EXTENSIONS OF GROUP THEORETICAL PROPERTIES

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If N is a normal subgroup of the group G, then G is called an extension of N by G/N. If $\mathfrak{f}$ and $\mathfrak{e}$ are any group theoretical properties, the question arises in which cases every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group. In Section 1 we show that for "many" factor inherited properties of finite groups this requirement upon $\mathfrak{f}$ and $\mathfrak{e}$ is equivalent to apparently weaker requirements. We quote the following typical

THEOREM. *If $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited properties of finite groups such that every finite elementary Abelian p-group is an $\mathfrak{f}$-group whenever the cyclic group of order p is an $\mathfrak{f}$-group then every extension of a ι $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group if and only if every product of a normal $\mathfrak{f}$-subgroup and an $\mathfrak{e}$-subgroup is an $\mathfrak{e}$-group* (*see Theorem* 1.3).

If $\mathfrak{e}$ is a saturated property (in the sense of W. Gaschütz) or if $\mathfrak{f}$ is a property of finite soluble groups, then this theorem may be improved; see Theorems 1.5 and 1.8.

In Section 2 we consider group theoretical properties $\mathfrak{f}$ and $\mathfrak{e}$ of Artinian groups such that $\mathfrak{e}$ has the following particular form. Let θ be a class of ordered pairs $(\mathfrak{x}, \mathfrak{y})$ of group theoretical properties $\mathfrak{x}$ and $\mathfrak{y}$. Then a group G is called a hyper-θ-group, if every non-trivial epimorphic image H of G possesses a non-trivial normal subgroup N such that for some pair $(\mathfrak{x}, \mathfrak{y})$ in θ, N is an $\mathfrak{x}$-group and $H/c_H N$ is a $\mathfrak{y}$-group. Let especially the class θ consist of one element only, and let $\mathfrak{u}$ denote the universal property of being a group. If $\mathfrak{a}$ denotes the class of Abelian groups, then hyper-$(\mathfrak{a}, \mathfrak{u})$ is the class of hyperabelian groups; if $\mathfrak{t}$ is the trivial class consisting of 1 only, then hyper-$(\mathfrak{u}, \mathfrak{t})$ is the class of hypercentral groups; if $\mathfrak{z}$ denotes the property of being a cyclic group, then hyper-$(\mathfrak{z}, \mathfrak{u})$ is the class of hypercyclic groups.

If the group G possesses a normal subgroup N and a subgroup U such that $G = NU$ and $N \cap U = 1$, then G is called a splitting extension of N by U. For Artinian and soluble hyper-θ-groups, the above mentioned theorem may be improved as follows.

THEOREM. *If $\mathfrak{f}$ and hyper-θ are factor inherited properties of Artinian and soluble groups, then every extension of an $\mathfrak{f}$-group by a hyper-θ-group is a hyper-θ-group if and only if every splitting extension of an $\mathfrak{f}$-group by a hyper-θ-group is a hyper-θ-group* (*see Theorem* 2.3).

Also the requirements that hyper-θ is a saturated property of Artinian and

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soluble groups and that products of two normal hyper-θ-groups are hyper-θ-groups are considered; see especially Theorem 2.6 and Section 3.

Notation.

$\mathfrak{c}_G N$ = centralizer of the subgroup N of the group G.
$\mathfrak{z}G$ = center of G.
ϕG = Frattini subgroup of G.
$G^{(0)} = G$.
$G^{(i+1)} = G^{(i)} \circ G^{(i)}$ = subgroup generated by the set of all $x^{-1}y^{-1}xy$ where x and y are elements in $G^{(i)}$.
$A \times B$ = direct product of the groups A and B.
factor = epimorphic image of a subgroup.
soluble group = group G with $G^{(i)} = 1$ for almost all i.
Artinian group = group with minimum condition on subgroups.
Noetherian group = group with maximum condition on subgroups.
almost Abelian group = group which possesses an Abelian normal subgroup of finite index.
A finite group is nilpotent if and only if it is hypercentral.
$\mathfrak{w}$-group = group with property $\mathfrak{w}$.
If $\mathfrak{w}$ is a group theoretical property[1], then it is assumed that there exist $\mathfrak{w}$-groups and that isomorphic images of $\mathfrak{w}$-groups are $\mathfrak{w}$-groups.
A group theoretical property $\mathfrak{w}$ is termed factor inherited [subgroup inherited/epimorphism inherited] if every factor [subgroup/epimorphic image] of a $\mathfrak{w}$-group is likewise a $\mathfrak{w}$-group.
A group theoretical property $\mathfrak{w}$ is termed saturated if every group G is a $\mathfrak{w}$-group whenever $G/\phi G$ is a $\mathfrak{w}$-group.

1. If N is a normal subgroup of the group G, then G is an extension of N by G/N. The following lemma contains a simple criterion for the requirement that every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.

LEMMA 1.1. *If $\mathfrak{f}$ is a subgroup inherited and $\mathfrak{e}$ is an epimorphism inherited property of Artinian groups, then every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group if and only if the following two conditions are satisfied:*

(a) *If the normal $\mathfrak{f}$-subgroup N of the group G is contained in ϕG and the quotient group G/N is an $\mathfrak{e}$-group, then G is an $\mathfrak{e}$-group.*

(b) *Every product of a normal $\mathfrak{f}$-subgroup and an $\mathfrak{e}$-subgroup is an $\mathfrak{e}$-group.*

Proof. If every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group, then especially (a) is valid. Now let $G = NU$, where N is a normal $\mathfrak{f}$-subgroup and U is an $\mathfrak{e}$-subgroup of the group G. Since $G/N = UN/N \simeq U/(U \cap N)$ and $\mathfrak{e}$ is epimorphism inherited, (b) is also valid.

[1] A group theoretical property is a non-empty class of groups closed under isomorphisms.

