THE AUTOMORPHISMS OF CERTAIN SUBGROUPS OF $PGL_n(V)$

BY
ROBERT SOLAZZI

Let M be an n-dimensional free module over the integral domain $\mathfrak{o}$, $n > 2$. One may define $GL_n(M)$ to be the group of matrices with entries from $\mathfrak{o}$ which have determinant a unit. $SL_n(M)$ is then the subgroup of $GL_n(M)$ of elements of determinant one. Denote by $TL_n(M)$ the group generated by the transvections in $SL_n(M)$. When $\mathfrak{o}$ is a field, the automorphisms of $GL_n(M)$ and $SL_n(M)$ were determined by Dieudonné [3] and Rickart [13].

When $\mathfrak{o}$ is an integral domain not a field, O'Meara [8] found the automorphisms of $GL_n(M)$, $SL_n(M)$, and $TL_n(M)$; the ideas he used were based on a study of double centralizers of involutions in $GL_n(M)$. He did not determine the automorphisms of the corresponding projective groups. Such an investigation would have encountered the familiar problem of distinguishing group theoretically between the projective involutions of the first and second kinds. When $\mathfrak{o}$ is a field, Dieudonné [3] was able to overcome this problem and find the automorphisms of the projective groups because he had available the structure theory of the classical groups over fields; of course no such structure theory exists for arbitrary integral domains. However in [10], O'Meara introduced involution-free methods in the automorphism theory of the general linear group. In this paper we adapt those ideas to the projective linear groups. The old difficulties associated in distinguishing between projective involutions of the first and second kind are avoided.

More specifically, for $n > 2$ let $GL_n(V)$ be the group of all non-singular linear transformations of a vector space V and let $RL(V)$ be its center. Let $^{-}$ denote the natural map of $GL(V)$ onto $GL(V)/RL(V)$. We say an element $\bar{\sigma}$ of $\bar{G}L(V)$ is a projective transvection if it is the image under the $^{-}$ map of a unique transvection σ. We let G be any subgroup of $\bar{G}L(V)$ such that for any line L and any hyperplane H with $L \subset H$ there is a non-trivial projective transvection in G having proper spaces $L \subset H$. We are able to show that an automorphism of G must carry any projective transvection of G into another projective transvection. In other words, an automorphism of G preserves all projective transvections (Theorem 1.14). A standard application of the fundamental theorem of projective geometry then shows that every automorphism of G has one of two familiar forms, $\bar{\Phi}_g$ or $\bar{\Psi}_h$. (Theorem 3.3).

The class of such subgroups G is very extensive. For any $\mathfrak{o}$ module M which is bounded in the sense of Section 4, the integral projective groups $PGL_n(M)$, $PSL_n(M)$, $PTL_n(M)$ satisfy the above condition; so do the projective congruence subgroups $PGL_n(M; \mathfrak{a})$, $PSL_n(M; \mathfrak{a})$, $PTL_n(M; \mathfrak{a})$ over

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any integral domain $\mathfrak{o}$. In Section 4 we discuss the special features of the automorphism theory of these groups. In Section 5 we give necessary and sufficient conditions for $\bar{\Phi}_g$ and $\bar{\psi}_h$ to be automorphisms of $PTL_n(M)$, $PGL_n(M)$, or $PSL_n(M)$, M a free module of finite rank.

0. Preliminaries

Let V be an n-dimensional vector space over the field F; V' denotes the dual space of V. We shall always assume our vector space V has dimension $\geqq 2$. We let "iff" denote "if and only if," and let $\dot{F}$ denote the non-zero elements of F. We let $\subseteq$ denote inclusion and $\subset$ proper inclusion. If a and b are members of a group G, let $[a; b]$ denote $aba^{-1}b^{-1}$ and DG denote the commutator subgroup of G. D^kG, for $k \geqq 1$, is defined by $D^1G = DG$ and $D^kG = D(D^{k-1}G)$.

Recall for $\sigma \in GL_n(V)$, the residual space R and fixed space P of σ are defined by

$$R = (\sigma - 1_V)V, \qquad P = \ker (\sigma - 1_V);$$

clearly $\sigma R = R$, and $\sigma P = P$. And we define res $\sigma = \dim R$; res σ is called the residual index of σ. By a transvection, we mean an element of $SL_n(V)$ whose fixed space contains a hyperplane. By the proper line of a transvection $\tau \neq 1_V$, we mean the residual space of τ; by the proper hyperplane of the transvection $\tau \neq 1_V$ we mean the fixed space of τ. We consider 1_V as a transvection having any line as proper line and having any hyperplane as proper hyperplane.

We denote the scalar transformations (radiations) of $GL_n(V)$ by $RL(V)$ or $RL_n(V)$, i.e., those elements of $GL_n(V)$ of the form $\alpha \cdot 1_V$ for some non-zero α in F. Following are four simple lemmas from [10] we use.

0.1. *Let σ_1 and σ_2 be in $GL(V)$. Put $\sigma = \sigma_1 \sigma_2$. Then $R \subseteq R_1 + R_2$, $P \supseteq P_1 \cap P_2$, res $\sigma_1\sigma_2 \leqq$ res $\sigma_1 +$ res σ_2. And σ_1 and σ_1^{-1} have the same residual and fixed spaces.*

0.2. *Let σ_1, σ_2 and σ be as in* 0.1. *Then*

(1) $V = P_1 + P_2 \Rightarrow R = R_1 + R_2$

(2) $R_1 \cap R_2 = 0 \Rightarrow P = P_1 \cap P_2$.

0.3. *Let σ_1 and σ_2 be non-trivial transvections in $GL_n(V)$. Then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ iff $R_1 \subseteq P_2$ and $R_2 \subseteq P_1$.*

0.4. *Let $\sigma \in GL_n(V)$ have residual space R. Then* $\det \sigma = \det \sigma \mid R$.

The next proposition is $1 \cdot 11$ of [8].

