LOCALLY ADJUNCTABLE FUNCTORS[1]

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Certain operations in mathematics, such as the algebraic closure of a field, construction of a universal covering space, or introduction of coordinates in geometry resemble, but fail to be, adjoint functors. We generalize the notion of adjoint functor to include such situations by lowering the degree of universality required of the adjunction morphisms. Certain group-valued functors arise which yield very familiar groups in the particular cases considered.

After the preliminaries are established, we give the proof that a "locally adjunctable" functor can be characterized in terms of a functor with genuine adjoint. This will be followed by a discussion of related results, special cases ("locally reflective" subcategories) and examples. We then show how a canonical group-valued functor may be associated with every locally adjunctable functor. We close with a discussion of some unresolved questions concerning the construction of locally adjunctable functors and the fibration of a locally adjunctable functor by a "best possible" genuine adjoint situation.

DEFINITION. Let $F : \mathbf{A} \to \mathbf{C}$ be a functor. We say F is locally left adjunctable if for any $f : X \to F(A)$ in $\mathbf{C}$ and $A \in \mathbf{A}$, there exists an object $f(X) \in \mathbf{A}$, an $\mathbf{A}$-morphism $f_1 : f(X) \to A$, and a $\mathbf{C}$-morphism $f_0 : X \to F(f(X))$ such that $f = F(f_1)f_0$. Moreover, if $f = F(h)g$ with $h : A' \to A$, then there exists *a unique* $t : f(X) \to A'$ such that $ht = f_1$ and $F(t)f_0 = g$, i.e. the following diagram commutes in $\mathbf{C}$:

$$\begin{array}{ccc} X & \xrightarrow{h} & \bar{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ F(A) & \xrightarrow[F(g)]{} & F(\bar{A}) \end{array}$$

If F has a left adjoint, say G, then F is clearly locally left adjunctable because the front adjunction for X serves as the "f_0" for every f with domain X and codomain in $F(\mathbf{A})$, and $G(X)$ serves as the "$f(X)$" for every such f.

If F and G are two functors with common codomain $\mathbf{D}$, (G, F) denotes the comma category [6]. (Its objects are triples $(B, A\ f,)$ where $B \in$ domain of G, $A \in$ domain of F, and $f : G(B) \to F(A)$ in $\mathbf{D}$; and $(h, g) : (B, A, f) \to (\bar{B}, \bar{A}, \bar{f})$

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in (G, F) if $\bar{f}G(h) = F(g)f$ in $\mathbf{D}$.) We shall write $(\mathbf{C}, F)$ for $(1_{\mathbf{C}}, F)$ and write $(\mathbf{C}, X)$ instead of $(\mathbf{C}, I_X)$ where I_X denotes the functor from the trivial one object category onto the object $X \in \mathbf{C}$. This category is sometimes called "the category of objects over X". Also if $\mathbf{A}$ is any category, $(\mathbf{A}, \mathbf{A})$ is just the morphism category of $\mathbf{A}$. If $F : \mathbf{A} \to \mathbf{C}$ is a functor there is an obvious functor $(F, A) : (\mathbf{A}, A) \to (\mathbf{C}, F(A))$. Then the definition may be stated in terms of universal elements (see [7] or [8]) as follows. Let A be any object of $\mathbf{A}$. Then for every object (X, A, f) of $(\mathbf{C}, F(A))$, there exists a universal element, namely the pair $(f_0, (f(X), A, f_1))$, for the functor $h_f \circ (F, A)$ where h_f denotes the covariant part of the $(\mathbf{C}, F(A))$-hom functor. Then as a universal element, the factorization of f is unique to within equivalence of (X). It is also clear that the functor F is locally left adjunctable iff each $f$functor (F, A) has a left adjoint. For reference we present the dual concept

Definition. A functor $F : \mathbf{A} \to \mathbf{C}$ is locally right adjunctable if for every $f : F(A) \to X$ in $\mathbf{C}$, there exist $f(X) \in \mathbf{A}$, $f_1 : A \to f(X)$ and $f_0 : F(f(x)) \to X$ in $\mathbf{C}$ such that $f = f_0 F(f_1)$. Moreover if $f = hF(g)$ with $g : A \to A'$ there exists a unique $t : A' \to f(X)$ such that $tg = f_1$ and $f_0 F(t) = h$.

