EXTENSION OF SPECTRAL MEASURES

BY

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The main object of the present paper is the study of the resolutions of the identity of the sum and the product of two commuting scalar type operators on an arbitrary Banach space. This is achieved by studying for an arbitrary Banach space the extension of a spectral measure defined on a field R of subsets of a set $\mathfrak{S}$ to the σ-field generated by R. A very special aspect of this study is the content of a recent paper by Kluvánek and Kovářiková [11]. These authors have restricted their attention to the case where the spectral measure to be extended is the product of two commuting spectral measures and the Banach space is weakly complete. Again their extension assumes a topological set up on $\mathfrak{S}$. Naturally the results of the present paper subsume those of [11].

The procedure of closely following the numerical analogue is used in the present paper; the referee has pointed out that the extension can also be obtained by reducing matters to the case of Hilbert space. In the latter method the extension rests on known results (Berberian [4]) which are however established for Hilbert spaces in [4] by a method altogether different from that adopted in this paper. The aesthetic satisfaction in sticking to the Banach space alone, as in this paper has justification on two counts: (i) the auxiliary notion of spectral outer measure introduced here seems to be interesting and worthwhile in itself, (ii) the fact that results follow from their numerical analogues is brought out vividly.

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1. Preliminaries

X will denote an arbitrary (complex) Banach space. A B. A. B of projections on X will be called complete (σ-complete) if B satisfies the condition of Definition 2.1 of Bade [2].

In this section, we recall some definitions and results from [14].

DEFINITION 1. By a $W^*(\|\cdot\|)$-algebra W on a Banach space X we mean a pair, consisting of an abelian subalgebra W of $B(X)$, generated by a σ-complete B.A. of projections on X in the weak operator topology, and some equivalent norm $\|\cdot\|$ on X such that every element S in W has a representation of the form $S = R + iJ$ where R and J satisfy the following conditions:

(i) $RJ = JR$ with R and J in W;

(ii) $R^m J^n$ $(m, n = 0, 1, 2, \cdots)$ are hermitian in the norm $\|\cdot\|$ (hermitian in the sense of Lumer [12]).

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(1.1) A $W^*(\|\cdot\|)$-algebra W is an abelian B^*-algebra in the operator norm $\|\cdot\|$ computed with respect to the Banach space norm $\|\cdot\|$ of X and hence the Gelfand map is an isometric isomorphism of W onto the space $\mathcal{C}(\mathfrak{M})$ of all complex-valued continuous functions on $\mathfrak{M}$, where $\mathfrak{M}$ is the maximal ideal space of W.

(1.2) The Banach algebra W generated weakly by a σ-complete B.A. B of projections on X is a $W^*(\|\cdot\|)$-algebra under a suitable equivalent norm $\|\cdot\|$ on X. (In fact, the norm $\|\cdot\|$ can be taken as that equivalent norm on X in which all the members of $\bar{B}^s$ are hermitian). If X is weakly complete, the hypothesis that B is bounded would suffice for the result to hold.

DEFINITION 2. An operator T on a Banach space X is called positive in the equivalent norm $\|\cdot\|$ on X (written as $T \geq 0$ in $(\|\cdot\|)$ if $[Tx, x] \geq 0$ for all x in X with $\|x\| = 1$, where $[\quad,\quad]$ is a semi-inner-product consistent with the norm $\|\cdot\|$ on X.

(1.3) If T is an operator belonging to a $W^*(\|\cdot\|)$-algebra then the following are equivalent:
- (i) $\sigma(T)$ is non-negative.
- (ii) The Gelfand function $T(m)$ in $\mathcal{C}(\mathfrak{M})$ is non-negative.
- (iii) T is positive in $\|\cdot\|$.

Thus from (1.3) we have:

(1.4) All the projections belonging to a $W^*(\|\cdot\|)$-algebra are positive in $\|\cdot\|$.

DEFINITION 3. For two operators T and T' on a Banach space X, we say that T is greater than T' in the equivalent norm $\|\cdot\|$ on X (written as $T \geq T'$ in $\|\cdot\|$) if
- (i) T, T' are hermitian in $\|\cdot\|$ and
- (ii) $T - T'$ is positive in $\|\cdot\|$.

(1.5) The set $H(W)$ of all elements in a $W^*(\|\cdot\|)$-algebra W, which are hermitian in $\|\cdot\|$, forms a conditionally complete lattice under the ordering $\geq$ in $\|\cdot\|$ given in Definition 3.

(1.6) Let T_α be a bounded (in the ordering sense) monotonic net in $H(W)$. Then for x in X, $\lim_\alpha T_\alpha x = \bigvee_\alpha T_\alpha x\,(\bigwedge_\alpha T_\alpha x)$ if T_α is increasing (decreasing), where the supremum (infimum) is taken in $H(W)$.

Let B be a σ-complete B.A. of projections on a Banach space X and W the $W^*(\|\cdot\|)$-algebra generated by B. Then $\bar{B}^s \subset H(W)$. Further, in view of (1.3), for two projections E_1 and E_2 in W, the relation $E_1 \geq E_2$ in $\|\cdot\|$ holds if and only if $E_1E_2 = E_2$. Thus we have:

(1.7) The usual ordering in $\bar{B}^s$ coincides with the ordering defined in $H(W)$.

2. Some lemmas

Throughout this section we shall assume that W is a $W^*(\|\cdot\|)$-algebra, generated by a σ-complete B. A. B of projections on a Banach space X.