0.5. *Let X be a subgroup of $GL_n(V)$ consisting entirely of transvections. Then all transvections in X have the same proper line or the same proper hyperplane.*

For any $a \in V$ and $\rho \in V'$ such that $\rho(a) = 0$, let

$$\tau_{a,\rho}(x) = x + (\rho x)a,$$

for all $x \in V$. $\tau_{a,\rho}$ is a transvection with hyperplane $\rho^{-1}(0)$ if $\rho \neq 0$ and proper line Fa if $a \neq 0$. Clearly $\tau_{a,\lambda\rho} = \tau_{\lambda a,\rho}$.

0.6. *Let $\sigma \neq 1_V$ be a transvection with spaces $L \subseteq H$. If ρ is a non-zero linear functional for which $\rho H = 0$, then there is an $a \in L$ such that $\sigma = \tau_{a,\rho}$.*

Proof. ρ is given. Pick $\mu \in V - H$ with $\rho\mu = 1$. Put $a = \sigma\mu - \mu \in L$. Then $\tau_{a,\rho}$ and σ agree on H. They also agree on μ since $\tau_{a,\rho}(\mu) = \mu + (\rho\mu)a = \mu + a = \sigma\mu$. Therefore $\tau_{a,\rho}$ and σ agree on all of the space V. Q.E.D.

By an elementary transvection with respect to a base $x_1, \cdots, x_n$ we mean a transvection of the form $\tau_{\lambda x_i,\alpha\rho_j}$ with λ, α in F, $i \neq j$. Here $\rho_1, \cdots, \rho_n$ denotes the dual base to $x_1, \cdots, x_n$. It is well known, and easy to verify, that $[\tau_{\lambda x_i,\rho_j} ; \tau_{\alpha x_j,\rho_k}] = \tau_{\alpha\lambda x_i,\rho_k}$ for distinct i, j, k and for α, λ in F.

Let S be a subgroup of $GL_n(V)$. As in [10], we say S is full of transvections if for any line L and hyperplane H such that $L \subseteq H$, there is a non-trivial transvection in S with spaces L and H. Observe that if S is full of transvections, then the center of S is $RL(V) \cap S$.

0.7. *Let $n \geqq 2$, τ a transvection in $SL_n(V)$. If $\alpha \in RL(V)$, and $\alpha\tau$ is also a transvection, then $\alpha = 1$.*

Proof. Apply 1.9 of [8]. Q.E.D.

Now denote by $^-$ the natural map (projection) of $GL(V)$ onto $GL(V)/RL(V) = PGL(V)$. For any subset $X \subseteq GL(V)$, $\bar{X}$ denotes the image of X in $PGL(V) = GL(V)/RL(V)$. For any $\sigma \in GL(V)$, $\bar{\sigma}$ denotes the coset of σ modulo $RL(V)$. Thus $\bar{G}L(V) = GL(V)/RL(V) = PGL(V)$. So $\bar{G}L(V)$ is the n-dimensional projective general linear group of V, $PGL(V)$.

DEFINITION. Given $\bar{\sigma} \in GL(V)/RL(V)$, we say $\bar{\sigma}$ is a (projective) transvection iff at least one coset representative of $\bar{\sigma}$ is a transvection.

Remark. It follows from 0.7 that if $\bar{\sigma} \in \bar{G}L_n(V)$, $n \geqq 2$, and $\bar{\sigma}$ is a (projective) transvection, then only one coset representative of $\bar{\sigma}$ can be a transvection. Thus if $\bar{\sigma} \in \bar{G}L(V)$ and $\bar{\sigma}$ is a (projective) transvection, we may define the proper line and proper hyperplane of $\bar{\sigma}$ to be the proper line and proper hyperplane of the unique coset representative of $\bar{\sigma}$ that is a transvection.

DEFINITION. Let G be a subgroup of $PGL(V)$. We will say G is *full of projective transvections* if for any line L and any hyperplane H such that $L \subseteq H$, there is a nontrivial projective transvection in G with proper spaces $L \subseteq H$. *In all that follows we will always assume G to be a subgroup of $PGL_n(V)$ that is full of projective transvections. It is our purpose to determine the automorphisms of G when $n > 2$.*

Put $\Delta = \{\sigma \in GL(V) \mid \bar{\sigma} \in G\}$. Then Δ is full of transvections, Δ is a subgroup of $GL(V)$, $RL(V) \subseteq \Delta$, and $\bar{\Delta} = G$. Whenever we refer to the group

Δ in Section 0 through Section 3 inclusive, we will always understand Δ to be defined as above. That is, Δ is the pre-image of G under the $^{-}$ map. We will use $C(X)$ to denote the centralizer in Δ of a non-void subset X of Δ. We use $C(\bar{X})$ to denote the centralizer in G of a non-void subset $\bar{X}$ of G. We use $C_V(X)$ to denote the centralizer of X in $GL_n(V)$.

Given $L \subseteq H$, define $\bar{T}(L, H)$, $\bar{T}(L)$, $\bar{T}(H)$ to be the group of all projective transvections in G having proper spaces $L \subseteq H$, having proper line L, having proper hyperplane H, respectively. These groups are non-trivial since G is full of projective transvections. Given $L \subseteq H$, define $T(L, H)$, $T(L)$, $T(H)$ to be the group of all transvections in $GL(V)$ having proper spaces $L \subseteq H$, having proper line L, having proper hyperplane H respectively. For any two subspaces U and W of V, we define

$$E(U, W) = \{\sigma \in GL_n(V) \mid R \subseteq U, P \supseteq W\}$$

where R and P are the residual and fixed spaces of σ.

0.8. *Let X be a subgroup of $PGL(V)$ consisting entirely of projective transvections. Let $n \geqq 3$. Then all elements of X have the same line or have the same hyperplane.*

Proof. Pull-back and apply 0.5. Q.E.D.

The following proposition is 3.2 of [10].

0.9. *Let σ be a non-involution in $\Delta \cap SL_n(V)$ with R a plane and $R \nsubseteq P$. Then $E(R, P) \cap \Delta \subseteq CDC(\sigma)$.*

DEFINITION. Let $\sigma_1, \sigma_2, \in GL_n(V)$; we say that σ_1 and σ_2 anti-commute iff $\sigma_1 \sigma_2 = r\sigma_2 \sigma_1$, where r is a radiation, $r \in RL_n(V)$, $r \neq 1_V$.