We shall deal primarily with locally left adjunctable functors and shall give a dual statement when it is not especially obvious. Unless otherwise noted, the functor $F : \mathbf{A} \to \mathbf{C}$ is fixed. There is a naturally induced functor $\bar{F} : (\mathbf{A}, \mathbf{A}) \to (\mathbf{C}, F)$ given by

$$\bar{F}(A_1, A_2, a) = (F(A_1), A_2, F(a))$$

on objects and $\bar{F}(h, g) = (F(h), g)$ on morphisms.

We can now state the characterization:

Theorem A. *F is locally left adjunctable iff $\bar{F}$ has a left adjoint.*

For the duration of the following lemmas, assume that F is locally left adjunctable, $f : X \to F(A)$ is a fixed $\mathbf{C}$-morphism, and $f = F(f_1)f_0$ as in the definition. Furthermore any use of the subscripts "0" and "1" is understood to be as in the definition.

Lemma 1. *Suppose that f_0 factors as $f_0 = F(h)g$. Then h is right invertible.*

Lemma 2. *$(f_0)_1$ is an equivalence.*

These two lemmas follow from the fact that $(f_0, (f(X), A, f_1))$ is a universal element.

Lemma 3. *Suppose $a : A \to A'$ in $\mathbf{A}$. Then there exists a unique equivalence*

$$e : f(X) \to (F(a)f)(X)$$

such that $(F(a)f)_1 e = f_1$ and $F(e)f_0 = (F(a)f)_0$.

Proof. Let $(F(a)f)(X) = \bar{A}$ and $(F(a)f)_1 = j_1$. The factorization $F(a)f = F(af_1)f_0$ through the object $F(f(X))$ induces a unique $t : \bar{A} \to f(X)$ such that

$$af_1 t = j_1 \quad \text{and} \quad F(t)(F(a)f)_0 = f_0.$$

Hence $F(f_1 t)(F(a)f)_0 = f$ so there exists a unique $e : f(X) \to \bar{A}$ such $f_1 te = f_1$ and $F(e)f_0$. Note $j_1 e = (af_1 t)e = af_1$ so e satisfies the two equations required.

We now show t is an inverse of e finishing the proof as e has already been uniquely determined. Consider first $te : f(X) \to f(X)$. Since $f_1 te = f_1$, we know $te = 1_{f(x)}$ if $F(te)f_0 = f_0$; but this last equation holds because

$$F(te)f_0 = F(t)(F(e)f_0) = F(t)(F(a)f)_0 = f_0.$$

On the other hand,

$$j_1 et = af_1 tet = af_1(te)t = af_1 t = j_1,$$

and

$$F(et)(F(a)f)_0 = F(e)F(t)(F(a)f)_0 = F(e)f_0 = (F(a)f)_0,$$

so $et = 1_{\bar{A}}$. Q.E.D.

We may suppress the equivalence appearing in Lemma 3 by rewriting the conclusion as $(F(a)f)_1 \cong f_1$ and $f_0 \cong (F(a)f)_0$ where "$\cong$" means "equal to within unique equivalence". We shall state and prove Lemma 4 using this convention.

LEMMA 4. *Suppose* $g : X' \to X$ *in* **C**. *Then* $(fg)_1 \cong f_1(f_0 g)_1$ *and* $(f_0 g)_0 \cong (fg)_0$.

Proof.

$$(fg)_1 = (F(f_1)f_0 g)_1 \cong f_1(f_0 g)_1 \quad \text{and} \quad (fg)_0 = (F(f_1)f_0 g)_0 \cong (f_0 g)_0. \quad \textit{Q.E.D.}$$

We shall now sketch the proof of Theorem A.

