MATRIX GROUPS OF THE SECOND KIND

BY
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A group, G, of matrices with entries from the field of complex numbers is said to be "of the second kind" if each matrix has real character, but G is not similar to a group of matrices with real entries. The single faithful irreducible representation of the quaternion group provides an example of such a group of matrices.

Two classical results are strengthened by Theorem 1 below: The first of these asserts that every non-trivial irreducible representation of a (finite) group of odd order involves complex characters. (We deal exclusively here with finite groups, and representations over the field of complex numbers.) Theorem 1 extends to the more general case of a group whose elements of odd order form a subgroup. The second classical result asserts that every matrix group of the second kind has even degree. Theorem 1 puts a constraint on this degree, leading to the easy corollary that a group whose order is not divisible by four cannot have an irreducible representation of the second kind.

Theorem 2 complements Theorem 1 by providing a set of circumstances under which we may assert that a group, G, *does* have a representation of the second kind.

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THEOREM 1. *Let G be a group whose elements of odd order form a subgroup, and suppose that $\rho(G)$ is an irreducible representation, of the second kind, of G. Then the order of G is divisible by twice the degree of $\rho(G)$.*

Proof. We may, without loss, assume that $\rho(G)$ is a faithful representation of G.

Let N be the subgroup of G which consists of all the elements of odd order in G. $N \lhd G$, and we may invoke Clifford's Theorem in considering $\rho(G) \downarrow N$, which has irreducible components $\sigma_i(N)$, with common multiplicity n. All of the $\sigma_i(N)$ are in the same family of irreducible representations of N.

Let f be the degree of $\rho(G)$, and suppose that the theorem is false, so that $2f$ does not divide $|G|$ (f, of course, does). Let P be a Sylow 2-group of G. It is trivial to show that G is a semi-direct product NP. Suppose that $|P| = 2^k$. Then $f = 2^k s$, with s odd. Suppose also that each $\sigma_i(N)$ has degree t (they are all in the same family of representations of N) and that z different irreducible representations of N appear in $\rho(G) \downarrow N$. Then $tzn = f = 2^k s$. Further,

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N has odd order, and hence the irreducibility of the $\sigma_i(N)$ implies that t is odd. It follows that 2^k, the order of P, divides zn.

Take the matrices for $\rho(G)$ in such a form that the matrices for $\rho(G) \downarrow N$ appear in reduced form, with repeated $\sigma_i(N)$ appearing consecutively. According to Clifford's Theorem, each matrix in $\rho(G)$ permutes the $\sigma_i(N)$ amongst themselves by conjugation. Let N_1 be the subgroup of G which "fixes" $\sigma_1(N)$ in this sense. That is, g is in N_1 if, and only if, $\chi^{\sigma_1}(ghg^{-1}) = \chi^{\sigma_1}(h)$ for each h in N.

The elements of G permute the $\sigma_i(N)$ transitively amongst themselves, by conjugation. Considering G as a permutation group, it follows immediately that the index of N_1 in G is z. Also, N_1 contains N, and so takes the form of a semi-direct product NQ, where Q is a 2-group.

It is our present purpose to show that, in $\rho(G)$, every element of P (and hence every 2-element) has character 0, except, of course, the identity. To that end, we consider a certain representation of N_1.

According to Clifford's Theorem, the matrices for the elements of N_1 have possibly non-zero elements in the first tn by tn block, and 0 entries in the row and column extensions of this block. Thus this block gives rise to a representation, $\beta(N_1)$, of N_1. But also, if g is an element of G not in N_1, then the matrix for g in $\rho(G)$ has 0's in the first tn by tn block. Now it is a theorem in group representations that $\rho(G)$ is irreducible if, and only if, the functions f_{ij} from G to the complexes given by $f_{ij}(g) = a_{ij}$ (the i, j entry in the matrix for g in $\rho(G)$) are linearly independent. Since $\rho(G)$ *is* irreducible, the set of functions arising from the first tn by tn block are linearly independent. But only the elements of N_1 make any contribution towards this independence. From this observation we may deduce that $\beta(N_1)$ is irreducible.

Now $\beta(N_1) \downarrow N$ is simply $\sigma_1(N)$ repeated n times. By Frobenius' Reciprocity Theorem, $\beta(N_1)$ appears n times in $\sigma_1(N) \uparrow N_1$, which has degree $|Q|\, t$. Thus we must have $|Q|\, t \geq (tn)n = tn^2$, or $|Q| \geq n^2$.

We already have that $|G| = z|N_1| = z|QN| = |PN|$ and so $|Q| = |P|/z$ which gives

$$|P| \geq zn^2 \geq zn \geq |P|$$

since $|P|$ divides zn.

But this implies that $n = 1$, and $|P| = z$, so that the $\sigma_i(N)$ are permuted by a group of order z, the number of $\sigma_i(N)$. It follows that only the identity of P fixes a $\sigma_i(N)$, and thence that in $\rho(G)$, all the non-identity elements of P have character 0.

From this last remark we obtain immediately that the identity representation, $I(P)$, of P occurs $f/|P|$ times in the induced representation $\rho(G) \downarrow P$. But then $\rho(G)$ appears $f/|P|$ times in the real representation $I(P) \uparrow G$. Since $f/|P|$ is odd, $\rho(G)$ could not possibly be of the second kind [1], and the theorem follows.