0.10. *Let V have dim n, $\sigma \in GL(V)$, $U \subseteq V$. Let σ act on U as a radiation. If $2 \cdot \dim (U) > n$, then no $\phi \in GL(V)$ can anti-commute with σ.*

Proof. Suppose $\phi\sigma = r\sigma\phi$, $r \in RL(V)$. Then we have $\phi\sigma\phi^{-1} = r\sigma$. We know σ acts on U as a radiation, i.e., $\sigma \mid U = \alpha \cdot 1_V \mid U$; and $\alpha \neq 0$ since $\sigma \in GL(V)$. Thus

$$\phi\sigma\phi^{-1} \mid \phi(U) = \alpha \cdot 1_V \mid \phi(U).$$

Since $\dim U + \dim \phi(U) = 2 \cdot \dim U > n$, there is a non-zero vector x in $U \cap \phi U$. For this x, we have $\alpha \cdot x = \phi\sigma\phi^{-1}(x) = r\sigma x = r\alpha \cdot x$. So $\alpha \cdot x = r\alpha \cdot x$, $x \neq 0$. Thus $\alpha = r\alpha$, or $r = 1_V$. Q.E.D.

0.11. *Let $n = 4$, and let σ be a non-involution in $SL(V)$ of residue 2. Then no $\Sigma \in GL(V)$ can anti-commute with σ.*

Proof. Suppose $\beta\Sigma\sigma\Sigma^{-1} = \sigma$, for some $\beta \in RL(V)$. We claim $P \cap \Sigma P \neq 0$, where P is the fixed space of σ.

For suppose $P \cap \Sigma P = 0$. Then $P \oplus \Sigma P = V$. Now $\sigma \mid P = 1_P$ and $\sigma \mid \Sigma P = \beta$. So σ has a diagonal matrix (α_{ij}) with $\alpha_{11} = 1 = \alpha_{22}$ and $\alpha_{33} = \beta =$

α_{44} in an appropriate base of V. Since det $\sigma = 1$, $\beta^2 = 1$ so $\beta = \pm 1$. But $\beta = \pm 1$ implies σ is an involution, a contradiction. So $P \cap \Sigma P \neq 0$.

Let x be a non-zero vector in $P \cap \Sigma P$. Then $x = \sigma x = \beta \Sigma \sigma \Sigma^{-1}(x) = \beta x$ which implies $\beta = 1$. Q.E.D.

0.12. (2.9, [10]). *Let $n \geqq 2$ and let σ_1 and σ_2 be non-trivial transvections in Δ having proper lines L_1, L_2 and proper hyperplanes H_1, H_2. Then $C(\sigma_1) = C(\sigma_2)$ if and only if $L_1 = L_2$ and $H_1 = H_2$.*

0.13. *Let $\tau_{a,\rho}$ be a non-trivial transvection with spaces $L_1 \subseteq H_1$. Let $\Sigma \in GL(V)$. Let $\chi(F) \neq 2$, and suppose $Fa = L_1 \nsubseteq \Sigma H_1$.* Then Σ does not commute with both $\tau_{a,\rho}\, \Sigma^{-1}\tau_{a,\rho}^{-1}$ and $\tau_{-a,\rho}\, \Sigma^{-1}\tau_{-a,\rho}^{-1}$.

Proof. A straight-forward computation which we omit. Q.E.D.

0.14. *With the same hypotheses as* 0.12, *$C(\bar{\sigma}_1) = C(\bar{\sigma}_2)$ iff $L_1 = L_2$ and $H_1 = H_2$.*

PROOF. By 0.7, $C(\bar{\sigma}_i) = C(\sigma_i)^-$ since σ_1 and σ_2 are transvections. So $C(\bar{\sigma}_1) = C(\bar{\sigma}_2)$ *iff* $C(\sigma_1)^- = C(\sigma_2)^-$ *iff* $C(\sigma_1) = C(\sigma_2)$ *iff* $L_1 = L_2$, $H_1 = H_2$ Q.E.D.

0.15 (3.1, [10]). *Let L be a line, H a hyperplane, and $L \subseteq H$. Suppose $n \geqq 3$, $\sigma \neq 1_V$, and $\sigma \in E(L, H) \cap \Delta$. Then $E(L, H) \cap DC(\sigma) \neq 1_V$.*

Now let $\sigma : V \to W$, where V and W are two vector spaces over the same field F. Define a map ${}^t\sigma : W' \to V'$ as follows:
${}^t\sigma(\rho)(x) = \rho\sigma(x)$, for all $\rho \in W'$, $x \in V$. Then ${}^t\sigma(\rho)$ is an element of V', and ${}^t\sigma$ is linear transformation from W' to V'. We have ${}^t(\sigma + \tau) = {}^t\sigma + {}^t\tau$, ${}^t(\alpha\sigma) = \alpha\, {}^t\sigma$, for scalars α. We have ${}^t\sigma = {}^t\tau$ iff $\sigma = \tau$, and ${}^t(\tau\sigma) = {}^t\sigma\, {}^t\tau$.

For any bijection σ, $\sigma \in GL_n(V)$, we use $\sigma^\vee$ for the contragredient, $({}^t\sigma)^{-1}$. So $\sigma^\vee$ is in $GL_n(V')$. We use X^0 for the annihilator in V' of any subset X of V. Note if $\sigma \in GL(V)$ then $\sigma^\vee$ has fixed space R^0 and residual space P^0.

0.16. *Let $\sigma \in GL(V)$ have residual space R. Then $\sigma^2 = 1_V$ iff $\sigma \mid R = -1_R$.*

Proof. $\sigma^2 = 1_V$ iff $\sigma(\sigma x - x) = -(\sigma x - x)$ for all x in V, iff $\sigma y = -y$ for all y in R. Q.E.D.

0.17. *Let $n \geqq 2$ and S be any subgroup of $GL_n(V)$ that is full of transvections. Then DS contains a σ with* res $\sigma = n$.

Proof. Apply 2.5 of [10]. Q.E.D.