First suppose F is locally left adjunctable. Then each object (X, A, f) in $(\mathbf{C}, F)$ determines an object $(f(X), A, f_1)$ of $(\mathbf{A}, \mathbf{A})$, which is determined to within a unique equivalence. (To define a functor from $(\mathbf{C}, F)$ to $(\mathbf{A}, \mathbf{A})$ we must make choices from among classes of equivalent objects and morphisms, but the problems which arise from this are artificial and will be ignored in the proof through the device of using "$=$" to sometimes mean "$\cong$" as used above.) Hence define $F^* : (\mathbf{C}, F) \to (\mathbf{A}, \mathbf{A})$ on objects by

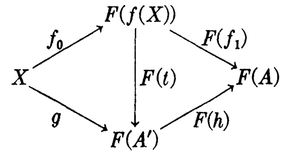
$$F^*(X, A, f) = (f(X), A, f_1).$$

If $(h, g) : (X, A, f) \to (\bar{X}, \bar{A}, \bar{f})$ in $(\mathbf{C}, F)$, define

$$F^*(h, g) = ((\bar{f}_0 h)_1, g).$$

Then $((\bar{f}_0 h)_1, g) : (f(X), A, f) \to (\bar{f}(\bar{X}), \bar{A}, \bar{f}_1)$ in $(\mathbf{A}, \mathbf{A})$ by application of

Lemmas 3 and 4 to the morphisms of the commutative diagram



It is then a routine matter to show F^* is a functor. To prove F^* is a left adjoint $\bar{F}$, let $(X, A, f) \in (\mathbf{C}, F)$. Then $\bar{F}F^*(X, A, f) = (F(f(X)), A, F(f_1))$ and the front adjunction for (X, A, f) is $(f_0, 1_A)$. We omit the details.

Now suppose a left adjoint G of $\bar{F}$ is given, and let $f : X \to F(A)$ in $\mathbf{C}$. Let the front adjunction associated with (X, A, f) be $(\bar{e}, e) : \bar{F}G(X, A, f) \to (X, A, f)$. Denote $G(X, A, f)$ by (B, B', g), so $F(g)\bar{e} = F(e)f$. Consider the morphism

$$(f, 1_A) : (X, A, f) \to (F(A), A, F(1_A))$$

which induces a unique $(\bar{\varphi}, \varphi) : (B, B', g) \to (A, A, 1_A)$ in $(\mathbf{A}, \mathbf{A})$ such that

$$(F(\bar{\varphi}), \varphi)(\bar{e}, e) = (f_1, 1_A).$$

Thus in particular $F(\bar{\varphi})\bar{e} = f$, and it can be shown, using the properties of adjunctions and right cancellability of φ that this factorization of f satisfies the second part of the definition of a local adjunction decomposition. This proves the theorem.

Note that by applying Lemma 3, if F is locally left adjunctable, then $F^*\bar{F}$ is naturally equivalent to the identity functor on $(\mathbf{A}, \mathbf{A})$, and F^* could actually be chosen to be a left inverse for $\bar{F}$. We could obviously have used Theorem A as a starting point, but in applications the functor on comma categories may not be especially clarifying, and it is usually the local adjunction morphisms which are of interest anyway.

If $F : X \to F(A)$ in $\mathbf{C}$, then a local solution set for f is a solution set for f with respect to the functor $(F, A) : (\mathbf{A}, A) \to (\mathbf{C}, F(A))$. We may then state the following.

THEOREM B (Local Adjoint Functor Theorem). *Let $F : \mathbf{A} \to \mathbf{C}$ be a functor where $\mathbf{A}$ is well powered, has intersections (generalized pullbacks) and equalizers. Then F is locally left adjunctable iff F preserves intersections and equalizers, and satisfies the local solution set condition.*

Proof. Apply the classical adjoint functor theorem to the functors (F, A), $A \in \mathbf{A}$, recalling that a product in the category $(\mathbf{A}, A)$ is precisely an intersection in $\mathbf{A}$. Q.E.D.