Conversely, let (a) and (b) be satisfied, and let N be a normal $\mathfrak{f}$-subgroup of the group G such that G/N is an $\mathfrak{e}$-group. Since N and G/N are Artinian, it is well known that also G is Artinian. Hence among the subgroups X of G which satisfy $G = NX$, there exists a minimal one U. Let V be any maximal subgroup of U. If the normal subgroup $N \cap U$ of U is not contained in V, then $U = (N \cap U)V$ and therefore $G = NU = N(N \cap U)V = NV$, contradicting the minimality of U. Thus, if there exists a maximal subgroup V of U, then $N \cap U$ is contained in V. It follows that $N \cap U$ is contained in the Frattini subgroup ϕU of U. Since $\mathfrak{f}$ is subgroup inherited, $N \cap U$ is a normal $\mathfrak{f}$-subgroup of U. Since $G/N = UN/N \simeq U/(N \cap U)$, the quotient group $U/(N \cap U)$ is an $\mathfrak{e}$-group, and application of (a) yields that U is an $\mathfrak{e}$-group. Now by applying (b) to $G = NU$ we may show that G is an $\mathfrak{e}$-group, and our assertion is proved.

If N is a normal subgroup and U is a subgroup of the group G such that $G = NU$ and $N \cap U = 1$, then G is a splitting extension of N by $G/N = UN/N \simeq U$. The following proposition will be useful.

PROPOSITION 1.2. *If $\mathfrak{f}$, $\mathfrak{e}$ and $\mathfrak{g}$ are group theoretical properties such that $\mathfrak{f}$ and $\mathfrak{g}$ are subgroup inherited and every finite elementary Abelian p-group is an $\mathfrak{f}$-group whenever the cyclic group of order p is an $\mathfrak{f}$-group, then the following conditions of $\mathfrak{f}$, $\mathfrak{e}$ and $\mathfrak{g}$ are equivalent:*

(I) *Every extension of a finite elementary Abelian p-group with property $\mathfrak{f}$ by an $\mathfrak{e}$-group is a $\mathfrak{g}$-group.*

(II) *Every splitting extension of a finite elementary Abelian p-group with property $\mathfrak{f}$ by an $\mathfrak{e}$-group is a $\mathfrak{g}$-group.*

Proof. Clearly, condition (II) is only a weak form of (I). Now let condition (II) be satisfied, and let N be a finite normal elementary Abelian p-group of the group G with property $\mathfrak{f}$ such that G/N is an $\mathfrak{e}$-group. We may apply a theorem of E. Artin to the extension G of the Abelian group N; see for instance W. Specht, p. 438, Satz 19*. Thus there exists a group H containing G, an Abelian normal subgroup A of H and a subgroup L of G with the following properties:

$$H = AL, \quad 1 = A \cap L; \qquad H = AG, \quad N = A \cap G;$$

A is the direct product of N and a free Abelian group of finite rank.

It follows that $L \simeq H/A = AG/A \simeq G/N$ is an $\mathfrak{e}$-group. The characteristic subgroup A^p of A is a normal subgroup of H with finite quotient group A/A^p, since A is a finitely generated Abelian group. Since N is a non-trivial elementary Abelian p-group with property $\mathfrak{f}$, our hypotheses imply that every finite elementary Abelian p-group is an $\mathfrak{f}$-group. Hence A/A^p is a normal $\mathfrak{f}$-subgroup of H/A^p. Application of Dedekind's Modular Law now yields

$$A \cap A^pL = A^p(A \cap L) = A^p \quad \text{and} \quad A/A^p \cap A^pL/A^p = 1,$$

so that $H/A^p = AL/A^p$ is a splitting extension of the $\mathfrak{f}$-group A/A^p by the $\mathfrak{e}$-

group $A^pL/A^p \simeq L$. Hence by (II), H/A^p and also its subgroup A^pG/A^p are $\mathfrak{g}$-groups. But we have

$$G \cap A^p = G \cap A \cap A^p = N \cap A^p = 1,$$

since N is a finite elementary Abelian p-group and A is the direct product of N and a free Abelian group. Therefore

$$A^pG/A^p \simeq G/(G \cap A^p) = G$$

is a $\mathfrak{g}$-group, and the validity of (I) is established.

THEOREM 1.3. *If $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited properties of finite groups, and if every finite elementary Abelian p-group is an $\mathfrak{f}$-group whenever the cyclic group of order p is an $\mathfrak{f}$-group, then the following properties of $\mathfrak{f}$ and $\mathfrak{e}$ are equivalent:*

(I) *Every product of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

(II) *Every product of a minimal normal subgroup with property $\mathfrak{f}$ and an $\mathfrak{e}$-subgroup is an $\mathfrak{e}$-group.*

Proof. Since $\mathfrak{e}$ is epimorphism inherited, every product of a normal $\mathfrak{f}$-subgroup and an $\mathfrak{e}$-subgroup is also an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, so that (I) implies (II).

Conversely, let (II) be satisfied and assume that (I) is not true. Then there exist extensions of $\mathfrak{f}$-groups by $\mathfrak{e}$-groups which are not $\mathfrak{e}$-groups, and all these groups are finite, since every $\mathfrak{f}$-group and every $\mathfrak{e}$-group is finite. If the group G is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, then every factor of G is also an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, since $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited. Hence among the finite groups which are extensions of $\mathfrak{f}$-groups by $\mathfrak{e}$-groups but not $\mathfrak{e}$-groups, there exists one W of minimal order with the following properties:

(1) W is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group but not an $\mathfrak{e}$-group; every proper factor of W is an $\mathfrak{e}$-group.

Since W is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group without being an $\mathfrak{e}$-group, there exists a normal $\mathfrak{f}$-subgroup $N \neq 1$ of W. Since every normal subgroup of W which is contained in N is likewise an $\mathfrak{f}$-group, we may conclude:

(2) There exists a minimal normal subgroup M of W which is an $\mathfrak{f}$-group.

