroup of the Symplectic group $Sp(2n, k)$.

5. In view of the constructions outlined in the previous section, we shall be interested in computing $H_1(G \cdot V, A)$ where $G = G_n$, $V = V_n$, and (I) $A = V_{n+1}$, (II) $A = S_{n+1}$, or (III) $A = A_{n+1}$. I claim that in each case the sequence

$$0 \to [V, A] \to A \to A_V \to 0 \tag{17}$$

splits as a G-sequence (but not as a V-sequence.) It follows from this by the usual edge homomorphism argument that

$$H_1(G \cdot V, A) \cong H_1(G, A_V) \oplus H_1(V, A)_G. \tag{18}$$

To demonstrate the splitting of (17), we exhibit a split sequence,

$$0 \longrightarrow [V, A] \longrightarrow A \underset{j}{\overset{h}{\rightleftarrows}} B \longrightarrow 0, \tag{17$'$}$$

in each case. ($B \cong A_V$.)

(I) Put $A = V_{n+1}$, $B = k$; define

$$h \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} = x_{n+1} \quad \text{and} \quad j(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ t \end{bmatrix}.$$

Notice $\operatorname{Ker} h = [V, A] \cong V_n$.

Remark. A certain amount of confusion can arise since V_n appears both as a subgroup of G_{n+1} and a submodule of V_{n+1}. This will be particularly annoying when we are dealing with explicit cycles. Rather than complicate the notation even more, we choose to try to keep our wits about us.

(II) Put $A = S_k^2(V_{n+1})$, $B = K$. Let $e_1, e_2, \cdots, e_{n+1}$ constitute the standard basis for V_{n+1}. Since $h = h_{\mathrm{I}}$ has already been defined, it makes sense to define $h = h_{\mathrm{II}}$ by $h(ab) = h(a)h(b)$, for a, b in V_{n+1}. Define $j = j_{\mathrm{II}}$ by $j(t) = t(e_{n+1})^2$. Put $T_n = \operatorname{Ker} h = [V, A]$.

As an auxilliary device, we consider also the following.

(IIa) Put $A = T_n$, $B = V_n$. Define h' by

$$h'(ab) = h(a)b + h(b)a.$$

Notice that T_n is spanned by products ab with a or b in V_n, and for such products $h'(ab)$ is in V_n. Of course, the above formula also defines a G_{n+1}-homomorphism $h' : S_{n+1} \to V_{n+1}$. Define j by $j(a) = ae_{n+1}$ for a in V_n. Notice

$$\operatorname{Ker} h' = [V, T_n] = [V, [V, S_{n+1}]] = S_n.$$

(III) Put $A = \wedge_k^2(V_{n+1})$, $B = V_n$. Define $h = h_{\mathrm{III}}$ by

$$h(a \wedge b) = h(a)b - h(b)a,$$

for a, b in V_{n+1}. (The image does in fact lie in V_n !) Define j by

$$j(a) = e_{n+1} \wedge a,$$

for a in V_n. Notice $\operatorname{Ker} h = [V, A] = A_n$.

One checks easily in all cases that $h(h')$ is a G_{n+1} homomorphism, but j is only a G_n-homomorphism.

We now turn our attention to the computation of $H_1(G \cdot V, A)$. Proceeding by induction, we reduce the computation to findings $H_1(V, A)_G$. The short exact V-sequence (17′) yields the long exact sequence

$$(19)\qquad \begin{aligned} &H_2(V, B) \xrightarrow{\delta_2} H_1(V, A') \longrightarrow H_1(V, A) \longrightarrow \\ &\qquad\qquad\qquad\qquad H_1(V, B) \xrightarrow{\delta_1} H_0(V, A') \end{aligned}$$

where we abbreviate $[V, A] = A'$.