Notation. If $\{E_\alpha\}_{\alpha \in A}$ is a set of projections in $\bar{B}^s$, then its suprema with respect to $\bar{B}^s$ and $H(W)$ will be denoted by $\bigvee_{\bar{B}^s} E_\alpha$ and $\bigvee_{H(W)} E_\alpha$ respectively. Similarly the corresponding infima are denoted.

LEMMA 1. *For projections* $\{E_\alpha\} \subseteq \bar{B}^s$,

$$\bigvee_{\bar{B}^s} E_\alpha = \bigvee_{H(W)} E_\alpha \quad and \quad \bigwedge_{\bar{B}^s} E_\alpha = \bigwedge_{H(W)} E_\alpha .$$

Proof. Since we may replace $\{E_\alpha\}$ by the net of its finite unions, we may suppose that it is an increasing net. If $x \in X$, then by Lemma 2.3 of Bade [2], $\lim_\alpha E_\alpha x = \bigvee_{\bar{B}^s} E_\alpha x$. But, by (1.7), $\{E_\alpha\}$ is also an increasing net in $H(W)$. Hence, by (1.6), $\lim_\alpha E_\alpha x = \bigvee_{H(W)} E_\alpha x$, for x in X. Thus $\bigvee_{\bar{B}^s} E_\alpha = \bigvee_{H(W)} E_\alpha$. Similarly, the result concerning the infimum is proved.

LEMMA 2. *Let $\{E_i\}$ be a sequence of projections in W. If $\sum_{i=1}^\infty E_i$ converges strongly, then $\sum_{i=1}^\infty E_i$ is an operator in W and is positive in $\|\cdot\|$. Further, $\sum_{i=1}^\infty E_i \geq \bigvee_{H(W)} E_i$ in $\|\cdot\|$.*

Proof. Since W is also strongly closed, the strong limit $\sum_{i=1}^\infty E_i$ belongs to W. Further, since each $E_i \geq 0$ in $\|\cdot\|$, for x in X with $\|x\| = 1$, we have

$$(1) \qquad [\textstyle\sum_{i=1}^\infty E_i x, x] = \sum_{i=1}^\infty [E_i x, x] \geq 0$$

where $[\ ,\ ]$ is a semi-inner-product on X, consistent with the norm $\|\cdot\|$. Hence $\sum_{i=1}^\infty E_i \geq 0$ in $\|\cdot\|$. The last statement of the lemma follows from (1).

Convention. If $\{E_i\}$ is a sequence of projections in W, then whenever we write $\sum_{i=1}^\infty E_i$, it is tacitly assumed that the series is strongly convergent and it denotes the positive operator in W, to which it converges strongly.

3. Spectral outer measures

In this section, we define a projection valued set function having properties similar to those of a numerical outer measure.

DEFINITION 4. Let $E^*(\cdot)$ be a projection-valued set function defined on the σ-field $B(\mathfrak{S})$ of all subsets of $\mathfrak{S}$, with its range contained in a $W^*(\|\cdot\|)$-algebra W on a Banach space X. Then $E^*(\cdot)$ is called a spectral outer measure on X if it has the following properties:

(i) $E^*(\emptyset) = 0$.
(ii) $E^*(\mathfrak{S}) = I$.
(ii) $E^*(\sigma_1) \geq E^*(\sigma_2)$ in $\|\cdot\|$ for σ_1, σ_2 in $B(\mathfrak{S})$ with $\sigma_1 \supseteq \sigma_2$.
(iv) $E^*(\bigcup_{i=1}^\infty \sigma_i) \leq \sum_{i=1}^\infty E^*(\sigma_i)$ in $\|\cdot\|$ for $\sigma_i \in B(\mathfrak{S})$, $i = 1, 2, \cdots$.

By our convention given in Section 2, condition (iv) has to hold only when $\sum_{i=1}^\infty E^*(\sigma_i)$ converges strongly. We note that $\sum_{i=1}^\infty E^*(\sigma_i)$ then belongs to W and is positive in $\|\cdot\|$, because of Lemma 2 of Section 2.

We shall prove below that a (strongly countably additive) spectral measure $E(\cdot)$, defined on a field R of subsets of a set $\mathfrak{S}$ can be extended to a spectral outer measure $E^*(\cdot)$ on the σ-field $B(\mathfrak{S})$. To this end, we need the following lemmas.

LEMMA 3. *Let $E(\cdot)$ be a spectral measure defined on a field R of subsets of a set $\mathfrak{S}$ with its range contained in a $W^*(\|\cdot\|)$-algebra W on X. Then $E^*(\cdot)$ defined by*

$$E^*(\sigma) = \wedge_{H(W)} \{\textstyle\sum_{i=1}^{\infty} E(\sigma_i) : \bigcup_{i=1}^{\infty} \sigma_i \supseteq \sigma, \sigma_i \in R\}$$

for $\sigma \subseteq \mathfrak{S}$, is a projection-valued set function on $B(\mathfrak{S})$ with its range contained in W.