If U is any maximal subgroup of W, then U is an $\mathfrak{e}$-group by (1). If M would not be contained in U, then $W = MU$ and W would be an $\mathfrak{e}$-group by (II). This contradicts (1), so that M must be contained in U. Thus we have shown:

(3) $M \subseteq \phi W$.

The Frattini subgroup of a finite group is nilpotent; see for instance B. Huppert, p. 270, Satz 3.6. Since the minimal normal subgroup M of W does not contain any proper characteristic subgroups, it follows that:

(4) M is an elementary Abelian p-group.

By (1), W/M is an $\mathfrak{e}$-group, and hence W is an extension of an elementary Abelian p-group with property $\mathfrak{f}$ by an $\mathfrak{e}$-group. Application of (II) and Proposition 1.2 yield that W is an $\mathfrak{e}$-group. This contradiction shows that (II) implies (I), and this proves the equivalence of the two conditions.

A group theoretical property $\mathfrak{w}$ is called *extension inherited*, if every extension of a $\mathfrak{w}$-group by a $\mathfrak{w}$-group is likewise a $\mathfrak{w}$-group. A special case of Theorem 1.3 says that a property $\mathfrak{e}$ of finite groups is extension inherited if and only if every product of a normal $\mathfrak{e}$-subgroup and an $\mathfrak{e}$-subgroup is an $\mathfrak{e}$-group.

Remark 1.4. Let $\mathfrak{e}$ be a subgroup inherited property of finite groups satisfied by all cyclic groups of prime order and which fulfills the following requirement:

(P) Every product of two commuting $\mathfrak{e}$-subgroups is likewise an $\mathfrak{e}$-group.

Then Theorem 1.3 implies that $\mathfrak{e}$ is extension inherited. Since all cyclic groups of prime order are $\mathfrak{e}$-groups, it then follows that all soluble groups and therefore also all cyclic groups are $\mathfrak{e}$-groups. Now the symmetric group of degree n is the product of a cyclic group of order n and a symmetric group of degree $n - 1$, and (P) implies that all finite symmetric groups are $\mathfrak{e}$-groups. Since $\mathfrak{e}$ is subgroup inherited, all finite groups are therefore $\mathfrak{e}$-groups. If $\mathfrak{s}$ denotes the property of being a finite soluble group, then $\mathfrak{s}$ is extension inherited and every finite cyclic group is an $\mathfrak{s}$-group. But if $\mathfrak{s}$ would also satisfy (P), then the above arguments would yield that every finite group is soluble. Thus $\mathfrak{s}$ is an extension inherited property which does not satisfy property (P).

If $\mathfrak{f}$ is a property of Artinian and soluble groups, then Theorem 1.3 may be improved as follows.

THEOREM 1.5. *If $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited group theoretical properties such that every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group satisfies the maximum condition on normal subgroups, and if $\mathfrak{f}$-groups are Artinian and soluble and every finite elementary Abelian p-group is an $\mathfrak{f}$-group whenever the cyclic group of order p is an $\mathfrak{f}$-group, then the following properties of $\mathfrak{f}$ and $\mathfrak{e}$ are equivalent:*

(I) *Every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

(II) *Every splitting extension of a minimal normal subgroup with property $\mathfrak{f}$ by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

Proof. Clearly condition (II) is only a weak form of (I). Conversely, let condition (II) be satisfied and assume that (I) is not true. Then there exist groups which are extensions of $\mathfrak{f}$-groups by $\mathfrak{e}$-groups but not $\mathfrak{e}$-groups. Since $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited, every factor of such a group is likewise an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group. If G is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, then G satisfies the maximum condition on normal subgroups. If G is not an $\mathfrak{e}$-group, then there exists therefore an epimorphic image W of G with the following properties:

(1) W is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group but not an $\mathfrak{e}$-group; every proper epimorphic image of W is an $\mathfrak{e}$-group.

Since W is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group without being an $\mathfrak{e}$-group, there exists a normal $\mathfrak{f}$-subgroup $N \neq 1$ of W. Since $\mathfrak{f}$-groups are Artinian, there exists a minimal normal subgroup of W which is contained in N. Since $\mathfrak{f}$ is subgroup inherited, this minimal normal subgroup is an $\mathfrak{f}$-group. Thus we have shown:

(2) There exists a minimal normal subgroup M of W which is an $\mathfrak{f}$-group.

As an $\mathfrak{f}$-group M is Artinian and soluble and by the minimality of M this implies that M is a finite elementary Abelian p-group. The quotient group W/M is an $\mathfrak{e}$-group by (1), so that W is an extension of a finite elementary Abelian p-group with property $\mathfrak{f}$ by an $\mathfrak{e}$-group. Application of (II) and Proposition 1.2 now yield that W is an $\mathfrak{e}$-group. This contradicts (1), and thus we have proved that the two conditions (I) and (II) are equivalent.

COROLLARY 1.6. *A factor inherited property $\mathfrak{e}$ of finite soluble groups is extension inherited if and only if every splitting extension of an $\mathfrak{e}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

Proof. This follows immediately from Theorem 1.5 if $\mathfrak{e} = \mathfrak{f}$.

Remark 1.7. If the group theoretical property $\mathfrak{f}$ is inherited by factors and Cartesian products, so that $\mathfrak{f}$ is a variety in the sense of Neumann, then a stronger result than Theorem 1.5 may be obtained; see R. Göbel, Satz 3.9. A special case of this theorem says that a factor inherited property $\mathfrak{e}$ of Noetherian groups is extension inherited if and only if every splitting extension of an $\mathfrak{e}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group. This improves Corollary 1.6.