Consider first δ_2. Denote the prime field by F_p. Since V acts trivially on B, the universal coefficient theorem (over the ring F_p) tells us that

$$H_2(V, B) \cong H_2(V, F_p) \otimes B.$$

(Initially the tensor product is taken over F_p, but of course this is the same as the tensor product over Z.) On the other hand, $H_2(V, F_p)$ is spanned as a vector space over F_p by the Pontryagin products $v \cap w$, $v, w \in H_1(V, F_p) = V$ *and* by elements $(v, p, 1)$, $v \in H_1(V, Z) = V$ *where* $(v, p, 1)$ is represented by the cycle $\sum_{i=0}^{p-1} \langle v, iv \rangle \otimes 1$ in $Z_2(V, F_p)$. (If V is written F_V/R_V with F_V free, then, expressing $H_2(V, F_p)$ in terms of relations as in section 3, we see that $v \cap w$ corresponds to $[x, y]$ and $(v, p, 1)$ corresponds to x^p. Here, as in Section 3, v and w are represented respectively by x and y.)

A routine computation shows

(20) $\delta_2(v \cap w \otimes b)$ is represented by

$$\langle v\rangle \otimes [w, j(b)] - \langle w\rangle \otimes [v, j(b)],$$

for v, w in V, b in B.

Also,

(21) $\delta_2((v, p, 1) \otimes b)$ is represented by

(21a) $$\langle v\rangle \otimes \tbinom{p}{2}[v, j(b)] \quad \text{in cases I, IIa, and III,}$$

and by

(21b) $$\langle v\rangle \otimes \{\tbinom{p}{2}\}[v, j(b)] + \tbinom{p}{3}[v, [v, j(b)]]\} \quad \text{in case II}$$

Thus, $\delta_2((v, p, 1) \otimes b)$ is zero in cases I, IIa, and III *provided* $p \neq 2$, and it is zero in case II *provided* $p \neq 2, 3$.

Consider next δ_1. We have $H_1(V, B) \cong V \otimes B$, and a routine calculation shows

(22) $\delta_1(v \otimes b)$ is represented by $-\langle\cdot\rangle \otimes [v, j(b)]$.

PROPOSITION 5.1. *For $p \neq 2$, and in case* II, $p \neq 2, 3$,

(23) $$0 \to \operatorname{Coker} \delta_2 \to H_1(V, A) \to \operatorname{Ker} \delta_1 \to 0$$

is a split exact G_n-sequence. It follows that

(24) $$H_1(V, A)_G \cong (\operatorname{Coker} \delta_2)_G \oplus (\operatorname{Ker} \delta_1)_G.$$

Proof. First consider cases I, IIa, and III. In these cases we have

$$[V, [V, A]] = 0 \quad \text{and} \quad H_0(V, A') = A'.$$

For $v \otimes b \in V \otimes B$, define $\beta(v \otimes b) \in C_1(V, A)$ by

(25) $$\beta(V \otimes b) = \langle v\rangle \otimes j(b) + (p - 1/2)\langle v\rangle \otimes [v, j(b)].$$

β so defined is additive in b, and a routine but laborious calculation shows that $\beta(v_1 \otimes b) + \beta(v_2 \otimes b) \sim \beta((v_1 + v_2) \otimes b)$. Hence β defines a homomorphism $\beta : V \otimes B \to C_1(V, A)/B_1(V, A)$. Since

$$d_1(\beta(v \otimes b)) = -\langle\cdot\rangle \otimes [v, j(b)],$$

it follows that $\sum v_i \otimes b_i$ is in the kernel of $\delta_1 : V \otimes B \to A'$ if and only if $\sum \beta(v_i \otimes b_i)$ is a cycle. Thus, β defines a homomorphism

$$\beta : \operatorname{Ker} \delta_1 \to H_1(V, A)$$

which splits the homomorphism $H_1(V, A) \to \operatorname{Ker} \delta_1 \subseteq H_1(V, B)$.

The argument in case II is very much the same except that we must make stronger use of the explicit nature of $A = S_{n+1}$. We have $[V, [V, [V, A]]] = 0$

and $B = k$. Define as above

$$(25') \quad \beta(v \otimes b) = [v] \otimes j(b) + ((p-1)/2)\langle v\rangle \otimes [v, j(b)] + (p-1)(p-2)/6\langle v\rangle \otimes [v, [v, j(b)]].$$

Also,

$$\delta_1 : V_n \otimes k \to A'/[V, A'] \cong T_n/S_n \cong V_n$$

is given by $\delta_1(v \otimes b) = -bv$. Suppose $\sum v_i \otimes b_i$ is in $\operatorname{Ker} \delta_1$. Then

$$\begin{aligned} d_1(\textstyle\sum \beta(v_i \otimes b_i)) &= -\textstyle\sum \{[v_i, b_i e_{n+1}^2] + ((p-1)/2)[v_i, b_i e_{n+1}^2]]\} \\ &= -\textstyle\sum \{(2b_i v_i e_{n+1} + b_i v_i^2) + ((p-1)/2)(2b_i v_i^2)\} \\ &= -2(\textstyle\sum b_i v_i)e_{n+1} \\ &= 0. \end{aligned}$$

Remark. In the explicit computations which follow the formulas (25) and (25′) which give the (25′) which give the splitting will be important.