0.18 (2.4, [10]). *If S is any subgroup of $GL_n(V)$, $n \geqq 3$, and S is full of transvections, then DS is full of transvections.*

0.19 (3.4, [10]). *Let $n \geqq 3$ and let S be any subgroup of $GL_n(V)$ that is full of transvections. Let τ be a nontrivial transvection in S with residual and fixed spaces R, P. Then $CC(\tau) \subseteq RL_n(V) \cdot E(R, P)$*

1. Action of an automorphism on projective transvections

DEFINITION. Let $\sigma \in SL_n(V)$. We say σ is a plane rotation if res $\sigma = 2$ and $R \cap P = 0$.

1.1. *Let $n > 3$. Let σ_2 be a non-involution in $\Delta \cap SL(V)$ with* res $\sigma_2 = 2$ *and $R_2 \nsubseteq P_2$. Then $(E(R_2, P_2) \cap \Delta)^- \subseteq CDC(\bar{\sigma}_2)$.*

Proof. Apply the $^-$ map to both sides of 0.9, noting that 0.10 and 0.11 imply $C(\sigma_2)^- = C(\bar{\sigma}_2)$. Thus

$$(E(R_2, P_2) \cap \Delta)^- \subseteq (CDC(\sigma_2))^- \subseteq CD(C(\sigma_2))^- = CDC(\bar{\sigma}_2). \quad \text{Q.E.D.}$$

1.2. *Let $n > 3$. Let σ be a non-trivial transvection in Δ with spaces $L \subseteq H$, Λ an automorphism of G. Then there is a plane rotation $\sigma_2' \in \Delta$ such that $\Lambda\bar{\sigma} \in CDC(\bar{\sigma}_2')$.*

Proof. Let $\bar{\Sigma} = \Lambda\bar{\sigma}$. Then there is a line $Fa = L_1$ in V such that $\Sigma L_1 \neq L_1$; for otherwise $\bar{\Sigma} = \bar{1}_V$ and we know $\bar{\sigma} \neq \bar{1}_V$. Pick a hyperplane H_1 of V such that $L_1 \subseteq H_1$, $\Sigma L_1 \subseteq H_1$, $\Sigma^{-1}L_1 \subseteq H_1$. Let $\tau_{a,\rho}$ be a non-trivial transvection in Δ with spaces $L_1 = Fa$ and $H_1 = \ker \rho$. If $\chi(F) \neq 2$, we can assume Σ and $\tau_{a,\rho}\,\Sigma^{-1}\tau_{a,\rho}^{-1}$ do not commute.

One also sees they cannot anti-commute either, by a dimension argument since $n > 3$. Put $T = \tau_{a,\rho}$; then $\bar{\Sigma}$ and $\bar{T}\bar{\Sigma}^{-1}\bar{T}^{-1}$ don't commute if $\chi(F) \neq 2$. Now let $\tau \in \Delta$ be such that $\Lambda\bar{\tau} = \bar{T}$, and put $\sigma_2 = \sigma\tau\sigma^{-1}\tau^{-1}$, $\sigma_2' = \Sigma T\Sigma^{-1}T^{-1}$. Clearly $\Lambda\sigma_2 = \sigma_2'$. One easily sees that the spaces of the transvections $\Sigma T\Sigma^{-1}$ and T^{-1} are $\Sigma L_1 \subseteq \Sigma H_1$ and $L_1 \subseteq H_1$. But $V = H_1 + \Sigma H_1$ so $R_2' = L_1 + \Sigma L_1$ by 0.2. And we have $P_2' = H_1 \cap \Sigma H_1$ by 0.2; clearly

$$R_2' \cap P_2' = [(L_1 + \Sigma L_1) \cap H_1] \cap \Sigma H_1 = L_1 \cap \Sigma H_1 = 0.$$

Thus σ_2' is a plane rotation. In fact σ_2 will turn out to be such that $\bar{\sigma} \in CDC(\bar{\sigma}_2)$. This will imply $\Lambda\bar{\sigma} \in CDC(\Lambda\bar{\sigma}_2) = CDC(\bar{\sigma}_2')$, which is our desired conclusion. So the rest of this proof is devoted to showing $\bar{\sigma} \in CDC(\bar{\sigma}_2)$.

Now $\Sigma I\Sigma^{-1} = \tau_{\Sigma a,\rho\Sigma^{-1}}$ and $T = \tau_{a,\rho}$. From these formulas it follows that $\Sigma T\Sigma^{-1}$ and T act on R_2'. Therefore $\Sigma T\Sigma^{-1}$ and T induce non-trivial transvections with different residual spaces on $R_2' = L_1 + \Sigma L_1$. So $\sigma_2' \mid R_2' \neq -1_{R_2'}$; so σ_2' is not an involution by 0.16. We also have that since $\sigma_2' \mid P_2' = 1_{P_2'}$, $(\sigma_2')^2 \neq \alpha$, α a radiation. So $(\bar{\sigma}_2')^2 \neq \bar{1}_V$. So σ_2 is not an involution.

Now let us compute the fixed and residual space of σ_2. We know $R_2 \subseteq L + \tau L$ and $P_2 \supseteq H \cap \tau H$, by 0.1. We will show

$$\tau H \neq H, \quad \tau L \neq L, \quad R_2 = L + \tau L, \quad P_2 = H \cap \tau H, \quad R_2 \nsubseteq P_2.$$

If $\chi(F) = 2$ all this follows from the fact that $\sigma_2^2 \neq 1_v$ and that transvections are involutions in characteristic 2. If $\chi(F) \neq 2$, we showed above that $\bar{\Sigma}$ and $\bar{T}\bar{\Sigma}^{-1}\bar{T}^{-1}$ do not commute. Hence, applying Λ^{-1}, $\bar{\sigma}$ and $\bar{\tau}\bar{\sigma}^{-1}\bar{\tau}^{-1}$ do not commute. So σ and $\tau\sigma^{-1}\tau^{-1}$ do not commute. Their spaces are $L \subseteq H$ and $\tau L \subseteq \tau H$ respectively. Hence $L \nsubseteq \tau H$ or $\tau L \nsubseteq H$ by 0.3; in particular $\tau L \neq L$

and $\tau H \neq H$. So $R_2 = L + \tau L$ and $P_2 = \tau H \cap H$ by 0.2. Finally $R_2 = L + \tau L \not\subseteq H \cap \tau H = P_2$. We have now shown that σ_2 satisfies the hypotheses of 1.1, so $\bar{\sigma} \in (E(R_2, P_2) \cap \Delta)^- \subseteq CDC(\bar{\sigma}_2)$ as required. Q.E.D.