A group theoretical property $\mathfrak{w}$ is called *saturated*, if every group G is a $\mathfrak{w}$-group whenever its Frattini quotient group $G/\phi G$ is a $\mathfrak{w}$-group. If G is a finite group such that $G/\phi G$ is a $\mathfrak{w}$-group, then G is an extension of a nilpotent group by a $\mathfrak{w}$-group, since the Frattini subgroup of a finite group is nilpotent.

THEOREM 1.8. *If $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited properties of Artinian groups such that every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group satisfies the maximum condition on normal subgroups and if $\mathfrak{e}$ is saturated, then the following conditions of $\mathfrak{f}$ and $\mathfrak{e}$ are equivalent:*

(I) *Every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

(II) *Every product of a minimal normal subgroup with property $\mathfrak{f}$ and an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

If $\mathfrak{f}$-groups are soluble, then (I) *is equivalent to*:

(II*) *Every splitting extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group.*

Proof. Since $\mathfrak{e}$ is epimorphism inherited, every product of a normal $\mathfrak{f}$-subgroup and an $\mathfrak{e}$-subgroup is also an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, so that (I) implies (II).

Conversely, let (II) (respectively (II*)) be satisfied and assume that (I) is not true. Then there exist groups which are extensions of $\mathfrak{f}$-groups by $\mathfrak{e}$-

groups without being $\mathfrak{e}$-groups. If G is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, then G is Artinian and the normal subgroups of G satisfy the maximum condition. Every factor of G is also an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, since $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited. Thus among the factors of G there exists one W with the following properties:

(1) W is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group without being an $\mathfrak{e}$-group; every proper subgroup and every proper epimorphic image of W is an $\mathfrak{e}$-group.

Since W is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group without being an $\mathfrak{e}$-group, there exists a normal $\mathfrak{f}$-subgroup $N \neq 1$ of W. Since N is Artinian there exists a minimal normal subgroup M of W which is contained in N. Since every normal subgroup of W contained in N is also an $\mathfrak{f}$-group, we may conclude:

(2) There exists a minimal normal subgroup M of W which is an $\mathfrak{f}$-group.

If U is a maximal subgroup of W, then U is an $\mathfrak{e}$-group by (1). If M were not contained in U, then $W = MU$ and the application of (II) (respectively (II*)) would yield that W is an $\mathfrak{e}$-group. (If $\mathfrak{f}$-groups are soluble, then M is abelian and $M \cap U = 1$, so that W is a splitting extension of M by U.) This is a contradiction, so that M must be contained in U. It follows that:

(3) $M \subseteq \phi W$.

Especially we have $\phi W \neq 1$, so that $G/\phi G$ is an $\mathfrak{e}$-group by (1). This implies that W is an $\mathfrak{e}$-group, since $\mathfrak{e}$ is saturated. Thus we have derived a contradiction to (1), and by that the equivalence of (I) and (II) is proved.

It is well known that in the class of finite groups "many" group theoretical properties are saturated. The following theorem shows that in the class of finite groups subgroup and extension inherited properties are also saturated.

THEOREM 1.9. *In the class of finite groups every subgroup and extension inherited group theoretical property* $\mathfrak{e}$ *is saturated.*

Proof. Let G be a finite group such that $G/\phi G$ is an $\mathfrak{e}$-group. Every prime divisor of the order of ϕG is also a prime divisor of the order of $G/\phi G$; see for instance G. Huppert, p. 270, Satz 3.8. Since $\mathfrak{e}$ is subgroup inherited, for every prime divisor p of the order of ϕG every cyclic group of order p is therefore an $\mathfrak{e}$-group. Since $\mathfrak{e}$ is extension inherited, it follows that every soluble group whose order is only divisible by prime divisors of the order of ϕG is an $\mathfrak{e}$-group. Hence especially ϕG (as a nilpotent group) is an $\mathfrak{e}$-group, and it follows that also the extension G of the $\mathfrak{e}$-group ϕG by the $\mathfrak{e}$-group $G/\phi G$ is an $\mathfrak{e}$-group.

The following theorem generalizes the well known fact that in the class of finite groups nilpotency is a saturated property. If k is a non-negative integer, let $\mathfrak{n}(k)$ denote the class of all groups G such that $G^{(k)}$ is nilpotent. If $\mathfrak{e}$ is any group theoretical property, let $\mathfrak{n}(k) \circ \mathfrak{e}$ be the class of all groups that are extensions of $\mathfrak{n}(k)$-groups by $\mathfrak{e}$-groups. Then we have:

THEOREM 1.10. *In the class of finite groups* $\mathfrak{n}(k) \circ \mathfrak{e}$ *is always saturated.*

Proof. Let G be a finite group such that $G/\phi G$ is a $\mathfrak{n}(k) \circ \mathfrak{e}$-group. Then there exists a normal subgroup N of G containing ϕG such that $N/\phi G$ is a $\mathfrak{n}(k)$-group and G/N is an $\mathfrak{e}$-group. Hence $(N/\phi G)^{(k)} = N^{(k)}\phi G/\phi G$ is nilpotent. Application of a theorem of W. Gaschütz shows the nilpotency of $N^{(k)}\phi G$ and $N^{(k)}$; see for instance B. Huppert, p. 270, Satz 3.5. Therefore G is a $\mathfrak{n}(k) \circ \mathfrak{e}$-group.

Remark 1.11. It can easily be derived from Theorem 1.10 that in the class of finite groups nilpotency is a saturated property. But the symmetric group S_3 of degree 3 contains a nilpotent normal subgroup with nilpotent quotient group without being nilpotent. Hence nilpotency is not extension inherited, and this shows that the converse of Theorem 1.9 is not true. Also Theorem 1.9 cannot be extended to Artinian groups, since Artinian groups need not possess maximal subgroups and thus may be equal to their Frattini subgroup. A simple example is the group $C = C_{p^\infty}$ of Prüfer's type for which $C/\phi C$ is trivial.