Proof. For any subset σ of $\mathfrak{S}$ as $\sigma \subseteq \mathfrak{S} \cup \emptyset \cup \cdots$ and $E(\mathfrak{S}) = I$, the set of operators defining $E^*(\sigma)$ is non-empty. Further, since $\sum_{i=1}^{\infty} E(\sigma_i)$ converges to an operator which is positive in $\|\cdot\|$ in $H(W)$, the definition of $E^*(\sigma)$ makes sense. Clearly, by definition, $E^*(\sigma)$ belongs to $H(W)$. The lemma will be proved, if we show that $E^*(\sigma)$ is a projection. If we set

$$E_1^*(\sigma) = \wedge_{H(W)} \{\textstyle\sum_{i=1}^{\infty} E(\delta_i), \bigcup_{i=1}^{\infty} \delta_i \supseteq \sigma, \delta_i \in R \text{ and } \delta_i \cap \delta_j = \emptyset, i \neq j\},$$

then clearly

$$E_1^*(\sigma) \geq E^*(\sigma) \quad \text{in } \|\cdot\|. \tag{2}$$

But, for any covering $\{\sigma_i\}$ of sets from R we can extract a covering $\{\delta_i\}$ of pair-wise disjoint sets of R such that $\bigcup_{i=1}^{\infty} \sigma_i = \bigcup_{i=1}^{\infty} \delta_i$ and $\delta_i \subseteq \sigma_i$. Thus $E(\sigma_i) \geq E(\delta_i)$ and hence from Lemma 2 of Section 2 we have

$$\textstyle\sum_{i=1}^{\infty} E(\sigma_i) \geq \vee_{H(W)} E(\delta_i) \quad \text{in } \|\cdot\|.$$

But since

$$E(\delta_i)E(\delta_j) = E(\delta_i \cap \delta_j) = 0 \quad \text{for } i \neq j,$$

$\vee_{H(W)} E(\delta_i) = \sum_{i=1}^{\infty} E(\delta_i)$ and hence

$$E_1^*(\sigma) \leq E^*(\sigma) \quad \text{in } \|\cdot\|. \tag{3}$$

From (2) and (3) it follows that $E_1^*(\sigma) = E^*(\sigma)$. Since each $\sum_{i=1}^{\infty} E(\delta_i)$ is a projection, by Lemma 1 of Section 2 it follows that $E_1^*(\sigma)$ is a projection. Hence $E^*(\sigma)$ is a projection.

COROLLARY. *The set function $E^*(\sigma)$ defined in the above lemma is also given by*

$$E^*(\sigma) = \wedge_{H(W)} \{\textstyle\sum_{i=1}^{\infty} E(\sigma_i) : \bigcup_{i=1}^{\infty} \sigma_i \supseteq \sigma, \sigma_i \in R \text{ and } \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j\}.$$

LEMMA 4. *Let $E^*(\cdot)$ be the set function defined in Lemma 3. Then, for $\sigma \subseteq \mathfrak{S}$, $E^*(\sigma)x = \lim_\alpha E_\alpha x$ for x in X, where $\{E_\alpha\}$ is some decreasing net of projections in W. Further, each E_α can be taken as the projection $\sum_{i=1}^{\infty} E(\sigma_{\alpha_i})$, where $\sigma \subseteq \bigcup_{i=1}^{\infty} \sigma_{\alpha_i}$, $\sigma_{\alpha_i} \in R$ and $\sigma_{\alpha_i} \cap \sigma_{\alpha_j} = \emptyset$ for $i \neq j$.*

Proof. By the corollary to Lemma 3,

$$(4)\quad E^*(\sigma) = \bigwedge_{H(W)} \{\textstyle\sum_{i=1}^{\infty} E(\sigma_i)\colon \bigcup_{i=1}^{\infty}\sigma_i \supseteq \sigma,\ \sigma_i \in R \text{ and } \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j\}.$$

But the family F of projections $\sum_{i=1}^{\infty} E(\sigma_i)$ defining $E^*(\sigma)$ in (4) is clearly closed for the formation of finite products. Hence we can suppose that the family F has been replaced by the net $\{E_\alpha\}$ of its finite intersections. Now the lemma follows from Lemma 2.3 of Bade [2].

THEOREM 1. *Let $E(\cdot)$ be a spectral measure defined on a field of subsets of a set $\mathfrak{S}$, with its range contained in a σ-complete B. A. B of projections in X. If W is a $W^*(\|\cdot\|)$-algebra generated by B in the weak operator topology, then the set function $E^*(\cdot)$, defined by*

$$E^*(\sigma) = \bigwedge_{H(W)} \{\textstyle\sum_{i=1}^{\infty} E(\sigma_i),\ \bigcup_{i=1}^{\infty}\sigma_i \supseteq \sigma,\ \sigma_i \in R\} \qquad \text{for } \sigma \subseteq \mathfrak{S},$$

is a spectral outer measure on the σ-field $B(\mathfrak{S})$. Further, $E^*(\cdot)$ is an extension of $E(\cdot)$.

Proof. First we shall prove that $E^*(\cdot)$ is an extension of $E(\cdot)$. If $\sigma \in R$, $\sigma = \sigma \cup \emptyset \cup \cdots$ so that $E^*(\sigma) \leq E(\sigma)$ in $\|\cdot\|$. But if $\{\sigma_i\}$ is a covering of σ by pairwise disjoint sets of R, then

$$E(\sigma) = \textstyle\sum_{i=1}^{\infty} E(\sigma_i \cap \sigma) \leq \sum_{i=1}^{\infty} E(\sigma_i) \quad \text{in } \|\cdot\|.$$

Thus by the corollary under Lemma 3, $E(\sigma) \leq E^*(\sigma)$ in $\|\cdot\|$. Hence $E^*(\sigma) = E(\sigma)$ for $\sigma \in R$.

Since $\emptyset \in R$, $E^*(\emptyset) = E(\emptyset) = 0$ and similarly, $E^*(\mathfrak{S}) = E(\mathfrak{S}) = I$. For two subsets σ_1, σ_2 of $\mathfrak{S}$ with $\sigma_1 \supseteq \sigma_2$, any covering of σ_1 is necessarily a covering of σ_2 and hence $E^*(\sigma_1) \geq E^*(\sigma_2)$ in $\|\cdot\|$.