6. The groups G_n, $n \geq 2$, $p \neq 2$. We have

$$H_2(G_{n+2}, Z) = H_2(G_{n+1}, Z) \oplus H_1(G_{n+1}, V_{n+1}) \oplus H_2(V_{n+1}, Z)_{G_{n+1}}.$$

Let T be an *ordered* F_p-basis for k which contains 1. For s in k, let $x_i(s)$, $i = 1, 2, \cdots, n$, be the square matrix of degree $n + 1$ with 1 on the diagonal, s in the i^{th} row and $(i+1)^{\text{th}}$ column, and 0 elsewhere. Let $e_i(s)$, $i = 1, 2, \cdots, n + 1$, be the column vector of degree $n + 1$ with s in the i^{th} position. The $x_i(t)$, t in T, constitute a set of generators of G_{n+1}, and the $e_i(t)$, t in T, constitute a set of generators of V_{n+1}.

Let F_G be the free group on generators $X_i(t)$, and let F_V be the free group on generators $E_i(t)$. Define the obvious epimorphisms of these groups onto G_{n+1} and V_{n+1} respectively. Let $F = F_G * F_V$, and define R_G, R_V, and R as in Section 3. Notice that the elements $x_i(t)$, $i = 1, 2, \cdots, n$, t in T, $e_{n+1}(t)$, t in T, constitute a minimal generating set for G_{n+2}. Let F_0 be the subgroup of F generated by the elements $X_i(t)$, $i = 1, 2, \cdots, n$, and $E_{n+1}(t) = X_{n+1}(t)$, t in T. We shall find a basis for $R \cap F'/[F, R]$ represented by elements of $R_0 = R \cap F_0$. Since F_0/R_0 is a minimal presentation of G_{n+2}, we will get a more convenient description of H_2 than that obtained by a direct application of the method of Section 3.

(a) Let $n \geq 1$. An F_p-basis for $H_2(V_{n+1}, Z)_{G_{n+1}}$ is given modulo $[G_{n+1}, H_2]$ by the elements

$$e_{n+1}(t) \cap e_{n+1}(t'), \quad t < t' \text{ in } T,$$
$$e_n(t) \cap e_{n+1}(1), \quad t \text{ in } T.$$

In terms of relations, these elements correspond to the elements of R,

$$[E_{n+1}(t), E_{n+1}(t')] \quad \text{and} \quad [E_n(t), E_{n+1}(1)].$$

Since $e_n(t) = [x_n(1), e_{n+1}(t)]$, we have

$$E_n(t) \equiv [X_n(1), E_{n+1}(t)] \mod R$$

so that

$$[E_n(t), X_{n+1}(1)] \equiv [[X_n(1), X_{n+1}(t)], X_{n+1}(1)] \mod [F, R].$$

This the contribution to an F_p-basis for $H_2(G_{n+2}, Z)$ from $H_2(V_{n+1}, Z)_{G_{n+1}}$ is given in terms of relations by the elements

$$(26) \qquad \begin{aligned} &[X_{n+1}(t), X_{n+1}(t')], \quad t < t' \text{ in } T, \\ &[X_{n+1}(1), [X_n(1), X_{n+1}(t)]], \quad t \text{ in } T. \end{aligned}$$