2. In the following some results of the first section will be improved for group theoretical properties of a particular form. For this the following definitions are needed.

We denote throughout by θ a class of ordered pairs $(\mathfrak{x}, \mathfrak{y})$ of group theoretical properties $\mathfrak{x}$ and $\mathfrak{y}$. The class of all group theoretical properties $\mathfrak{x}$, for which there exists a group theoretical property $\mathfrak{y}$ such that $(\mathfrak{x}, \mathfrak{y})$ is contained in θ, is called the *first component* θ_1 of θ. Similarly, the class of all group theoretical properties $\mathfrak{y}$, for which there exists a group theoretical property $\mathfrak{x}$ such that $(\mathfrak{x}, \mathfrak{y})$ is contained in θ, is called the *second component* θ_2 of θ. A component θ_i of θ is factor inherited [subgroup inherited], if for every $\mathfrak{x}$ in θ_i factors (subgroups) of $\mathfrak{x}$-groups are likewise $\mathfrak{x}$-groups. We say that θ is factor inherited (subgroup inherited) if both components of θ are factor inherited (subgroup inherited).

The normal subgroup N of the group G is called a *θ-normal-subgroup*, in symbols $N\,\theta\,G$, if there exists a pair $(\mathfrak{x}, \mathfrak{y})$ in θ such that N is an $\mathfrak{x}$-group and the automorphism group induced by G in N, which is essentially the same as $G/\mathfrak{c}_G N$, is a $\mathfrak{y}$-group. The group G is called a *θ-group*, if $G\,\theta\,G$. G is called a *hyper-θ-group*, if every epimorphic image, not 1, of G possesses a θ-normal-subgroup, not 1.

The following facts about hyper-θ-groups have already been mentioned in B. Amberg [1]; see especially p. 103, Lemma 1.3.

LEMMA 2.1 (a) *The direct product of an arbitrary number of hyper-θ-groups is likewise a hyper-θ-group.*

(b) *If the first component of θ is subgroup inherited and if the second com-*

ponent of θ is factor inherited, then for every normal subgroup $N \neq 1$ of the hyper-θ-group G there exists a θ-normal-subgroup E of G such that $1 \subset E \subseteq N$.

(c) *If θ is factor inherited, then every factor of a hyper-θ-group is likewise a hyper-θ-group.*

(d) *If θ is factor inherited, then the Artinian and almost Abelian group G is a hyper-θ-group if and only if its finite subgroups are hyper-θ-groups.*

Proof. (a) is a special case of B. Amberg [1, Lemma 1.2, p. 102]. The proof of (b) is essentially the same as the proof of R. Baer [3, Lemma 3.2, p. 17]. The proof of (c) is essentially the same as the proof of R. Baer [2, Satz 4.4 (a), p. 358]. (d) is identical with B. Amberg [1, Hilfssatz 3.5, p. 112].

The following proposition will be useful.

PROPOSITION 2.2. *If θ is factor inherited, if the minimal normal subgroup M of the Artinian and soluble group G is contained in the Frattini subgroup ϕG of G, and if every proper subgroup of G is a hyper-θ-group, then M is a θ-normal-subgroup of G.*

Proof. As an Artinian and soluble group G is almost Abelian and locally finite; see for instance R. Baer [3, Lemma 3.3, p. 18] or B. Amberg [2]. If G is not finitely generated, then every finitely generated subgroup of G is a finite hyper-θ-group, and this implies that G is likewise a hyper-θ-group; see Lemma 2.1(d). As a minimal normal subgroup of a hyper-θ-group M is a θ-normal-subgroup; see Lemma 2.1(b). If G is finitely generated, then G is even finite and soluble and the minimal normal subgroup M of G is an elementary Abelian p-group. Now a theorem of R. Baer shows the existence of a subgroup S of G with the following properties:

$$M \cap S = 1, \qquad G = S\mathfrak{c}_G M;$$

see for instance B. Huppert, p. 688, Aufgabe 12, or R. Baer [1]. Assume $G = SM$; then the minimality and commutativity of M imply that S is a maximal subgroup of G; see for instance R. Baer [1, Lemma 1, p. 642]. From $M \subseteq \phi G$ we deduce $S \subset G = SM = S$, which is impossible. Hence $SM \subset G$, so that SM is a hyper-θ-group. Since $G = S\mathfrak{c}_G M$, both S and G induce the same group of automorphisms in M, and M is also a minimal normal subgroup of SM, so that M is a θ-normal-subgroup of SM and G; see Lemma 2.1(b).

For Artinian and soluble hyper-θ-groups Theorem 1.5 may be improved as follows.

THEOREM 2.3. *If θ is factor inherited, and if $\mathfrak{f}$ and hyper-θ are properties of Artinian and soluble groups such that $\mathfrak{f}$ is factor inherited, then the following conditions of $\mathfrak{f}$ and hyper-θ are equivalent:*

(I) *Every extension of an $\mathfrak{f}$-group by a hyper-θ-group is a hyper-θ-group.*

(II) *Every splitting extension of a uniquely determined minimal normal subgroup with property $\mathfrak{f}$ by a hyper-θ-group is a hyper-θ-group.*

Proof. Clearly, condition (II) is only a weak form of (I). Conversely, let (II) be satisfied, and assume that (I) is not true. Then there exists a group G which is an extension of an $\mathfrak{f}$-group by a hyper-θ-group without being a hyper-θ-group. As an extension of an Artinian and soluble group by an Artinian and soluble group G is likewise Artinian and soluble. Since θ is factor inherited, also hyper-θ is factor inherited; see Lemma 2.1(c). Since $\mathfrak{f}$ and hyper-θ are inherited by factors, every factor of G is an extension of an $\mathfrak{f}$-group by a hyper-θ-group. As an Artinian and soluble group G is also almost Abelian; see for instance R. Baer [3, Lemma 3.3, p. 18] or B. Amberg [2]. If every finite subgroup of G were a hyper-θ-group, then G would also be a hyper-θ-group; see Lemma 2.1(d). Thus among the finite factors of G which are extensions of $\mathfrak{f}$-groups by hyper-θ-groups without being hyper-θ-groups, there exists one W of minimal order with the following properties:

(1) W is not a hyper-θ-group, but every proper factor of W is a hyper-θ-group.