To prove the property (iv) of the spectral outer measure for $E^*(\cdot)$, let $\{\sigma_i\}$ be a sequence of subsets of $\mathfrak{S}$. Then

$$E^*(\sigma_i) = \bigwedge_{H(W)} \{\textstyle\sum_{j=1}^{\infty} E(\sigma_{ij}) : \bigcup_{j=1}^{\infty}\sigma_{ij} \supseteq \sigma_i .\ \sigma_{ij} \in R \text{ and } \sigma_{ij} \cap \sigma_{ij'} = \emptyset \text{ for } j \neq j'\}$$

by the corollary under Lemma 3. Hence if we denote this family of projections over which infimum is taken by F_i, then by Lemma 4 we can assume F_i to be a decreasing net of projections so that, for x in X,

$$E^*(\sigma_i)x = \lim_{P_{i_\alpha} \in F_i} P_{i_\alpha} x.$$

Thus, for x in X with $\|x\| = 1$, we have

$$[E^*(\sigma_i)x, x] = \lim_{P_{i_\alpha} \in F_i} [P_{i_\alpha} x, x]$$

where $[\ ,\ ]$ is a semi-inner-product on X consistent with the norm $\|\cdot\|$ Since $E^*(\sigma_i) \leq P_{i_\alpha}$ in $\|\cdot\|$, we have

$$[E^*(\sigma_i)x, x] \leq [P_{i_\alpha} x, x].$$

Hence for such a fixed x and an $\epsilon > 0$,

$$[E^*(\sigma_i)x, x] + \epsilon/2^i \geq [P_{i_{\alpha_i}} x, x] \quad \text{for some } P_{i_{\alpha_i}} \in F_i .$$

Thus

$$\begin{aligned}[\textstyle\sum_{i=1}^{\infty} E^*(\sigma_i)x, x] + \varepsilon &= \textstyle\sum_{i=1}^{\infty} ([E^*(\sigma_i)x, x] + \varepsilon/2^i) \\ &\geq \textstyle\sum_{i=1}^{\infty} [P_{i_{\alpha_i}} x, x] \\ &\geq [E^*(\bigcup_{i=1}^{\infty} \sigma_i)x, x],\end{aligned}$$

the last inequality following from the definition of $E^*(\cdot)$. Since ε is arbitrary, it follows that

$$[\textstyle\sum_{i=1}^{\infty} E^*(\sigma_i)x, x] \geq [E^*(\bigcup_{i=1}^{\infty} \sigma_i)x, x].$$

Now as x is arbitrary but for $\| x \| = 1$, it follows that

$$\textstyle\sum_{i=1}^{\infty} E^*(\sigma_i) \geq E^*(\bigcup_{i=1}^{\infty} \sigma_i) \quad \text{in } \|\cdot\|.$$

This completes the proof of the theorem.

4. Extension of spectral measures

As in the case of the numerical measure, we use the Caretheodory's inequality to define $E^*(\cdot)$-measurable sets and obtain the extension of a spectral measure.

DEFINITION 5. Let $E^*(\cdot)$ be a spectral outer measure on the σ-field $B(\mathfrak{S})$ of all subsets of $\mathfrak{S}$, with its range contained in a $W^*(\|\cdot\|)$-algebra W on X. Then a subset σ of $\mathfrak{S}$ is said to be $E^*(\cdot)$-measurable if

$$E^*(\delta) \geq E^*(\sigma \cap \delta) + E^*(\sigma' \cap \delta) \quad \text{in } \|\cdot\|$$

for all sets $\delta \in B(\mathfrak{S})$.

Remark. In view of the subadditivity of $E^*(\cdot)$, for $E^*(\cdot)$-measurable sets σ of $B(\mathfrak{S})$ we have

$$E^*(\sigma) = E^*(\sigma \cap \delta) + E^*(\sigma' \cap \delta);$$

where $\delta \in B(\mathfrak{S})$.

THEOREM 2. *If $E^*(\cdot)$ is a spectral outer measure on $B(\mathfrak{S})$ and if $\bar{S}$ is the class of all $E^*(\cdot)$-measurable sets then $\bar{S}$ is a σ-field of subsets of $\mathfrak{S}$. Further, every set σ for which $E^*(\sigma) = 0$ belongs to $\bar{S}$. Also, the set function $\bar{E}(\cdot)$, defined for $\sigma \in \bar{S}$ by $\bar{E}(\sigma) = E^*(\sigma)$, is a complete spectral measure on $\bar{S}$ (i.e. if $\sigma \in \bar{S}$ and $\bar{E}(\sigma) = 0$, then all subsets of σ belong to $\bar{S}$).*