(b) We have

$$H_1(G_{n+1}, V_{n+1}) \cong H_1(G_n, k) \oplus H_1(V_n, V_{n+1})_{G_n}.$$

Also, $H_1(G_n, k) \cong G_n/G'_n \otimes k$, and an F_p-basis for this group is given by the representative k-cycles $\langle x_i(t)\rangle \otimes t'$, $i = 1, 2, \cdots, n-1$, t, t' in T, which lift to the V_{n+1}-cycles $\langle x_i(t)\rangle \otimes e_{n+1}(t')$. Let $[a, b]'$ denote the commutator computed in $G_{n+1} * V_{n+1}$. Then the latter cycles correspond as in Section 3 to the elements

$$[x_i(t), e_{n+1}(t')]'[x_i(t), e_{n+1}(t')]^{-1} = [x_i(t), e_{n+1}(t')]'.$$

In R, these correspond to the elements

$$(27) \qquad [X_i(t), X_{n+1}(t')], \quad i = 1, 2, \cdots, n-1, t, t' \text{ in } T.$$

(c) $H_1(V_n, V_{n+1}) = \operatorname{Coker} \delta_2 \oplus \operatorname{Ker} \delta_1$ (a G_n-direct sum.) $H_1(V_n, V_n) = V_n \otimes V_n$, and formula (20) tells us $\operatorname{Im} \delta_2$ is generated by all $v \otimes sw - w \otimes sv$, v, w, in V_n, s in k. Hence,

$$\operatorname{Coker} \delta_2 \cong S_k^2(V_n), \quad \text{and} \quad (\operatorname{Coker} \delta_2)_{G_n} \cong k$$

with F_p-basis represented by the elements $e_n(1) \otimes e_n(t)$, t in T. In terms of cycles, we are led to the elements $\langle x_n(1)\rangle \otimes e_n(t)$, or, in terms of relations, the elements $[X_n(1), E_n(t)]$. As in (a), modulo $[F, R]$ the latter elements are congruent to the relations

$$(28) \qquad [X_n(1), [X_n(1), X_{n+1}(t)]], \quad t \text{ in } T.$$

(d) $\delta_1 : V_n \otimes k \to V_n$ is defined by $v \otimes s \mapsto -sv$, and one sees easily that

$$(\operatorname{Ker} \delta_1)_{G_n} \cong \operatorname{Ker} \{(\delta_1)_{G_n} : k \otimes k \to k\}.$$

An F_p-basis is given by the elements represented by the cycles

$$\langle x_n(t)\rangle \otimes t' - \langle x_n(1)\rangle \otimes tt', \quad t, t' \text{ in } T, \ t \neq 1.$$

The splitting map defined by formula (25) yields the V_{n+1}-cycles

$$\langle x_n(t)\rangle \otimes e_{n+1}(t') - \langle x_n(1)\rangle \otimes e_{n+1}(tt') \\ + ((p-1)/2)(\langle x_n(t)\rangle \otimes e_n(tt') - \langle x_n(1)\rangle \otimes e_n(tt')).$$

However, the expression in parentheses represents an element of Coker δ_2 and, after an appropriate change of basis, we are left with the first part of the expression. In $G_{n+1}*V_{n+1}$, this expression corresponds to

$$[x_n(t), e_{n+1}(t')]'[x_n(t), e_{n+1}(t')]^{-1}[x_n(1), e_{n+1}(tt')]([x_n(1), e_{n+1}(tt')]')^{-1}$$
$$= [x_n(t), e_{n+1}(t')]'([x_n(1), e_{n+1}(tt')]')^{-1}.$$

If $s = \sum_{r\epsilon T} a_r r$, $a_r \epsilon F_{p\,T}$ write

$$X_{n+1}(s) = \prod_r X_{n+1}(r)^{a_r}.$$

Then we are finally led to the relations

$$(29) \qquad [X_n(t), X_{n+1}(t')][X_n(1), X_{n+1}(tt')]^{-1}, \quad t. t' \text{ in } T, t \neq 1.$$

Induction now yields

PROPOSITION 6. *Let $n \geq 1$, $p \neq 2$. An F_p-basis for $H_2(G_{n+1}, Z)$ is given in terms of the generators $X_i(t)$, $i = 1, 2, \cdots, n$, by the relations*

(i) $[X_i(t), X_i(t')]$, $i = 1, 2, \cdots, n, t, t'$ *in* T,

(ii) $[X_i(t), X_j(t')]$, $1 \leq i < j \leq n, j \neq i + 1, t, t'$ *in* T,

(ii) $[X_i(1), [X_i(1), X_{i+1}(t)]]$ *and* $[X_{i+1}(1), [X_i(1), X_{i+1}(t)]]$,
$i = 1, 2, \cdots, n - 1, t$ *in* T,

(iv) $[X_i(t), X_{i+1}(t')][X_i(1), X_{i+1}(tt')]^{-1}$,
$i = 1, 2, \cdots, n - 1, t, t'$ *in* T, $t \neq 1$.