1.3. *Let $\sigma \in \Delta$ be a plane rotation with spaces R and P. Let $n > 4$. Then $CD^kC(\bar{\sigma}) \subseteq (GL_2(R) \oplus RL_{n-2}(P))^-$, for any $k \geqq 0$.*

Proof. Let Δ_P be the subgroup of $GL_{n-2}(P)$ defined by

$$\Delta_P = \{\varphi \in GL_{n-2}(P) \mid 1_R \oplus \varphi \in \Delta\}.$$

By definition $1_R \oplus \Delta_P \subseteq \Delta$; and Δ_P is full of transvections since Δ is: just restrict the transvections in Δ to P. Now

$$1_R \oplus D^k\Delta_P \subseteq D^k(1_R \oplus \Delta_P) \subseteq D^kC(\sigma).$$

So

$$(1_R \oplus D^k\Delta_P)^- \subseteq D^kC(\sigma)^- \subseteq D^kC(\bar{\sigma}).$$

Now since Δ_P is full of transvections, and $\dim P \geqq 3$, 0.18 implies $D^k\Delta_P$ is full of transvections.

So for any line L in P, we have a transvection τ in $D^k\Delta_P$ with line L. Then $1_R \oplus \tau$ is a transvection, and

$$(1_R \oplus \tau)^- \in (1_R \oplus D^k\Delta_P)^- \subseteq D^kC(\bar{\sigma}).$$

Now let $\bar{\Sigma} \in CD^kC(\bar{\sigma})$. Then Σ commutes with $1_R \oplus \tau$ by 0.10 and so $\Sigma L = L$. Hence Σ acts on every line of P, and so $\Sigma \mid P = \alpha \cdot 1_P$ for some $\alpha \in \dot{F}$.

Now choose a base $x_1, \cdots, x_{n-2}$ for P, and define the hyperplanes H_i of V by $H_i = R \oplus \hat{x}_i$, where

$$\hat{x}_i = Fx_1 + \cdots + F_{x_{i-1}} + Fx_{i+1} + \cdots + Fx_{n-2}.$$

Then $\bigcap_{i=1}^{n-2} (H_i) = R$. Since $D^k\Delta_P$ is full of transvections, we can choose nontrivial transvections $\tau_i \in D^k\Delta_P$ such that $1_R \oplus \tau_i$ is a transvection with hyperplane H_i. We see

$$(1_R \oplus \tau_i)^- \in (1_R \oplus D^k\Delta_P)^- \subseteq D^kC(\bar{\sigma});$$

so Σ commutes with $1_R \oplus \tau_i$, and so $\Sigma H_i = H_i$. So $\Sigma R = \Sigma \bigcap_i H_i = \bigcap_i \Sigma H_i = R$. Thus Σ acts on R. Q.E.D.

Recall from linear algebra that $\sigma \in GL(V)$ is called unipotent iff $\sigma - 1_V$ is nilpotent. It then follows that σ is unipotent iff there is a base for V in which the matrix of σ is upper triangular with all 1's on the main diagonal iff all characteristic roots of σ are 1.

Propositions 1.4 to 1.6 can now be easily proved.

1.4. *Let $n \geqq 2$. If σ and $\alpha\sigma$ are both unipotent, and $\alpha \in RL(V)$, then $\alpha = 1$.*

1.5. *Let $n \geqq 2$. If σ is unipotent and $\Sigma \in GL(V)$, then $\Sigma\sigma\Sigma^{-1}$ is unipotent.*

1.6. *Let $n \geqq 2$. If σ is unipotent then no Σ in $GL(V)$ can anti-commute with σ.*

We say a co-plane is a subspace of V of codimension 2. Now we have the following theorem, basic to our approach to the automorphism question.

1.7. *Let $n > 4$, and let Λ be an automorphism of G. Let τ be a non-trivial transvection in Δ, and Σ a coset representative of $\Lambda\bar{\tau}$. (i.e., $\bar{\Sigma} = \Lambda\bar{\tau}$). Then there is a plane $R \subseteq V$, and a coplane $P \subseteq V$, such that $\Sigma R = R$, $\Sigma P = P$, $R \cap P = 0$, and $\Sigma \mid P = \alpha \cdot 1_P$ for some $\alpha \in \dot{F}$.*

Proof. Apply 1.2 and 1.3. Q.E.D.

1.8. *Let $n \geqq 2$. Let Σ be an element of Δ such that $\Sigma \in DC(\Sigma)$. Then $\Sigma^{n!}$ is unipotent.*

Proof. Apply 4.1 of [10]. Q.E.D.

DEFINITION. Let $\bar{\Sigma} \in G$. We say $\bar{\Sigma}$ is projectively unipotent if at least one coset representative of $\bar{\Sigma}$ is unipotent. It follows from 1.4 that at most one coset representative of $\bar{\Sigma}$ is unipotent, for any $\bar{\Sigma} \in PGL_n(V)$.