Proof. By arguing as in the proof of Theorem A of Section 11 of Halmos [9] it can be shown that $\bar{S}$ is a ring of sets. Since

$$E^*(\sigma) = E^*(\sigma \cap \mathfrak{S}) + E^*(\sigma \cap \mathfrak{S}')$$

for all subsets σ of $\mathfrak{S}$, $\mathfrak{S} \in \bar{S}$ and hence $\bar{S}$ is a field of sets. That $\bar{S}$ is a σ-field of sets, $\bar{E}(\cdot)$ is strongly countably additive and $\bar{E}(\cdot)$ is complete on $\bar{S}$ can be proved by giving arguments similar to those of Theorems B and C of Section 11 of Halmos [9]. We shall complete the proof of the theorem, by proving that $\bar{E}(\cdot)$ is multiplicative on $\bar{S}$. Let $\sigma, \delta \in \bar{S}$. Then

$$\bar{E}(\sigma \cup \delta) = \bar{E}(\sigma) + \bar{E}(\sigma' \cap \delta) = \bar{E}(\delta) + \bar{E}(\delta' \cap \sigma)$$

since $\bar{E}(\cdot)$ is additive on $\bar{S}$. Since $\bar{E}(\sigma \cup \delta)$ etc. are commuting projections

$$\bar{E}(\sigma \cup \delta) = \bar{E}(\sigma)\bar{E}(\delta) + \bar{E}(\sigma' \cap \delta) + \bar{E}(\delta' \cap \sigma) \tag{5}$$

since, for disjoint sets $\sigma_1, \sigma_2 \in \bar{S}$ we have

$$\bar{E}(\sigma_2) \leq \bar{E}(\sigma'_1) = I - E(\sigma_1)$$

so that $\bar{E}(\sigma_1)\bar{E}(\sigma_2) = 0$. But, again by the additivity of $\bar{E}(\cdot)$ on $\bar{S}$ we have

$$\bar{E}(\sigma \cup \delta) = \bar{E}(\sigma \cap \delta') + \bar{E}(\sigma' \cap \delta) + \bar{E}(\sigma \cap \delta). \tag{6}$$

Thus from (5) and (6) we have $\bar{E}(\sigma \cap \delta) = \bar{E}(\sigma)\bar{E}(\delta)$. This completes the proof of the theorem.

Thus from Theorem 2 we see that a spectral outer measure $E^*(\cdot)$ induces a spectral measure $\bar{E}(\cdot)$. Also by Theorem 1, a spectral measure $\bar{E}(\cdot)$ induces a spectral outer measure $E^*(\cdot)$. The relation between $E^*(\cdot)$ and $\bar{E}(\cdot)$ is given in the following theorem.

THEOREM 3. *Let $E(\cdot)$ be a spectral measure on a field R of subsets of $\mathfrak{S}$, with its range contained in a σ-complete B. A. B of projections on X. Let W be the $W^*(\|\cdot\|)$-algebra generated by B in the weak operator topology. Then every set in the σ-field $S(R)$ of subsets generated by R is $E^*(\cdot)$-measurable, where $E^*(\cdot)$ is the spectral outer measure induced by $E(\cdot)$ on $B(\mathfrak{S})$.*

Proof. Let $\sigma \in R$, $\delta \in B(\mathfrak{S})$ and $\epsilon > 0$. Then by Lemma 4 of Section 3, for a fixed x in X with $\|x\| = 1$ there exists a sequence $\{\sigma_i\}$ of pairwise disjoint sets in R such that $\delta \subseteq \bigcup_{i=1}^{\infty} \sigma_i$ and

$$\begin{aligned}[E^*(\delta)x, x] + \varepsilon &\geq \textstyle\sum_{i=1}^{\infty} [E(\sigma_i)x, x] \\ &= \textstyle\sum_{i=1}^{\infty} [E(\sigma_i \cap \sigma)x, x] + \sum_{i=1}^{\infty} [E(\sigma_i \cap \sigma')x, x] \\ &\geq [E^*(\sigma \cap \delta)x, x] + [E^*(\sigma' \cap \delta)x, x\end{aligned}$$

where $[\ ,\ ]$ is a semi-inner-product consistent with the norm $\|\cdot\|$ on X. Since ε is arbitrary the above inequality implies

$$[E^*(\delta)x, x] \geq [(E^*(\sigma \cap \delta) + E^*(\sigma' \cap \delta))x, x].$$

Since x is arbitrary but for $\|x\| = 1$, it follows that

$$E^*(\delta) \geq E^*(\sigma \cap \delta) + E^*(\sigma' \cap \delta) \quad \text{in } \|\cdot\|$$

and hence $\sigma \in \bar{S}$. Thus $R \subseteq \bar{S}$. Since $\bar{S}$ is a σ-field, $S(R) \subseteq \bar{S}$.

The following theorem deals with the extension of spectral measures.

THEOREM 4. *Let $E(\cdot)$ be a spectral measure on a field R of subsets of $\mathfrak{S}$, with its range contained in a σ-complete Boolean algebra B of projections on a Banach space X. Then there is a unique spectral measure $\bar{E}(\cdot)$ on the σ-field $S(R)$ of subsets generated by R such that, for σ in R, $\bar{E}(\sigma) = E(\sigma)$. Further, the range of $\bar{E}(\cdot)$ is contained in $\bar{B}^s$.*

Proof. The existence of $\bar{E}(\cdot)$ has been proved in Theorems 2 and 3. To prove uniqueness, let $E_1(\cdot)$ and $E_2(\cdot)$ be two spectral measures on $S(R)$ such that $E_1(\sigma) = E_2(\sigma)$ for all $\sigma \in R$. Let M be the class of all sets $\sigma \in S(R)$ for which $E_1(\sigma) = E_2(\sigma)$. If $\{\sigma_n\}$ is a monotone sequence from M, then

$$\lim_{n\to\infty} E_i(\sigma_n)x = E_i(\lim_n \sigma_n)x, \qquad i = 1, 2,$$

for $x \in X$. Thus $\lim_n \sigma_n \in M$. Thus M is a monotone class and since $M \supseteq R$, it follows from Theorem B of Section 6 of Halmos [9] that M contains $S(R)$.