(*Convention.* $X_i(s) = \prod X_i(t)^{a_t}$ *where* $s = \sum a_t t$.)

7. The groups J_n, $n \geq 2$, $p \neq 2, 3$.

$$H_2(J_{n+1}, Z) = H_2(G_{n+1}, Z) \oplus H_1(G_{n+1}, S_{n+1}) \oplus H_2(S_{n+1}, Z)_{G_{n+1}}.$$

Let F_G be as in section 6. The elements $e_i(1)e_j(t)$, $1 \leq i \leq j \leq n + 1$, t in T, generate S_{n+1}. Let F_S *be the free group on generators* $E_{ij}(t)$, $1 \leq i \leq j \leq n + 1$, t in T, and map it onto S_{n+1} in the obvious way. Otherwise, proceed as in Section 6. Notice that the elements $X_i(t)$, $i = 1, 2, \cdots, n$, t in T, $X_{n+1}(t) = E_{n+1,n+1}(t)$, t in T, form a minimal generating set for J_{n+1}.

(a) An F_p-basis for $H_2(S_{n+1}, Z)_{G_{n+1}}$ consists of the elements represented by

$$e_{n+1}(1)e_{n+1}(t) \cap e_{n+1}(1)e_{n+1}(t'), \quad t < t' \text{ in } T,$$
$$e_n(1)e_{n+1}(t) \cap e_{n+1}(1)^2, \quad t \text{ in } T.$$

Arguing as in Section 6a, we obtain (modulo $[F, R]$) the relations

$$(30) \qquad [X_{n+1}(t), X_{n+1}(t')], \quad t < t' \text{ in } T,$$

and

$$\begin{aligned}
[E_{n,n+1}(t),\, X_{n+1}(1)] &\equiv \tfrac{1}{2}([[X_n(1),\, X_{n+1}(t)],\, X_{n+1}(1)] - [E_{n,n}(t),\, X_{n+1}(1)]) \\
&\equiv -\tfrac{1}{2}[X_{n+1}(1),\, [X_n(1),\, X_{n+1}(t)]].
\end{aligned}$$

Thus, we may take for the contribution to a basis from the latter terms the relations

$$[X_{n+1}(1),\, [X_n(1),\, X_{n+1}(t)]], \quad t \text{ in } T. \tag{31}$$

(b) $H_1(G_{n+1}, S_{n+1}) = H_1(G_n, k) \oplus H_1(V_n, S_{n+1})_{G_n}$, and, as in Section 6, $H_1(G_n, k)$ contributes to an F_p-basis the relations

$$[X_i(t),\, X_{n+1}(t')], \quad i = 1, 2, \cdots, n-1,\ t,\, t' \text{ in } T. \tag{32}$$

(c) To compute $H_1(V_n, S_{n+1})$ we first take a small detour. Consider the commutative, exact diagram of G_{n+1}-modules

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & V_n \arrow[r] \arrow[u] & V_{n+1} \arrow[r, "h"] \arrow[u] & k \arrow[r] & 0 \\
0 \arrow[r] & T_n \arrow[r] \arrow[u, "h'"'] & S_{n+1} \arrow[r, "h"] \arrow[u, "h'"'] & k \arrow[r] \arrow[u, "1"'] & 0 \\
 & S_n \arrow[u] \arrow[r, equal] & S_n \arrow[u] & & \\
 & 0 \arrow[u] & 0. \arrow[u] & &
\end{tikzcd} \tag{33}$$