It should be noted that locally adjunctable functors do not preserve general limits. We shall now turn to a particular type of locally adjunctable functor.

DEFINITION. If **A** is a subcategory of **C**, we say **A** is locally reflective in **C** if the inclusion functor is locally left adjunctable.

Notice that the morphism category of any category **A** is a fibration of **A** (see [3]) via the domain functor, and similarly (**C**, F) is a fibration of **C**. Hence theorem A exhibits the functor $\bar{F}$ as a fibration of the functor F. (The dual of this theorem gives an opfibration of locally right adjunctable functors.) In the case of locally reflective subcategories there is a kind of converse to theorem A. We assume (for convenience only) that the fibrations mentioned henceforth are split and normal.

THEOREM C. *Suppose* $P : \mathbf{E} \to \mathbf{C}$ *is a fibration and* $\mathbf{B} = P^{-1}(\mathbf{A})$ *is a full reflective subcategory of* **E** *where* **A** *is a full subcategory of* **C**. *Then* **A** *is locally reflective in* **C**. *Conversely, by Theorem* A, *every locally reflective subcategory arises as the image under a fibration of a reflective subcategory.*

Proof. Let $f : X \to P(B)$ be a **C**-morphism with $B \in \mathbf{B}$. Consider

$$\theta_f = \theta_f(B) : f^*(B) \to B$$

in **E** (notation of [3]). Let $e : f^*(B) \to R$ be the reflection map of $f^*(B)$ into **B**. Then there exists a unique $\bar{\theta}_f : R \to B$ such that $\bar{\theta}_f e = \theta_f$. We shall show the factorization $f = P(\bar{\theta}_f)P(e)$ through $P(R) \in \mathbf{A}$ is a local reflection decomposition for f.

Suppose $f = gh$ with $h : X \to \bar{B}$ and $\bar{B} \in \mathbf{B}$. Consider

$$\theta_g = \theta_g(B) : g^*(B) \to B \quad \text{and} \quad \theta_h = \theta_h(g^*(B)) : f^*(B) \to g^*(B).$$

Again there is a unique $\bar{\theta}_h : R \to g^*(B)$ such that $\bar{\theta}_h e = \theta_h$. Then clearly $P(\bar{\theta}_h)P(e) = h$. Also $gP(\bar{\theta}_h) = P(\bar{\theta}_f)$ because $\theta_g \bar{\theta}_h = \bar{\theta}_f$ as $\bar{\theta}_g \bar{\theta}_h e = \theta_g \theta_h = \bar{\theta}_f e$. Hence it remains to show $P(\bar{\theta}_h)$ is unique with this property.

To this end assume $m : P(R) \to B$ satisfies $mP(e) = h$ and $gm = P(\bar{\theta}_f)$. Consider

$$\theta'' = \theta_{P(\theta_f)} : B'' = P(\bar{\theta}_f)^*(B) \to B.$$

Since P is a fibration there is a unique $\theta' : R \to B''$ with $\theta'' \theta' = \bar{\theta}_f$ and $P(\theta') = 1_{P(R)}$. Finally, let $\theta_m = \theta_m(g^*(B)) : B'' \to g^*(B)$. To show $P(\bar{\theta}_h) = m$, it suffices to show $\bar{\theta}_h e = \theta_m \theta' e$ because e is a reflection map. However, we claim this equation holds because P is a fibration: Since $\theta_g \bar{\theta}_h e = \theta_f$, there exists a unique morphism r such that $P(r) = h$ and $\theta_g r = \theta_g \bar{\theta}_h e$, so $\bar{\theta}_h e = r$. But also

$$\theta_g(\theta_m \theta' e) = (\theta_g \theta_m)\theta' e = \theta'' \theta' e = \bar{\theta}_f e = \theta_f \quad \text{and} \quad P(\theta_m \theta' e) = m 1_{P(R)} P(e) = h,$$

so $\theta_m \theta' e = r$ also, proving the claim. Q.E.D.