From Lemma 4 of Section 3 it follows that for $\sigma \in B(\mathfrak{S})$, $E^*(\sigma)$ belongs to $\bar{B}^s$, being the strong limit of projections belonging to $\bar{B}^s$. Hence $\bar{E}(\sigma) \in \bar{B}^s$ for $\sigma \in S(R)$.

In the following theorem we show that the spectral outer measure induced by $\bar{E}(\cdot)$ on $S(R)$ and that induced by $\bar{E}(\cdot)$ on $\tilde{S}$ are the same.

THEOREM 5. *If $\sigma \in B(\mathfrak{S})$, then*

$$E^*(\sigma) = \wedge_{H(W)} \{\bar{E}(\delta) : \sigma \subseteq \delta \in \tilde{S}\}$$
$$= \wedge_{H(W)} \{\bar{E}(\delta) : \sigma \subseteq \delta \in S(R)\}.$$

Proof. The proof is similar to that of Theorem B of Section 12 of Halmos [9] and hence omitted.

DEFINITION 6. If $\delta \in B(\mathfrak{S})$ and $\sigma \in S(R)$, then we shall call σ a measurable cover of δ if δ is contained in σ and if for every τ in $S(R)$ for which $\tau \subseteq \sigma - \delta$, we have $\bar{E}(\tau) = 0$.

THEOREM 6. *If $E(\cdot)$ is a spectral measure on a field R of subsets of $\mathfrak{S}$ with its range contained in a σ-complete B. A. of projections B in X and if $\bar{E}(\cdot)$ is the extended spectral measure of $E(\cdot)$ on $S(R)$, then for every $\delta \in B(\mathfrak{S})$ there exists a set σ in $S(R)$ such that $E^*(\delta) = \bar{E}(\sigma)$ and such that σ is a measurable cover of δ.*

Proof. Let W be the $W^*(\|\cdot\|)$-algebra generated by B. Take a fixed $x \in X$ with $\|x\| = 1$. Then by Lemma 4 of Section 3, there exists a sequence $\{\sigma_i\}$ of pairwise disjoint sets belonging to R such that

$$[E^*(\delta)x, x] + 1/n \geq [\textstyle\sum E^*(\sigma_i)x, x]$$
$$= [\bar{E}(\textstyle\bigcup_{i=1}^{\infty} \sigma_i)x, x],$$

where [ ] is a s.i.p. on x, consistent with the norm $\|\cdot\|$. Set $\bigcup_{i=1}^{\infty} \sigma_i = \delta_n$.

Let $\sigma = \bigcap_{n=1}^{\infty} \delta_n$. Then clearly $\delta \subseteq \sigma \in S(R)$. Further,

$$[E^*(\delta)x, x] \leq [\bar{E}(\sigma)x, x] \leq [\bar{E}(\delta_n)x, x] \leq [E^*(\delta)x, x] + 1/n.$$

Since n is arbitrary,

$$[E^*(\delta)x, x] = [\bar{E}(\sigma)x, x]$$

for $\| x \| = 1$. Hence by Theorem 5 of Lumer [12] we have $E^*(\delta) = \bar{E}(\sigma)$.

Further, if $\tau \in S(R)$ and $\tau \subseteq \sigma - \delta$, then $\delta \subseteq \sigma - \tau$. Hence $E^*(\delta) \leq \bar{E}(\sigma) - \bar{E}(\tau)$ in $\| \cdot \|$. But $E^*(\delta) = \bar{E}(\sigma)$. Thus $\bar{E}(\tau) = 0$.

This completes the proof of the theorem.

Since the above result on measurable covers holds for spectral measures, the following results can be proved here, as in the numerical case. For brevity we state these results without proof.

THEOREM 7. *Let $E(\cdot)$ be a spectral measure on a field R of subsets of $\mathfrak{S}$ with its range contained in a σ-complete B. A. of projections on X. Let $E^*(\cdot)$ be the spectral outer measure induced by $E(\cdot)$ on $B(\mathfrak{S})$ and $\bar{E}(\cdot)$ the extended spectral measure on $S(R)$. Then we have the following*:

(i) If $\sigma \in B(\mathfrak{S})$ *and τ a measurable cover of σ then $E^*(\sigma) = \bar{E}(\tau)$. If both τ_1 and τ_2 are measurable covers of σ, then $\bar{E}(\tau_1 \triangle \tau_2) = 0$.*

(ii) *If $\bar{E}(\cdot)$ is completed, then the measure $E_1(\cdot)$ obtained is defined exactly on the class of all $E^*(\cdot)$-measurable sets and $E_1(\cdot)$ agrees with $E^*(\cdot)$ there.*

Note. The definition of a spectral inner measure can be correspondingly given and properties, similar to the numerical case, can be studied.

The reduction of matters to the Hilbert space mentioned at the outset can now be explained as follows. Standard facts from [2] show that a $W^*(\| \cdot \|)$-algebra $\mathfrak{A}$ is the uniformly closed algebra generated by the set B of all the idempotent members and these idempotents form a complete B. A. of projections. By regarding B as a strongly countably additive spectral measure on the Borel sets of its stone space one obtains from Theorem 3 of [7] that there is a $*$-isomorphism φ of $\mathfrak{A}$ onto a von Neuman algebra of operators on some Hilbert space, with φ strongly and weakly bicontinuous on bounded sets. Further, φ is norm-preserving if $\mathfrak{A}$ is given the operator norm corresponding to the equivalent norm $\| \cdot \|$ on X since φ is a $*$-isomorphism. The outcome of the foregoing remarks is that the extension theorem (Theorem 4) can be obtained as a direct consequence of Theorems 6 and 7 on pages 14, 15 of Berberian [4] after reducing to the Hilbert space setting.