On the level of homology, (33) induces the following diagram

$$\begin{tikzcd}
H_2(V, k) \arrow[r, "\delta_2"] & V_n \otimes V_n \arrow[r] & H_1(V, V_{n+1}) & & \\
H_2(V, k) \arrow[r, "\delta_2'"] \arrow[u, "2"'] & H_1(V, T_n) \arrow[r] \arrow[u] & H_1(V, S_{n+1}) \arrow[r] \arrow[u] & V_n \otimes k \arrow[r, "\delta_1"] & V_n \\
 & V_n \otimes S_n \arrow[r, equal] \arrow[u, "i"'] & V_n \otimes S_n \arrow[u] & & \\
 & H_2(V, V_n) \arrow[r] \arrow[u] & H_2(V, V_{n+1}) \arrow[r] \arrow[u] & H_2(V, k) \arrow[r, "\delta_2"] & V_n \otimes V_n \\
 & & H_2(V, S_{n+1}) \arrow[r] \arrow[u] & H_2(V, k) \arrow[r, "\delta_2'"] \arrow[u] & H_1(V, T_n). \arrow[u]
\end{tikzcd} \tag{34}$$

Diagram chasing shows that

$$H_1(V, T_n) = \operatorname{Im} \delta_2' + \operatorname{Im} i. \tag{35}$$

Also, our explicit knowledge of δ_2 and δ_2' shows us that Ker δ_2 is the k-subspace of $H_2(V, k)$ spanned by all $(v, p, 1)$, v in V; hence $\operatorname{Ker} \delta_2 \subseteq \operatorname{Ker} \delta_2'$. That fact and some more diagram chasing shows that $\operatorname{Im} \delta_2' \cap \operatorname{Im} i = (0)$. It follows that the sum in (35) is direct and hence the sequence

$$0 \to \operatorname{Im} i \to H_1(V, S_{n+1}) \to \operatorname{Ker} \delta_1 \to 0 \tag{36}$$

is exact. However, by Proposition 5.1, we already know that it splits. Thus

$$H_1(V, S_{n+1})_{G_n} = (\operatorname{Im} i)_{G_n} \oplus (\operatorname{Ker} \delta_1)_{G_n}. \tag{37}$$

Returning now to the computation, we note that

$$\operatorname{Im} i = \operatorname{Coker} (H_2(V_n, V_n) \to V_n \otimes S_n)$$

which by formula (20) is isomorphic to $\mathrm{S}_k^3(V_n)$. Also, $(\mathrm{S}_k^3(V_n))_{G_n} \cong k$, and, in terms of cycles in S_n, an F_p-basis is given by the elements

$$\langle x_n(1)\rangle \otimes e_n(1)e_n(t), \quad t \text{ in } T.$$

Arguing as in Section 6c, we are led (modulo $[F, R]$, and except for a factor of 2,) to the relations

$$[X_n(1), [X_n(1), [X_n(1), X_{n+1}(t)]]], \quad t \text{ in } T. \tag{38}$$

(d) One argues as in section 6d. The splitting map β defined by formula (25′) is somewhat more complicated, and the calculation in $G_{n+1} * S_{n+1}$ is considerably more complicated. Eventually, after an appropriate change of basis, one obtains as the contribution to an F_p-basis from $(\operatorname{Ker} \delta_1)_{G_n}$ the relations

$$\begin{aligned} &[X_n(t), X_{n+1}(t')][X_n(1), X_{n+1}(tt')]^{-1}[X_n(t), [X_n(1), X_{n+1}(tt'/2)]]^{-1} \\ &\qquad \cdot [X_n(1), [X_n(1), X_{n+1}(tt'/2)]], \quad t, t' \text{ in } T, t \neq 1. \end{aligned} \tag{39}$$

If we combine Proposition 6 with what has been done in this section, we obtain

PROPOSITION 7. *Let $n \geq 1$, $p \neq 2, 3$. An F_p-basis for $H_2(J_{n+1}, Z)$ is given in terms of the generators $X_i(t)$, $i = 1, 2, \cdots, n+1$, t in T, by the relations*

(i) $[X_i(t), X_i(t')]$, $\quad i = 1,2, \cdots, n+1$, $t < t'$ *in* T,

(ii) $[X_i(t), X_j(t')]$, $\quad 1 \leq i < j \leq n+1$, $j \neq i+1$, t, t' *in* T,