COROLLARY. *Let G be a group of order $2n$, n odd. Then G has no irreducible representations of the second kind.*

Proof. Observe first that the elements of odd order in G form a normal subgroup, and so we may apply Theorem 1 to assert that any irreducible representation of G of the second kind has odd degree. But [Feit] any matrix group of the second kind has even degree, proving the corollary.

THEOREM 2. *Let G be a finite group containing exactly one involution. Then G possesses a representation of the second kind if, and only if, G does not have a non-trivial direct factor which is a cyclic group of order a power of two.*

Proof. Suppose first that G has a non-trivial direct factor of order 2^s, $s > 0$. Then G can be written as $G = C \times N$, where C has order 2^s, and N has odd order. (Since G has only one involution.) Every irreducible representation of G is the tensor product of irreducible representations of C and N. But this leads, in all cases, to representations of odd degree (since C is abelian, and N has odd order) precluding the possibility of a representation of the second kind [Feit].

Suppose, then, that G does not possess such a direct factor. Let $\rho(G)$ be an irreducible representation of G, and let χ^ρ be the associated character. It is proved in [Feit] that

$$\sum_G \chi^\rho(g^2) = c(\rho)|G| \tag{1}$$

where $c(\rho) = 1, 0,$ or -1 according as $\rho(G)$ is of the first kind (real matrices), third kind (complex character), or second kind (real character, complex matrices), respectively. If t is the number of involutions in G then

$$t + 1 = \sum_\rho c(\rho)\chi^\rho(1)$$

where the sum is over the inequivalent irreducible representations of G. Here, $t = 1$, and so we have

$$2 = \sum_\rho c(\rho)\chi^\rho(1). \tag{2}$$

If Theorem 2 fails for G, then every term on the right-hand side of (2) is non-negative. We proceed by induction, assuming Theorem 2 for groups having the stated properties of G, but smaller order.

Let z be the single involution of G, and consider the factor group $\bar{G} = G/\langle z\rangle$. If $\bar{G}$ is odd, then $|G| = 2n$, n odd, and G decomposes as a semi-direct product NC, with $|N| = n$, and $|C| = 2$. But since z is in the centre of G, $C = \langle z\rangle$, and this semi-direct product is a direct product, with a factor a cyclic group of order a power of two, contrary to assumption. Thus $\bar{G}$ is even. The remainder of the proof will be divided into two cases.

Case I. Suppose that $\bar{G}$ has exactly one involution. We would conclude by induction that $\bar{G}$ (and hence G) has a representation of the desired kind unless $\bar{G}$ has a direct factor which is cyclic, of order a power of 2. Assume, then, that $\bar{G}$ can be decomposed as $\bar{G} = \bar{A} \times \bar{B}$, where $\bar{A}$ has order a power of 2.

Choose $\bar{A}$ to be as small as possible consistent with this decomposition taking place with $\bar{A}$ non-trivial.

If, now, $\bar{B} = \bar{1}$, then $\bar{G}$ is cyclic and so G is generated by a single element together with the element z, which lies in its centre. Thus G is abelian, which is impossible, for then it surely contains a cyclic direct factor of order a power of 2. Hence we may assume that $\bar{B} \neq \bar{1}$.

Let $\bar{g}$ be a generator of $\bar{A}$, and let $\bar{A} = 2^s$, so that $\bar{g}^{2^s} = \bar{1}$. Let g be an inverse image of $\bar{g}$ in G. Since G has only one involution, we must have $g^{2^s} = z$.

Let B be the inverse image of $\bar{B}$ in G. Since $\bar{G}$ has only one involution, $\bar{B}$ is odd, and $B = 2m$, m odd. But since z is in the centre of B, B decomposes as a direct product $B_0 \times \langle z \rangle$, where B_0 has odd order. Case I will be disposed of when we have demonstrated the contradition that G is the direct product of the subgroups $\langle g \rangle$ and B_0. Indeed, since

$$G = \langle g, B \rangle = \langle g, B_0 \rangle \quad \text{and} \quad \langle g \rangle \cap B_0 = 1,$$

it will suffice to show that g and B_0 commute. Let h be any element of B_0. If $h^g \neq h$ then, because the elements of $\bar{A}$ and $\bar{B}$ commute, $h^g = hz$. But $|h|$ is odd, and h^g, which is conjugate to h, has order $2|h|$, a contradiction. Thus G is a direct product with a cyclic factor of order a power of 2, contrary to assumption. This disposes of Case I.

Case II. Suppose now that $\bar{G}$ contains at least 2 involutions.

Let $\beta(\bar{G})$ be an irreducible representation of $\bar{G}$ (and, consequently, of G). Using the expression preceding (2), and noting that t is now greater than 1, it follows that

$$3 \leq \textstyle\sum_{\beta} c(\beta)\chi^{\beta}(\bar{1}).$$

However, by assumption, the G-sum

$$\textstyle\sum_{\rho} c(\rho)\chi^{\rho}(1)$$

which is composed of non-negative terms, and contains the sum

$$\textstyle\sum_{\beta} c(\beta)\chi^{\beta}(\bar{1})$$

is equal to 2.

This is impossible, and Theorem 2 is proved.

Reference

W. Feit, *Characters of finite groups*, W. A. Benjamin, New York, pp. 20, 61, 68.

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