5. Some characterisations of extendable spectral measures

In this section we give two characterisations for a spectral measure $E(\cdot)$ on a field R of subsets of $\mathfrak{S}$ to be extendable to a unique spectral measure on the σ-field $S(R)$ generated by R.

THEOREM 8. *The necessary and sufficient condition for a spectral measure*

$\sigma(\cdot)$ on a field R of subsets of $\mathfrak{S}$ to be extendable to a unique spectral measure on the σ-field $S(R)$ generated by R is that the range of $E(\cdot)$ be contained in a σ-complete B. A. of projections on the Banach space X.

Proof. The sufficient part of the theorem follows from Theorem 4 of Section 4. The condition is also necessary. For, if $\bar{E}(\cdot)$ is an extended spectral measure of $E(\cdot)$ on $S(R)$ then as $\bar{E}(\cdot)$ is strongly countably additive on the σ-field $S(R)$ its range B is a σ-complete B. A. of projections on X. Since $E(\sigma) = \bar{E}(\sigma)$ for $\sigma \in R$, the range of $E(\cdot)$ on R is contained in B. Hence the theorem.

We give below another characterisation theorem which generalizes Theorem 8 of Kluvánek and Kováříková [11]. To this end we need the following lemma.

LEMMA 5. *Let B be a B. A. of projections on a Banach space X. Then B is embedded in a σ-complete B. A. of projections if for each x in X,*

$$N(x) = \{Ex : E \in B\}$$

is relatively weakly compact in X.

Proof. Since $N(x)$ is relatively weakly compact for every $x \in X$, $N(x)$ is bounded and hence by the principle of uniform boundedness B is (uniformly) bounded. Therefore by remarks in Section III of Lumer [13], there exists an equivalent norm $\|\cdot\|$ on X in which all the members of B are hermitian. Hence by Theorem 1.3 of [14] the uniformly closed algebra $A(B)$ generated by B is a B^*-algebra in the operator norm $\|\cdot\|$, computed with respect to the norm $\|\cdot\|$ on X so that $A(B)$ is isometrically isomorphic to $\mathfrak{C}(\mathfrak{M})$ the space of all complex-valued continuous functions on the maximal ideal space $\mathfrak{M}$ of $A(B)$. Then each projection $E \in B$ corresponds to the characteristic function of a unique clopen set $\gamma(E)$ of $\mathfrak{M}$.

By Theorem 18 of Dunford [5], there exists an X-spectral measure $F(\cdot)$ on the family Σ of Borel sets of $\mathfrak{M}$ such that for $x \in X$, $x' \in X'$

$$x'Tx = \int_{\mathfrak{M}} T(m)xF(dm)x', \qquad T \in A(B).$$

Hence, in particular, for $E \in B$ we have

$$x'Ex = \int_{\mathfrak{M}} E(m)xF(dm)x' = \int_{\mathfrak{M}} \chi_{\gamma(E)}^{(m)}\, xF(dm)x' = xF(\gamma(E))x'$$

where $\chi_{\gamma(E)}$ denotes the characteristic function of $\gamma(E)$. Thus

$$E' = F(\gamma(E)) \tag{7}$$

where E' denotes the adjoint of E.

Let M be the class of all sets $\tau \in \Sigma$ such that $F(\tau)$ is the adjoint of some projection $G(\tau) \in B(X)$ and such that $G(\tau)x \in \overline{N(x)}$ for every x in X where the closure is taken in the weak topology. Then clearly M contains the field $\mathfrak{C}$ of all clopen sets of $\mathfrak{M}$, since by (7) we have $E = G(\gamma(E))$ for $E \in B$.

We claim that M is monotone. This is proved by giving an argument similar to that of [9]. If $\{\tau_n\}$ is a monotone sequence of sets from M and $\tau = \lim_n \tau_n$, then for $x \in X$, $x' \in X'$

$$(8) \qquad \lim_n xF(\tau_n)x' = xF(\tau)x'$$

since $F(\cdot)$ is an X-spectral measure on Σ. But, as $\tau_n \in M$, $xF(\tau_n)x' = x'G(\tau_n)x$, so that $\{G(\tau_n)x\}$ is weakly fundamental for fixed x. Since $\{G(\tau_n)x\}$ belongs to the weakly compact set $\overline{N(x)}$, by the Eberlein-Šmulian theorem [6, V. 6.1], $\{G(\tau_n)x\}$ is weakly convergent to a unique element $G(\tau)x$ in $\overline{N(x)}$. Clearly $G(\tau) \in B(X)$. But $x'G(\tau)x = \lim_n x'G(\tau_n)x = \lim_n xF(\tau_n)x' = xF(\tau)x'$ from (8). Hence $G(\tau)' = F(\tau)$. Thus $\tau \in M$.

Since M is monotone and contains the field $\mathfrak{C}$ of all clopen sets, it also contains the σ-field $S(\mathfrak{C})$ generated by $\mathfrak{C}$ by Theorem B of Section 6 of Halmos [9]. Thus over the σ-field $S(\mathfrak{C})$ we have the projection-valued set function $G(\cdot)$ with its range in $B(X)$. Since $F(\cdot)$ is X-countably additive on Σ, $G(\cdot)$ is X'-countably additive on $S(\mathfrak{C})$. Hence by a theorem of Pettis this implies that $G(\cdot)$ is strongly countably additive on the σ-field $S(\mathfrak{C})$ and hence its range B_1 is a σ-complete B. A. of projections on X. Since B is contained in B_1 the lemma follows.