Since W is an extension of an $\mathfrak{f}$-group by a hyper-θ-group without being a hyper-θ-group, there exists a normal $\mathfrak{f}$-subgroup $N \neq 1$ of W. Since every normal subgroup of W which is contained in N is also an $\mathfrak{f}$-group, it follows that:

(2) There exists a minimal normal subgroup M of W which is an $\mathfrak{f}$-group.

Assume there exists another minimal normal subgroup $L \neq M$ of W. Then we have $L \cap M = 1$, and W is isomorphic to a subgroup of the direct product $W/L \times W/M$. By (1), W/L and W/M and hence also $W/L \times W/M$ and its subgroup W are hyper-θ-groups; see Lemma 2.1 (a) and (c). This contradicts (1), and we have shown:

(3) M is the only minimal normal subgroup of W.

If U is a maximal subgroup of W, then by (1), U is a hyper-θ-group. If M were not contained in U, then $W = MU$ and $M \cap U$ were a normal subgroup of W contained in N. The minimality of M together with $M \nsubseteq U$ imply $M \cap U = 1$, so that W is a splitting extension of M by U. The application of (II) shows that W is a hyper-θ-group, which contradicts (1). Hence M must be contained in U, and we have shown:

(4) $M \subseteq \phi W$.

We may now apply Proposition 2.2, from which it follows that M is a θ-normal-subgroup of W. But since by (1), W/M is a hyper-θ-group, W is a hyper-θ-group. This contradiction finally proves the equivalence of (I) and (II).

For many choices of θ, hyper-θ is a saturated property in the class of finite groups, and we are now going to characterize these properties.

THEOREM 2.4. *If θ is factor inherited, then hyper-θ is a saturated property in the class of finite groups if and only if the following condition is satisfied:*

(E) *If N is a uniquely determined minimal normal subgroup of the finite group G which is contained in ϕG and whose quotient group G/N is a hyper-θ-group, then N is a θ-normal-subgroup of G.*

Proof. If the minimal normal subgroup N of the group G is contained in ϕG and if G/N is a hyper-θ-group, then $G/\phi G$ is also a hyper-θ-group, since hyper-θ is epimorphism inherited. If hyper-θ is a saturated property, then also G is a hyper-θ-group and the minimal normal subgroup N of G is a θ-normal-subgroup; see Lemma 2.1(b). This shows the necessity of condition (E).

Assume there exist finite groups which are not hyper-θ-groups, though their Frattini quotient groups are hyper-θ-groups, and let G be one of minimal order. Since $G/\phi G$ is a hyper-θ-group, we have $\phi G \neq 1$. If $H = G/N \neq 1$ is an epimorphic image of G, then $\phi G N/N \subseteq \phi(G/N)$; see for instance B. Huppert, p. 269, Hilfssatz 3.4(a). Since $G/\phi G$ is a hyper-θ-group and since hyper-θ is epimorphism inherited also $G/N\phi G \simeq (G/N)/(N\phi G/N)$ and $(G/N)/\phi(G/N)$ are hyper-θ-groups. Now the minimality of G implies that $H = G/N$ is a hyper-θ-group, and we have shown:

(1) Every proper epimorphic image of G is a hyper-θ-group.

Assume there exist two different normal subgroups A and B of G. Then $A \cap B = 1$, and G is isomorphic to a subgroup of the direct product $G/A \times G/B$. By (1), G/A and G/B and therefore also $G/A \times G/B$ and its subgroup G are hyper-θ-subgroups; see Lemma 2.1(a) and (c). This contradiction shows:

(2) There exists one and only one minimal normal subgroup M of G.

By (1), G/M is a hyper-θ-group. Since $\phi G \neq 1$, the minimal normal subgroup M of G must be contained in ϕG. Application of (E) shows that M is a θ-normal-subgroup of G. But since G/M is a hyper-θ-group, also G is a hyper-θ-group. This contradiction finally proves that the property hyper-θ must be saturated in the class of finite groups.

Remark 2.5. If the minimal normal subgroup N of the finite group G is contained in ϕG, then N is an elementary Abelian p-group, since the Frattini subgroup of a finite group is nilpotent; see for instance B. Huppert, p. 270, Satz 3.6. Since p divides the order of ϕG, p also divides the order of $G/\phi G$; see for instance B. Huppert, p. 270, Satz 3.8. Therefore, if $G/\phi G$ is a hyper-θ-group and hyper-θ is subgroup inherited, the cyclic group of order p must be a θ-group. This shows that condition (E) of Theorem 2.4 follows from the following two conditions:

(a) If the cyclic group of order p is a θ-group, then every elementary Abelian p-group is a θ-group.

(b) If N is a uniquely determined minimal normal subgroup of the finite group G which is contained in ϕG, if N is a θ-group and G/N is a hyper-θ-group, then the group of automorphisms induced in N by G is a θ_2-group.

THEOREM 2.6. *If θ is factor inherited and if hyper-θ is a property of soluble groups, then the Artinian group G is a hyper-θ-group if and only if the following condition is satisfied:*

(F) *If E is a finitely generated subgroup of G, then $E/\phi E$ is a hyper-θ-group.*

Proof. If G is a hyper-θ-group, then by Lemma 2.1(c), also every factor of G is a hyper-θ-group, so that especially (F) is satisfied.