Since the composition of fibrations is a fibration, Theorem C combined with Theorem A implies that the image of a locally reflective subcategory under a fibration is locally reflective. We now show that the fibration of a locally reflective subcategory guaranteed by Theorem A has a lifting property very reminiscent of lifting properties of topological fibrations. Here $D : (\mathbf{C}, \mathbf{A}) \to \mathbf{C}$ is the domain functor.

THEOREM D. *Let* $\mathbf{A}$ *be a full locally reflective subcategory of* $\mathbf{C}$, $P : \mathbf{E} \to \mathbf{C}$ *be a functor such that* $\mathbf{B} = P^{-1}(\mathbf{A})$ *is a full reflective subcategory of* $\mathbf{E}$. *Then*:

(1) P *"lifts" to a functor* $\bar{P} : \mathbf{E} \to (\mathbf{C}, \mathbf{A})$ *such that* $D\bar{P} = P$;

(2) *If* $P' : \mathbf{E} \to (\mathbf{C}, \mathbf{A})$ *agrees with* $\bar{P}$ *on* $\mathbf{B}$ *and if* $DP' = P$, *then there is a natural transformation* $\eta : P' \to \bar{P}$ *such that* $B(\eta_E) = 1_{P(E)}$ *for all* $E \in \mathbf{E}$;

(3) *If, moreover,* P *is a fibration, then each component of* η *is right invertible.*

Proof. (1) For $E \in \mathbf{E}$, define $\bar{P}(E) = (P(E), PR(E), P(e))$ where $e : E \to R(E)$ is the reflection map of E into $\mathbf{B}$, and for $t : E \to E'$ in $\mathbf{E}$, define $\bar{P}(t) = (P(t), PR(t))$, where $R : \mathbf{E} \to \mathbf{B}$ is the reflector. Then $\bar{P}$ is clearly a functor such that $D\bar{P} = P$.

(2) Note that since the subcategories involved are full, the reflectors may be normalized so that they leave the objects and morphisms of the reflective subcategories fixed. Again, let $e : E \to R(E)$ be a reflection map. Then $P'(R(E)) = \bar{P}(R(E)) = 1_{PR(E)}$. Now, abusing the notation of comma categories somewhat, we may assume $P'(E) : P(E) \to A$ for $A \in \mathbf{A}$, and for $t : E \to E'$ in $\mathbf{E}$, $P'(t) = (P(t), P'(t)_r)$ where $P(t)_r$ denotes the codomain end of $P'(t)$. Thus

$$P'(e) = (P(e), P'(e)_r) : P'(E) \to 1_{\bar{P}R(E)},$$

so $P'(e)_r P'(E) = P(e)$. Thus we can define for each $E \in \mathbf{E}$, a morphism

$$\eta_E = (1_{P(E)}, P'(e), P'(e)_r) : P'(E) \to \bar{P}(E).$$

It is straightforward to show that $\{\eta_E\}_{E \in \mathbf{E}}$ is a natural transformation from P' to $\bar{P}$, and clearly $D(\eta_E) = 1_{P(E)}$.

(3) Finally, if P is a fibration which maps $\mathbf{B}$ onto $\mathbf{A}$, the images of reflection maps are the local reflection maps, i.e., the local front adjunctions. Also, the codomain end of each η_E is an $\mathbf{A}$-morphism and the left factor of a local front adjunction, so by Lemma 1, η_E is right invertible. Q.E.D.

Note there is no guarantee that the collection of right inverses in (3) comprise a natural transformation from $\bar{P}$ to P'. We now give a last result for general locally left adjunctable functors which seems to be of interest.