(iii) $[X_i(1), [X_i(1), X_{i+1}(t)]]$, $\quad i = 1, 2, \cdots, n-1$, t *in* T,

and

$[X_{i+1}(1), [X_i(1), X_{i+1}(t)]]$, $\quad i = 1, 2, \cdots, n$, t *in* T,

(iiia) $[X_n(1), [X_n(1), [X_n(1), X_{n+1}(t)]]]$, $\quad t$ *in* T,

$$
\text{(iv)} \quad [X_i(t), X_{i+1}(t')][X_i(1), X_{i+1}(tt')]^{-1}, \quad i = 1, 2, \cdots, n-1, t, t' \textit{ in } T, t \neq 1,
$$

$$
\text{(iva)} \quad [X_n(t), X_{n+1}(t')][X_n(1), X_{n+1}(tt')]^{-1}[X_n(t), [X_n(1), X_{n+1}(tt'/2]]^{-1} \cdot [X_n(1), [X_n(1), X_{n+1}(tt'/2)]], \quad t, t' \textit{ in } T, t \neq 1
$$

8. The groups H_n, $n \geq 2$, $p \neq 2$.

$$H_2(H_{n+1}, Z) = H_2(G_{n+1}, Z) \oplus H_1(G_{n+1}, A_{n+1}) \oplus H_2(A_{n+1}, Z)_{G_{n+1}}.$$

Let F_G be as before. The elements $e_i(1) \wedge e_j(t)$, $1 \leq i < j \leq n+1$, t in T, generate A_{n+1}. Let $E'_{ij}(t)$, $1 \leq i < j \leq n+1$, t in T, be free generators of a group F_A, and map it onto A_{n+1} in the obvious way. Then $X_i(t)$, $i = 1, 2, \cdots, n$, t in T, $X_{n+1}(t) = E'_{n,n+1}(t)$, t in T, represent a minimal generating set for H_{n+1}.

(a) The elements of an F_p-basis for $H_2(A_{n+1}, Z)_{G_{n+1}}$ are represented by

$$e_n(1) \wedge e_{n+1}(t) \cap e_n(1) \wedge e_{n+1}(t'), \quad t < t' \text{ in } T,$$
$$e_{n-1}(1) \wedge e_{n+1}(t) \cap e_n(1) \wedge e_{n+1}(1), \quad t \text{ in } T,$$

which yield, modulo $[F, R]$, in terms of relations

$$
\text{(40)} \quad \begin{aligned} &[X_{n+1}(t), X_{n+1}(t')], \quad t < t' \text{ in } T, \\ &[[X_{n-1}(1), X_{n+1}(t)], X_{n+1}(1)], \quad t \text{ in } T. \end{aligned}
$$

(b) $H_1(G_{n+1}, A_{n+1}) = H_1(G_n, V_n) \oplus H_1(V_n, A_{n+1})G_n$. $H_1(G_n, V_n)$ has been computed in Sections 6b, 6c, and 6d. Cycles in V_n representing and F_p-basis are

$$\langle x_i(t)\rangle \otimes e_n(t'), \quad i = 1, 2, \cdots, n-2, t, t' \text{ in } T,$$
$$\langle x_{n-1}(1)\rangle \otimes e_{n-1}(t), \quad t \text{ in } T,$$
$$\langle x_{n-1}(t)\rangle \otimes e_n(t') - \langle x_{n-1}(1)\rangle \otimes e_n(tt'), \quad t, t' \text{ in } T, t \neq 1.$$

Lifting to cycles in A_{n+1} yields the elements

$$\langle x_i(t)\rangle \otimes e_{n+1}(1) \wedge e_n(t'),$$
$$\langle x_{n-1}(1)\rangle \otimes e_{n-1}(1) \wedge e_{n-1}(t),$$
$$\langle x_{n-1}(t)\rangle \otimes e_{n+1}(1) \wedge e_n(t') - \langle x_{n-1}(1)\rangle \otimes e_{n+1}(1) \wedge e_n(tt'),$$

which modulo $[F, R]$ yields as previously the relations

$$
\text{(41)} \quad \begin{aligned} &[X_i(t), X_{n+1}(t')], \quad i = 1, 2, \cdots, n-2, t, t' \text{ in } T, \\ &[X_{n-1}(1), [X_{n-1}(1), X_{n+1}(t)]], \quad t \text{ in } T, \\ &[X_{n-1}(t), X_{n+1}(t')][X_{n-1}(1), X_{n-1}(tt')]^{-1}, \quad t, t' \text{ in } T, t \neq 1. \end{aligned}
$$