Conversely, let G satisfy condition (F). If E is any finitely generated subgroup of G, then E possesses maximal subgroups and the Frattini subgroup ϕE of E is a proper characteristic subgroup of E. As a factor of an Artinian group $E/\phi E$ is likewise Artinian, so that $E/\phi E$ is a finitely generated torsion group. But by (F) $E/\phi E$ is a hyper-θ-group and therefore soluble, and this implies that $E/\phi E$ is finite. Since E is finitely generated, ϕE is also finitely generated; see for instance B. Huppert, p. 141, Satz 19.10. By repeated use of these arguments we may construct a properly descending chain of characteristic subgroups E_i of E with the following properties:

$$E_0 = E, \quad E_{i+1} = \phi E_i\,, \quad E_{i+1} \subset E_i\,, \quad E/E_i \text{ is finite.}$$

Since E is Artinian, it follows $E_i = 1$ for almost all i, so that especially E is finite. The Frattini subgroup of the finite group E is nilpotent and hence also soluble; see for instance B. Huppert, p. 270, Satz 3.6. By (F) also $E/\phi E$ is soluble, and this implies the solubility of E. Thus we have shown:

(1) G is locally finite and soluble.

Since G is an Artinian group satisfying (1), it is soluble and almost Abelian; see for instance R. Baer [3, Lemma 3.3, p. 18] or B. Amberg [2]. Assume G is not a hyper-θ-group. If all finite subgroups of G are hyper-θ-groups, then G itself is a hyper-θ-group; see Lemma 2.1(d). Thus there exist finite factors of G which are not hyper-θ-groups. If F is any finite factor of G, then there exists a finitely generated subgroup U of G and a normal subgroup N of U such that $F = Y/N$. By (F), $Y/\phi U$ is a hyper-θ-group and hence also $(U/N)/(N\phi U/N) \simeq U/N\phi U$ is a hyper-θ-group, since hyper-θ is epimorphism inherited. Then also $F/\phi F = (U/N)/\theta(U/N)$ is a hyper-θ-group, since $N\phi U/N \simeq \phi(U/N)$; see for instance B. Huppert, p. 269, Hilfssatz 3.4(a). We have shown that condition (F) is inherited by every finite factor of G. Thus among the finite factors of G that are not hyper-θ-groups there exists one W of minimal order with the following properties:

(2) W is not a hyper-θ-group, but every proper factor of W is a hyper-θ-group.

Since W is not a hyper-θ-group, we have $W \neq 1$, and there exists a minimal normal subgroup $N \neq M$ of W. Then $M \cap N = 1$, and W is isomorphic to a subgroup of the direct product $W/N \times W/M$. By (2), W/N and W/M and therefore also $W/N \times W/M$ and its subgroup W are hyper-θ-groups; see Lemma 2.1(a) and (c). This contradicts (2), and we have shown:

(3) There exists one and only one minimal normal subgroup M of W.

Since $W/\phi W$ is a hyper-ϕ-group, but W is not a hyper-θ-group, we have $\phi W \neq 1$. Thus by (3), M is contained in ϕW. Application of Proposition 2.2 shows that M is a θ-normal-subgroup of W. But since by (2), W/M is a hyper-θ-group, W must be a hyper-θ-group. By this contradiction we finally have shown that every Artinian group satisfying (F) is a hyper-θ-group.

Remark 2.7. There exist factor inherited classes θ of pairs of group theoretical properties for which finite soluble groups G exist that are not hyper-θ-groups, though their Frattini quotient group $G/\phi G$ is a hyper-θ-group; see B. Amberg [1, Bemerkung 3.3, p. 111]. This shows that not every property of finite soluble groups of the form hyper-θ is saturated. It also shows that for the validity of Theorem 2.6 condition (F) must be subgroup inherited.

3. A group theoretical property $\mathfrak{w}$ is called *strongly* (respectively *weakly*) *product inherited*, if products of arbitrarily (respectively finitely) many normal $\mathfrak{w}$-subgroups have likewise the property $\mathfrak{w}$. It is easy to show that for Noetherian groups these two notations coincide and that every factor and extension inherited group theoretical property is always weakly product inherited. However, the converse of the last statement is not true in general, since in the class of finite groups the property $\mathfrak{n}$ of being a nilpotent group is product inherited, but the symmetric group S_3 of degree 3 is an extension of a nilpotent group by a nilpotent group without being nilpotent.

THEOREM 3.1. *If $\mathfrak{f}$ and $\mathfrak{e}$ are group theoretical properties, and if $\mathfrak{f}$ is strongly product inherited and $\mathfrak{e}$ is epimorphism inherited and weakly product inherited, then the class of all extensions of $\mathfrak{f}$-groups by $\mathfrak{e}$-groups is weakly product inherited.*

Proof. Let the group $P = LJ$ be the product of its normal subgroups L and J, which both are extensions of $\mathfrak{f}$-groups by $\mathfrak{e}$-groups. Since $\mathfrak{f}$ is strongly product inherited, the product L^* of all normal $\mathfrak{f}$-subgroups of L is a characteristic $\mathfrak{f}$-subgroup of L and hence a normal $\mathfrak{f}$-subgroup of P. The group L is an extension of a normal $\mathfrak{f}$-subgroup K of L by an $\mathfrak{e}$-group L/K. Since K is contained in L^* and $\mathfrak{e}$ is epimorphism inherited, also L/L^* is an $\mathfrak{e}$-group. Because of similar reasons there exists a normal f-subgroup J^* of P such that J/J^* is an $\mathfrak{e}$-group. Since L^* and J^* are normal $\mathfrak{f}$-subgroups of P, also $F = L^*J^*$ is a normal $\mathfrak{f}$-subgroup of P. Now $L^* \subseteq F \cap L$, and as an epimorphic image of the $\mathfrak{e}$-group L/L^* also $FL/F \simeq L/(F \cap L)$ is an $\mathfrak{e}$-group. In the same way it may be shown that FJ/F is also an $\mathfrak{e}$-group. Therefore $P/F = (FL/F)(FJ/F)$ is an $\mathfrak{e}$-group, since it is the product of two normal $\mathfrak{e}$-subgroups. Hence P is an extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group, and the assertion is proved.