THEOREM E. *Let* $F : \mathbf{A} \to \mathbf{C}$ *be locally left adjunctable. Then*:

(1) *For each* $(X, A, f) \in (\mathbf{C}, F)$ *there is a group of automorphisms of* $f(X)$, $\Pi_F(x, A, f)$, *such that if* $a, a' : f(X) \to A'$ *in* $\mathbf{A}$ *satisfy* $F(a)f_0 = F(a')f_0$, *then there exists a unique* $\alpha \in \Pi_F(X, A, f)$ *such that* $a\alpha = a'$ *and* $F(\alpha)f_0 = f_0$.

(2) $\Pi_F : (\mathbf{C}, F) \to$ Groups *is a contravariant functor.*

Proof. (1) Let $F(a)f_0 = s = F(a')f_0$. Then by Lemma 2, using "$\cong$" as before, $(f_0)_0 \cong f_0$ and $(f_0)_1 \cong 1_{f(X)}$. By Lemma 3, $s_0 \cong (f_0)_0$ and

$s_1 = (F(a)f_0)_1 \cong a(f_0)_1 \cong a$. Similarly $s' \cong f_0$ and $s' \cong a'$ where s' also denotes the morphism $F(a)f_0$. Hence we have two factorizations of s through the local front adjunction f_0, so there exists a unique equivalence $\alpha : f(X) \to f(X)$ such that $a\alpha = a'$ and $F(\alpha)f_0 = f_0$. Then (1) follows clearly.

(2) Suppose $(r, b) : (X, A, f) \to (Y, B, g)$ in $(\mathbf{C}, F)$. Let $(g_0 r)_1 = t$ and $(g_0 r)_0 = p$ so $g_0 r = F(t)p$. Now let $\alpha \in \Pi_F(Y, B, g)$ and consider αt and t, both morphisms from $(g_0 r)(X)$ to $g(Y)$. Now

$$F(\alpha t)p = (F(\alpha)g_0)r = g_0 r = F(t)_p$$

so there exists a unique $\bar{\alpha} \in \Pi_F(X, g(Y), g_0 r)$ such that $(\alpha t)\bar{\alpha} = t$. Since $F(b)f = gr$, there is a unique equivalence $e : f(X) \to (g_0 r)(X)$ preserving commutativity, hence a group isomorphism between $\Pi_F(X, g(Y), g_0 r)$ and $\Pi_F(X, A, f)$ given by $\bar{\alpha}^{-1} \to e^{-1}\bar{\alpha}^{-1}e$. There is thus a well defined function

$$\Pi_F(r, b) : \Pi_F(Y, B, g) \to \Pi_F(X, A, f)$$

given by $\Pi_F(r, b)(\alpha) = e^{-1}\bar{\alpha}^{-1}e$. To show $\Pi_F(r, b)$ is a morphism of groups suppose $\alpha, \beta \in \Pi_F(Y, B, g)$. Now by the previously mentioned group isomorphism it suffices to show $\alpha\beta = t\bar{\alpha}^{-1}\bar{\beta}^{-1}$ where $\bar{\alpha}$ and $\bar{\beta}$ are determined as above. But β satisfies $\beta t\bar{\beta} = t$ and α satisfies $\alpha t\bar{\alpha} = t$, so $\alpha\beta t\bar{\beta}\bar{\alpha} = t$. It is now a straightforward but messy argument using the uniquenesses available to show Π_F is a functor. Q.E.D.

The group $\Pi_F(X, A, f)$ may be more simply characterized as the group of automorphisms α of $f(X)$ such that $F(\alpha)f_0 = f_0$.

The assignment $F \rightsquigarrow \Pi_F$ can be shown to be a functor between appropriately defined categories. Note that if F is locally right adjunctable, then one gets a covariant functor from $(F, \mathbf{C})$ to groups. We now turn to some examples.

(1) The category of simply connected spaces is locally coreflective in the category of "well connected" spaces, i.e., those spaces which admit simply connected covering spaces: if $f : S \to X$ is a map and $p : \bar{X} \to X$ is a simply connected covering of X, then any lifting of f yields a local coreflection map. Moreover, the associated automorphism group is the deck transformation group, which for such spaces, is of course isomorphic to the fundamental group of X. Notice that the same situation in the corresponding categories of pointed spaces is a genuine coreflection. The base point forgetting functor is a (discrete) opfibration. More generally the dual of Theorem C says that any coreflective subcategory of a category of pointed spaces gives rise to a corresponding locally coreflective subcategory when base points are removed.