1.9. *Let $n \geqq 2$. Let $\bar{\Sigma} \in \Delta$ be such that $\bar{\Sigma} \in DC(\bar{\Sigma})$. Then $\bar{\Sigma}^{n \cdot n!}$ is projectively unipotent.*

Proof. $\bar{\Sigma} = \prod_{\text{fin}} [\bar{A}_i, \bar{B}_i]$, $\bar{A}_i, \bar{B}_i \in C(\bar{\Sigma})$. Put $\sigma = \prod_{\text{fin}} [A_i, B_i]$. We have $A_i \sigma A_i^{-1} = (\alpha_i 1_V) \cdot \sigma$ for $\alpha_i \in F$. Since $\det \sigma = 1$, $\alpha_i^n = \det(\alpha_i \cdot 1_V) = 1$. So $A_i \sigma^n A_i^{-1} = \sigma^n$. Hence $A_i \in C(\sigma^n)$. Similarly $B_i \in C(\sigma^n)$. So

$$\sigma^n = (\textstyle\prod_{\text{fin}} [A_i, B_i])^n \in DC(\sigma^n).$$

So 1.8 implies $(\sigma^n)^{n!}$ is unipotent. Hence $\bar{\sigma}^{n \cdot n!} = \bar{\Sigma}^{n \cdot n!}$ is projectively unipotent. Q.E.D.

1.10. *Let Λ be an automorphism of G, suppose $n \geqq 3$, and $L \subseteq H$ is given. Then there is a non-trivial projective transvection $\bar{\tau} \in G$ with spaces $L \subseteq H$ such that $\Lambda\bar{\tau}$ is projectively unipotent.*

Proof. First suppose $\chi(F) = 0$ and $n \geqq 3$. Fix a non-trivial transvection τ in $E(L, H) \cap \Delta$. By 0.15 there is a non-trivial transvection Σ in $E(L, H) \cap DC(\tau)$. But $C(\Sigma) = C(\tau)$ by 0.12. So $\Sigma \in DC(\Sigma)$. Hence $\bar{\Sigma} \in DC(\bar{\Sigma})$. Hence $\Lambda\bar{\Sigma} \in DC(\Lambda\bar{\Sigma})$. So 1.9 implies $(\Lambda\bar{\Sigma})^{n \cdot n!} = \Lambda\bar{\Sigma}^{n \cdot n!}$ is projectively unipotent. So $\bar{\Sigma}^{n \cdot n!}$ does the job.

Now suppose $\chi(F) = p > 0$ and $n > 4$. Take a non-trivial transvection σ in $E(L, H) \cap \Delta$; $\sigma^p = 1_V$. So $\Lambda\bar{\sigma}^p = \bar{1}_V$. Put $\bar{\Sigma} = \Lambda\bar{\sigma}$. Then $\Sigma^p = \alpha \cdot 1_V$, $\alpha \in F$. By 1.7, we know Σ is a radiation on some co-plane U of the space V. So $\Sigma \mid U = \beta \cdot 1_U$, $\beta \in F$. So $\beta^p = \alpha$. Hence $\Sigma^p = (\beta \cdot 1_V)^p$, or $(\Sigma - \beta)^p = 0$. Thus $\Sigma \mid \beta$ is unipotent. So $(\Sigma \mid \beta)^{-} = \Lambda\bar{\sigma}$ is projectively unipotent.

Now suppose $p > 0$ and $p \neq 3$ and $n = 3$; or $p > 0$ and $p \neq 2$ and $n = 4$. By 0.18, Δ full of transvections implies $D\Delta$ full of transvections. Thus $(D\Delta)^{-}$ $(= D\bar{\Delta} = DG)$ is full of projective transvections. So choose a non-trivial projective transvection $\bar{\tau} \in DG$ with proper spaces $L \subseteq H$. So $\Lambda\bar{\tau} \in DG$, and $(\Lambda\bar{\tau})^p = \bar{1}_V$. Thus

$$\Lambda\bar{\tau} = \prod\nolimits_{\text{fin}} [\bar{A}_i, \bar{B}_i], \quad \bar{A}_i, \bar{B}_i \in G.$$

Put $\sigma = \prod_{\text{fin}} [A_i, B_i]$; $\bar{\sigma} = \Lambda\bar{\tau}$, so $\bar{\sigma}^p = \bar{1}_V$; hence $\sigma^p = \alpha\cdot 1_V$, $\alpha \in F$. But

$$\alpha^n = \det \alpha\cdot 1_V = (\det \sigma)^p = 1.$$

Hence $\sigma^{pn} = 1_V$. So $(\sigma^n - 1_V)^p = 0$. So $\Lambda\bar{\tau}^n = (\sigma^n)^-$ is projectively unipotent, and surely $\bar{\tau}^n$ is a non-trivial projective transvection with proper spaces $L \subseteq H$. ($\bar{\tau}^n \neq \bar{1}_V$ since $\chi(F)$ does not divide n.)

Finally let $p = 3$ and $n = 3$; or $p = 2$ and $n = 4$. As above, choose a non-trivial projective transvection $\bar{\tau} \in DG$ with spaces $L \subseteq H$. So $\Lambda\bar{\tau} \in DG$ and $(\Lambda\bar{\tau})^p = \bar{1}_V$. Now $\Lambda\bar{\tau} = \prod_{\text{fin}} [\bar{A}_i, \bar{B}_i]$. Put $\sigma = \prod_{\text{fin}} [A_i, B_i]$. Then $\bar{\sigma} = \Lambda\bar{\tau}$, $\bar{\sigma}^p = \bar{1}_V$, or $\sigma^p = \alpha\cdot 1_V$, $\alpha \in F$. But $\alpha^n = \det \alpha\cdot 1_V\, \sigma^p = 1$. Hence $\sigma^{pn} = 1$; $(\sigma - 1)^{pn} = \sigma^{pn} - 1 = 0$, since pn is 8 when $p = 2$ and 9 when $p = 3$. So σ is unipotent, and $\Lambda\bar{\tau} = \bar{\sigma}$ is projectively unipotent. Q.E.D.

1.11. *Let S be any subgroup of $GL_n(V)$ that is full of transvections, with $n \geqq 2$. If σ is a unipotent element of $C_V(DS)$, then $\sigma = 1_V$.*

Proof. Apply 2.8 of [10]. Q.E.D.