(c) $\operatorname{Im}\{\delta_2 : H_2(V_n, V_n) \to V_n \otimes A_n\}$ is the subgroup of $V_n \otimes A_n$ generated by all $u \otimes v \wedge w - v \otimes u \wedge w$, u, v, w in V_n. (Formula (20).) But, modulo this subgroup,

$$u \otimes v \wedge w \equiv v \otimes u \wedge w = -v \otimes w \wedge u \equiv -w \otimes v \wedge u$$
$$= w \otimes u \wedge v \equiv u \otimes w \wedge v = -u \otimes v \wedge w.$$

Hence $2u \otimes v \wedge w \equiv 0$, and since $p \neq 2$, Coker $\delta_2 = (0)$.

Notice that in this case we can dispense with Proposition 5.1.

(d) According to (c),

$$H_1(V_m, A_{n+1}) \cong \operatorname{Ker}\{\delta_1 : V_n \otimes V_n \to \wedge_k^2 V_n\}$$

where $\delta_1(u \otimes v) = -u \wedge v$. An F_p-basis for this kernel consists of the elements

$$e_i(t) \otimes e_j(t') + e_j(t) \otimes e_i(t'), \quad 1 \leq i < j \leq n, \ t, t' \text{ in } T,$$
$$e_i(t) \otimes e_i(t'), \quad i = 1, 2, \cdots, n, \ t, t' \text{ in } T,$$
$$e_i(t) \otimes e_j(t') + e_j(1) \otimes e_i(tt'), \quad 1 \leq i < j \leq n, \ t, t' \text{ in } T, \ t \neq 1.$$

An F_p-basis for $(\operatorname{Ker} \delta_1)_{G_n}$ consists of the elements represented by $e_n(t) \otimes e_n(t')$, t, t' in T, which yield in terms of cycles the elements, $\langle x_n(t)\rangle \otimes e_{n+1}(1) \wedge e_n(t')$, or, in terms of relations

$$[X_n(t), X_{n+1}(t')], \quad t, t' \text{ in } T. \tag{42}$$

The results of this section, together with Proposition 6, yield

PROPOSITION 8. *Let $n \geq 2$, $p \neq 2$. An F_p-basis for $H_2(H_{n+1}, Z)$ is given in terms of the generators $X_i(t)$, $i = 1, 2, \cdots, n + 1$, t in T, by the relations*

(i) $[X_i(t), X_i(t')]$, $i = 1, 2, \cdots, n + 1$, $t < t'$ *in* T,

(ii) $[X_i(t), X_j(t')]$, $1 \leq i < j \leq n$, $j \neq i + 1$, t, t' *in* T,

(iia) $[X_i(t), X_{n+1}(t')]$, $i = 1, 2, \cdots, n - 2$, $i = n$, t, t' *in* T,

(iii) $[X_i(1), [X_i(1), X_{i+1}(t)]]$ *and* $[X_{i+1}(1), [X_i(1), X_{i+1}(t)]]$, $i = 1, 2, \cdots, n$, t *in* T,

(iiia) $[X_{n-1}(1), [X_{n-1}(1), X_{n+1}(t)]]$ *and* $[X_{n+1}(1), [X_{n-1}(1), X_{n+1}(t)]]$, t *in* T.

(iv) $[X_i(t), X_{i+1}(t')][X_i(1), X_{i+1}(tt')]^{-1}$, $i = 1, 2, \cdots, n$, t, t' *in* T, $t \neq 1$.

(iva) $[X_{n-1}(t), X_{n+1}(t')][X_{n-1}(1), X_{n+1}(tt')]^{-1}$, t, t' *in* T, $t \neq 1$.

REFERENCES

1. M. BARR AND G. S. RINEHART, *Cohomology as the derived functor of derivations*, Trans. Amer. Math. Soc., vol. 122 (1966), pp. 416–426.
2. N. BLACKBURN, *Some homology groups of wreathe products*, Illinois J. Math., vol. 16 (1972), pp. 116–129 (this issue).
3. L. EVENS, *Terminal p-groups*, Illinois J. Math., vol. 12 (1968), pp. 682–699.

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