It is clear in such cases how the associated automorphism groups arise.

For another such example see [4] or [5].

(2) The category of algebraically closed fields is locally reflective in the category of fields (with unitary homomorphisms). If $f : k \to L$ is an em-

bedding into an algebraically closed field, the local reflection of k with respect to f is the closure of $f(k)$ in L. Then the associated automorphism group in this instance is precisely the group of $f(k)$-automorphisms of the closure of $f(k)$.

Another related example is the subcategory of separably closed fields.

(3) In [2], Joseph D'Atri points out the existence of a pair of functors arising in Chapter II of Artin's book *Geometric algebra* [1] and demonstrates, in effect, that they fail to be an adjoint pair because of the freedoms involved in choosing coordinates for a geometry. We shall show in fact that each of these functors is locally adjunctable. We summarize the situation first.

(a) Artin showed that any (not necessarily commutative) field k yields a Desarguian plane geometry k^2, whose points are pairs of elements of k and whose lines are those subsets of k^2 satisfying linear equations with coefficients in k. On the other hand, he showed that one could associate with any Desarguian plane geometry G a field $\phi(G)$ which could then be used to coordinatize the geometry. This will be used in 3(b). The first construction sets up a functor into a category of Desarguian plane geometries. The objects of this category are pairs of sets (P, L) with an incidence relation which satisfies the axioms for a Desarguian plane geometry (see [1, Chapter II]).

A morphism is an incidence relation and parallelism preserving pair of functions of the respective pairs of sets. (It can be shown such maps are all injective.)

Then the functor from the category of fields to the category of geometries is locally right adjunctable: if $f : k^2 \to G$ is a morphism in the category of geometries, then there exists a field $\phi(G)$ such that there is an isomorphism $f_0 : \phi(G)^2 \to G$–this is merely a coordinatization of G with the origin and coordinate directions received from k^2 via f. Then the morphism $f_0^{-1}f : k^2 \to \phi(G)^2$ is clearly of the form $(f_1)^2$. The automorphism group is trivial here.

It is apparent that other coordinatization procedures may be viewed in a similar way.

(b) Artin has also showed how to associate with any Desarguian plane geometry G, a (not necessarily commutative) field $\phi(G)$ of direction preserving endomorphisms of the group of translations of G (a translation is an automorphism of G with no fixed points). He demonstrated that one may represent any element of this field as an inner automorphism $\theta \cdots \theta^{-1} \cdots \theta$ where θ is an automorphism (in the category of geometries, of course) leaving a prescribed point of G fixed. Now a morphism in the category of geometries $G \to G'$ induces functorially a homomorphism of the respective automorphism groups of G and G', which in turn, by the above representation, induces functorially a field homomorphism $\phi(s) : \phi(G) \to \phi(G')$. This functor ϕ from geometries to fields is locally left adjunctable: if $f : k \to \phi(G)$ in the category of fields, define $f_1 : G_f = \phi(f(k))^2 \to G$ to be the restriction of the coordinatization isomorphism between $\phi(G)^2$ and G with respect to any fixed choice of coordinates. Define $f_0 : k \to \phi(G_f)$ by $f_0 = \phi(f_1)^{-1}f$. The automorphism group in this case is the multiplicative group of non-zero elements of $\phi(G)$.

Similar procedures may be applied to projective geometry and vector spaces, yielding the general linear group.

We should note that the existence of a "best" fibration of a locally adjunctable functor is still somewhat unresolved as is the related problem of the construction of a locally adjunctable functor from minimal data.

In general those constructions realizable as adjoint functors are candidates for generalization by replacing "adjoint" by "local adjoint." For instance local limits may be defined in this way. We plan to explore the above questions further in a later paper.

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