1.12. *Let $n > 2$. Let $L \subseteq H$ be given, and let Λ be an automorphism of G. Then there is a non-trivial projective transvection $\bar{\sigma}$ in G having spaces $L \subseteq H$ such that $\Lambda\bar{\sigma}$ is a projective transvection.*

Proof. First assume that $n > 4$. By 1.10 we may choose a non-trivial transvection σ in $E(L, H) \cap \Delta$ such that $\Lambda\bar{\sigma}$ is projectively unipotent. Let $\bar{\Sigma} = \Lambda\bar{\sigma}$, Σ being unipotent. By 1.7, there is a plane R and coplane P such that $\Sigma P = P$, $\Sigma R = R$, $R \cap P = 0$, and $\Sigma \mid P = \alpha\cdot 1_P$, $\alpha \in F$. Since $\Sigma \mid P$ is unipotent, $\alpha = 1$. Now $\Sigma \mid R$ is unipotent, and is therefore a transvection. So Σ is a transvection.

Now let $n = 3$. By 1.10 we may again choose a non-trivial transvection σ in $E(L, H) \cap \Delta$ such that $\Lambda\bar{\sigma}$ is projectively unipotent. Let $\bar{\Sigma} = \Lambda\bar{\sigma}$, Σ being unipotent; we may assume Σ is not a transvection so the residual space R of Σ is a plane. By 0.15 there is a non-trivial transvection τ in $E(L, H) \cap DC(\sigma)$. We have $\bar{\tau} \in DC(\bar{\sigma})$ so $\Lambda\bar{\tau} \in DC(\bar{\Sigma})$ or $\Lambda\bar{\tau} = \prod_{\text{fin}} [\bar{A}_i, \bar{B}_i]$ where $\bar{A}_i, \bar{B}_i \in C(\bar{\Sigma})$ and in fact $A_i, B_i \in C(\Sigma)$ by 1.6. Put $f = \prod_{\text{fin}} [A_i, B_i]$; then $f \in DC(\Sigma)$. It is clear that $\Sigma R = R$, $A_i(R) = R$, $B_i(R) = R$, and that $\Sigma \mid R \in SL_2(R) - RL_2(R)$. We know $f \in DC(\Sigma)$; therefore $fR = R$ and we see $f \mid R \in DC_R(\Sigma \mid R)$, where C_R denotes $C_{GL(R)}$. By 2.6 of [10], $DC_R(\Sigma \mid R) = 1_R$. Thus $f \mid R = 1_R$ and $\operatorname{res} f = 1$. But $\det f = 1$ since $f \in DC(\Sigma)$ and thus f is a transvection and $\Lambda\bar{\tau} = \bar{f}$ is a projective transvection.

Finally let $n = 4$. Using 1.10 we can choose a nontrivial transvection $\tau \in \Delta$ with spaces $L \subseteq H$ such that $\Lambda\bar{\tau}$ is projectively unipotent. By 1.2 there is a plane rotation $\sigma \in \Delta$ such that $\Lambda\bar{\tau} \in CDC(\bar{\sigma})$. We may write $\bar{\Sigma} = \Lambda\bar{\tau}$, Σ unipotent. Let σ have spaces R, P both planes and $R \oplus P = V$. Let Δ_P be the subgroup of $GL_2(P)$ defined by

$$\Delta_P = \{\varphi \in GL_2(P) \mid 1_R \oplus \varphi \in \Delta\}.$$

It is clear $1_R \oplus \Delta_P \subseteq \Delta$ and that Δ_P is full of transvection in $GL_2(P)$. (Just restrict the transvections in Δ to P.) Also

$$1_R \oplus D\Delta_P \subseteq D(1_R \oplus \Delta_P) \subseteq DC(\sigma).$$

So $(1_R \oplus D\Delta_P)^- \subseteq DC(\sigma)^- \subseteq DC(\bar{\sigma})$. Now $D\Delta_P$ contains an element φ of residual index 2 by 0.17. Hence $(1_R \oplus \varphi)^- \subseteq DC(\bar{\sigma})$. So $\bar{\Sigma}$ and $(1_R \oplus \varphi)^-$ commute. Since Σ is unipotent, Σ and $1_R \oplus \varphi$ cannot anti-commute by 1.6. So Σ and $1_R \oplus \varphi$ commute. R is the fixed space of $1_R \oplus \varphi$ and P is its residual space. So $\Sigma R = R$, $\Sigma P = P$. Since

$$(1_R \oplus D\Delta_P)^- \subseteq DC(\bar{\sigma}),$$

we have $\bar{\Sigma} \in C(1_R \oplus D\Delta_P)^-$. Since Σ is unipotent, 1.6 implies $\Sigma \in C(1_R \oplus D\Delta_P)$. Since $\Sigma R = R$ and $\Sigma P = P$, $\Sigma \mid P \in C_P(D\Delta_P)$. But $\Sigma \mid P$ is unipotent so $\Sigma \mid P = 1_P$ by 1.11. Now $\Sigma \mid R$ is a transvection since it is unipotent and $\dim R = 2$. Since $V = R \oplus P$, Σ is a transvection. Q.E.D.

1.13. *Let $n \geqq 3$ G full of projective transvections and $\bar{\tau}$ a non-trivial projective transvection in G with proper spaces $L \subseteq H$. Then $CC(\bar{\tau}) = \bar{T}(L, H)$.*

Proof. Write $\bar{\tau}$ so that τ is a transvection, $\tau \in \Delta$. First we show $CC(\bar{\tau}) \subseteq CC(\tau)^-$. Let $\bar{\Sigma} \in CC(\bar{\tau})$ and L_1 a line in H. Choose a hyperplane H_1 containing L and L_1, and a transvection σ_1 in Δ with proper spaces $L_1 \subseteq H_1$. By 0.3, $\sigma_1 \in C(\tau)$ and so $\bar{\sigma}_1 \in C(\bar{\tau})$. Thus $\bar{\Sigma}$ and $\bar{\sigma}_1$ commute and 0.10 implies Σ and σ_1 commute. Thus $\Sigma L_1 = L_1$ and Σ stabilizes all lines of H. So $\Sigma \mid H = \alpha$, $\alpha \in RL(H)$. Let $\sigma \in C(\tau)$. It is enough to show Σ and σ commute, since $\bar{\Sigma} \in CC(\bar{\tau})$, $\bar{\Sigma}$ and $\bar{\sigma}$ commute. But $\Sigma \mid H = \alpha$; so 0.10 implies Σ and σ commute. Thus $CC(\bar{\tau}) \subseteq CC(\tau)^-$. But by 0.19 $CC(\tau)^- \subseteq \bar{T}(L, H)$. So $CC(\bar{\tau}) \subseteq \bar{T}(L, H)$.