If G is a hyper-θ-group, then the *θ-center* $\mathfrak{Z}_\theta\, G$ of G is the product of all normal θ_1-subgroups X of G such that the group of automorphisms induced by G in X is a θ_2-group. A component θ_i of θ is called *strongly product inherited* if every group theoretical property $\mathfrak{x}$ in θ_i is strongly product inherited. θ is *strongly product inherited* if both components of θ are strongly product inherited.

PROPOSITION 3.2. *If θ is strongly product inherited and if the second component θ_2 of θ is factor inherited, then the θ-center $\mathfrak{Z}_\theta\, G$ of the hyper-θ-group G is a characteristic θ-normal-subgroup of G.*

Proof. Clearly $\mathfrak{Z}_\theta\, G$ is a characteristic subgroup of G, and as a product of normal θ_1-subgroups it is a θ_1-group. If $\theta_2\, G$ denotes the intersection of all normal subgroups X of G such that G/X is a θ_2-group, then $\theta_2\, G$ is a characteristic subgroup of G and $G/\theta_2\, G$ is isomorphic to a subgroup of the Cartesian product of the θ_2-groups G/X. Therefore $G/\theta_2\, G$ is a θ_2-group, since θ_2 is strongly product inherited and subgroup inherited. Since $\mathfrak{Z}_\theta\, G$ is a product of normal subgroups that are centralized by $\theta_2\, G$, it is also centralized by $\theta_2\, G$. But since $G/\theta_2\, G$ is a θ_2-group, the group of automorphisms induced by G in $\mathfrak{Z}_\theta\, G$ is also a θ_2-group, and thus our assertion is proved.

The next theorem gives a sufficient condition that hyper-θ is weakly product inherited.

THEOREM 3.3. *If θ is factor inherited and strongly product inherited, then hyper-θ is weakly product inherited.*

Proof. Since θ is factor inherited, every factor of a hyper-θ-group is likewise a hyper-θ-group; see Lemma 2.1(c). If the group G is the product of two normal hyper-θ-subgroups and if $H \neq 1$ is an epimorphic image of G, then $H = AB$ is likewise the product of two normal hyper-θ-groups A and B. If for example $A = 1$, then $H = B$ is a hyper-θ-group. If $A \neq 1$ and $B \neq 1$, then their θ-centers are also different from 1. As characteristic subgroups of normal subgroups $\mathfrak{Z}_\theta\, A$ and $\mathfrak{Z}_\theta\, B$ are non-trivial normal subgroups of H. If for example $B \cap \mathfrak{Z}_\theta\, A = 1$, then $\mathfrak{Z}_\theta\, G$ is a non-trivial normal θ_1-subgroup of H which is centralized by B, so that H and A induce the same group of automorphisms, a θ_2-group, in $\mathfrak{Z}_\theta\, A$; see Proposition 3.2. In this case we have $1 \subset \mathfrak{Z}_\theta\, A \subseteq \mathfrak{Z}_\theta\, H$. If $B \cap \mathfrak{Z}_\theta\, A \neq 1$, then there exists a non-trivial θ-normal-subgroup of B which is contained in $B \cap \mathfrak{Z}_\theta\, A$; see Lemma 2.1(b). This implies $\mathfrak{Z}_\theta\, B \cap \mathfrak{Z}_\theta\, A \neq 1$, and in this normal θ_1-subgroup of H both A and B induce a θ_2-group of automorphisms. Since θ_2 is product inherited, $H = AB$ induces a θ_2-group of automorphisms in $\mathfrak{Z}_\theta\, A \cap \mathfrak{Z}_\theta\, B$. Thus H possesses a non-trivial θ-normal-subgroup, and G is a hyper-θ-group.

A group G is called hypercentral, if every non-trivial epimorphic image of G possesses a non-trivial center. These are exactly the hyper-θ-groups where $\theta = \{(\mathfrak{u}, \mathfrak{t})\}$ and $\mathfrak{u}$ denotes the universal class of all groups and $\mathfrak{t}$ the trivial class which consists of 1 only.

THEOREM 3.4. *If θ is factor inherited and if every cyclic group is a θ-group, then every product of a hypercentral normal subgroup and a normal hyper-θ-group is a hyper-θ-group.*

Proof. If $H \neq 1$ is an epimorphic image of such a group, then there exist normal subgroups A and B of H with $H = AB$ such that A is hypercentral and B is a hyper-θ-group. If $A = 1$, then $H = B$ is a hyper-θ-group and therefore possesses a non-trivial θ-normal-subgroup. Assume next $A \neq 1$. Since A is hypercentral, $\mathfrak{z}A$ is a non-trivial characteristic subgroup of A and thus a non-trivial normal subgroup of H. If $B \cap \mathfrak{z}A = 1$, then $\mathfrak{z}A$ is centralized by B and A and hence also by $H = AB$, so that $1 \subset \mathfrak{z}A \subseteq \mathfrak{z}H$. Then there exists a non-trivial cyclic normal subgroup of H which is contained in the center of H, and our hypotheses imply that this is a θ-normal-subgroup of H. Assume finally $K = B \cap \mathfrak{z}A \neq 1$. Then K is a non-trivial normal subgroup of the hyper-θ-group B, and this implies the existence of a θ-normal-subgroup Z of B such that $1 \subset Z \subseteq K$; see Lemma 2.1(b). Since Z is centralized by A and normalized by B, it is normalized by $H = AB$, and B and H induce the same group of automorphisms in Z. Thus Z is a non-trivial θ-normal-subgroup of H, and G is a hyper-θ-group.

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