THEOREM 9. *Let $E(\cdot)$ be a spectral measure on a field R of subsets of a set $\mathfrak{S}$ with its range in $B(X)$. Then $E(\cdot)$ can be extended to a unique spectral measure $\bar{E}(\cdot)$ on the σ-field $S(R)$ generated by R if and only if for $x \in X$,*

$$N(x) = \{E(\sigma)x : \sigma \in R\}$$

is relatively weakly compact in the Banach space X.

Proof. Suppose $E(\cdot)$ is extendable to a unique spectral measure $\bar{E}(\cdot)$ on the σ-field $S(R)$. Then for each $x \in X$, $\bar{E}(x)$ is a vector measure on the σ-field $S(R)$ and hence by Theorem 2.9 of [3], $N_1(x) = \{\bar{E}(\tau)x : \tau \in S(R)\}$, is relatively weakly compact in X. Thus $\overline{N(x)}$ being the closed subset of the compact set $\overline{N_1(x)}$ is itself weakly compact. In other words $N(x)$ is relatively weakly compact in X.

Conversely, if for $x \in X$, $N(x)$ is relatively weakly compact, then by Lemma 5, there exists a σ-complete B. A. B_1 of projections on X such that $B \subseteq B_1$. That is, B is embedded in the σ-complete B. A. B_1. Now the conclusion follows from the sufficient part of Theorem 8.

This completes the proof of the theorem.

6. An application to spectral operators

For weakly complete Banach spaces Foguel proved in [8] that the sum $S_1 + S_2$ and the product $S_1 S_2$ of two (bounded) commuting scalar type operators is again scalar type if and only if the B. A. determined by the resolutions of the identity of S_1 and S_2 is bounded. Further he obtained the resolutions of the identity of $S_1 + S_2$ and $S_1 S_2$ in terms of those of S_1 and S_2, under some restrictions on the boundary of the Borel sets. Later, in [10] Kantorovitz improved this result removing the additional conditions imposed on the boundary of the Borel sets. Recently in [11] Kluvánek and Kovářiková proved the improved result of Kantorovitz, for weakly complete Banach spaces applying the theory of extension of the product of two spectral measures. Motivated by the method of proof in [11] we have the following generalization of this result, even for arbitrary Banach spaces.

THEOREM 10. *Let S_1 and S_2 be two commuting (not necessarily bounded) spectral operators of scalar type of class X'* [1], [5] *with $E_1(\cdot)$ and $E_2(\cdot)$ as their respective resolutions of the identity and further suppose that the B. A. B determined by $E_1(\cdot)$ and $E_2(\cdot)$ can be embedded in a σ-complete B. A. of projections on X. Then $S_1 + S_2$ and $S_1 S_2$ are scalar type of class X'. Also, their respective resolutions of the identity $G_1(\cdot)$ and $G_2(\cdot)$ are given by*

$$G_1(\delta)x = \int E_2(\delta - \lambda)\, dE_1(\lambda)x$$

and

$$G_2(\delta)x = \int E_2(\delta/\lambda)\, dE_1(\lambda)x$$

for each $x \in X$ and for each Borel set δ of the complex plane.

Proof. Since the σ-field Σ of Borel sets of $p \times p$, where p denotes the complex plane, is generated by the field

$R = \{\bigcup_{i=1}^{n} \sigma_i \times \delta_i : \sigma_i, \delta_i$ Borel sets of p and $\sigma_i \times \delta_i$ are mutually disjoint$\}$,

$\Sigma = S(R)$. The $B(X)$-valued set function G_0 on R given by

$$G_0(\tau) = \sum_{i=1}^{n} E_1(\sigma_i)E_2(\delta_i)$$

where $\tau = \bigcup_{n=1}^{\infty} \sigma_i \times \delta_i$ where σ_i, δ_i are Borel sets of p and $\sigma_i \times \delta_i$ are mutually disjoint is clearly well defined and is a spectral measure on R.

Since by hypothesis, the range of $G_0(\cdot)$ is contained in a σ-complete B. A. of projections, by Theorem 4 of Section 4, $G_0(\cdot)$ can be extended to a unique spectral measure $G(\cdot)$ on $S(R) = \Sigma$. Thus if $f(\lambda, \mu)$ is any Borel measurable function on $p \times p$ then (by Lemma 6 of Dunford [5] if f is bounded or by Theorem 4 of Panchapagesan [14] if f is $G(\cdot)$-essentially unbounded), the operator $f(S_1, S_2)$ given by

$$f(S_1, S_2)x = \int_{p\times p} f(\lambda, \mu)\, dG(\lambda, \mu)x$$

for $x \in D(f(S_1, S_2))$ is a scalar type operator and its resolution of the identity is given by

$$G_f(\sigma) = G\{(\lambda, \mu) : f(\lambda, \mu) \in \sigma\}$$

where σ is a Borel set of p.

Now the theorem follows by applying the above result to the particular case of the Borel measurable functions $\lambda + \mu$ and $\lambda\mu$.

Note. In weakly complete Banach spaces, the boundedness of a B. A. B of projections implies $\bar{B}^s$ is complete and $\bar{B}^s$ is hence σ-complete. This follows from Corollary 2.10 of Bade [2]. Hence in view of this result, it suffices to assume in all the theorems above that the B. A. in question is bounded, if the Banach space X is weakly complete.

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