Now we prove the reverse inclusion; let $\bar{\Sigma} \in \bar{T}(L, H)$. We can assume Σ is a non-trivial transvection with proper spaces $L \subseteq H$. By 0.14, $C(\bar{\Sigma}) = C(\bar{\tau})$. So $\bar{\Sigma} \in CC(\bar{\Sigma}) = CC(\bar{\tau})$. Thus $\bar{T}(L, H) \subseteq CC(\bar{\tau})$ and $CC(\bar{\tau}) = \bar{T}(L, H)$. Q.E.D.

1.14. *Let $n > 2$, and let Λ be an automorphism of G. Let $\bar{\tau} \in G$ be any non-trivial projective transvection. Then $\Lambda\bar{\tau}$ is also a projective transvection.*

Proof. Using 1.12 and 0.14 we see that there is a projective transvection $\bar{\sigma} \in G$ such that $C(\bar{\sigma}) = C(\bar{\tau})$ and such that $\Lambda\bar{\sigma}$ is a projective transvection. Then $\bar{\tau} \in CC(\bar{\tau}) = CC(\bar{\sigma})$. Hence $\Lambda\bar{\tau} \in CC(\Lambda\bar{\sigma})$ and by 1.13 every element of $CC(\Lambda\bar{\sigma})$ is a projective transvection. Q.E.D.

2. The automorphisms $\bar{\phi}_g$ and $\bar{\psi}_h$

Let g be a semi-linear isomorphism of V onto V; for $\sigma \in GL_n(V)$ let $\phi_g(\sigma) = g\sigma g^{-1}$. Let h be a semi-linear isomorphism of V onto its dual space

V'; for $\sigma \in GL_n(V)$ let $\psi_h(\sigma) = h^{-1}\sigma^{\vee}h$, where $\sigma^{\vee}$ denotes the contragredient of σ. One sees ϕ_g and ψ_h are automorphisms of $GL_n(V)$. We have $\phi_g \circ \psi_h = \psi_{hg^{-1}}$; and $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1g_2}$. Note $\phi_{g_{-1}} = \phi_{g^{-1}}$ and $\psi_h^{-1} = \psi_{h^t}$ where h^t is the transpose of h [4, p. 13].

Since ϕ_g and ψ_h act on the center of $GL_n(V)$, we may define in a natural way automorphisms $\bar{\phi}_g$ and $\bar{\psi}_h$ of $PGL_n(V)$. The defining equation of $\bar{\phi}_g$ is $\bar{\phi}_g(\bar{\sigma}) = \phi_g(\sigma)^-$. Similarly for $\bar{\psi}_h$. Observe $\bar{\phi}_g^{-1} = \bar{\phi}_{g^{-1}}$ and $\bar{\psi}_h^{-1} = \bar{\psi}_{h^t}$.

We shall need to know the residual spaces and fixed spaces of the elements $\phi_g(\sigma)$, $\sigma^{\vee}$, and $\psi_h(\sigma)$. These are (gR, gP), (P^0, R^0) and $(h^{-1}P^0, h^{-1}R^0)$ respectively, where (R, P) are the residual and fixed spaces of σ. In Section 2, S denotes any subgroup of $PGL_n(V)$ that is full of transvections.

2.1. *Let S be full of transvections, $n \geqq 3$. Then the equation $\phi_g(\sigma)^- = \psi_h(\sigma)^-$ for all $\sigma \in S$ is impossible.*

Proof. Take non-trivial transvections τ and τ' in S having the same line but different hyperplanes. Then $\phi_g(\tau)$ and $\phi_g(\tau')$ have the same line, while $\psi_h(\tau)$ and $\psi_h(\tau')$ do not. But now $\phi_g(\tau) = \alpha\psi_h(\tau)$ and $\phi_g(\tau') = \beta\psi_h(\tau')$, where $\alpha, \beta \in RL(V)$. Therefore $\alpha = 1 = \beta$, by 0.7, since $n \geqq 2$ (τ and $\alpha\tau$ cannot both be transvections when $n \geqq 2$, unless $\alpha = 1$, by 0.7).

But then $\psi_h(\tau)$ and $\psi_h(\tau')$ have the same line, a contradiction. So the equation $\phi_g(\sigma)^- = \psi_h(\sigma)^-$ for all $\sigma \in S$ is indeed impossible. Q.E.D.

2.2. *Suppose that $n \geqq 2$. Then the following are equivalent*

(1) $\phi_{g_1}(\sigma)^- = \phi_{g_2}(\sigma)^-$ *for all* $\sigma \in S$

(2) $\phi_{g_1}(\sigma) = \phi_{g_2}(\sigma)$ *for all* $\sigma \in S$

(3) $g_1 = rg_2$ *for some* $r \in RL_n(V)$.

Proof. First we prove (1) $\Rightarrow$ (3). Consider any line L in V. Since S is full of transvections, there is a non-trivial transvection τ in S with proper line L. Then $\phi_{g_1}(\tau)^- = \phi_{g_2}(\tau)^-$, so $\phi_{g_1}(\tau) = \alpha\phi_{g_2}(\tau)$, $\alpha \in RL(V)$. But $\phi_{g_1}(\tau)$ and $\phi_{g_2}(\tau)$ are transvections. Thus $\phi_{g_2}(\tau)$ is a transvection and $\alpha\phi_{g_2}(\tau)$ is also a transvection; thus $\alpha = 1$. So

$$\tau = \phi_{g_2^{-1}}\phi_{g_1}(\tau) = \phi_{g_2^{-1}g_1}(\tau).$$

So $g_2^{-1} \circ g_1 = r$, $r \in RL(V)$. Thus $g_1 = g_2 \cdot r = r^{\mu} \cdot g_2$, μ the field automorphism of g_2. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. Q.E.D.

2.3. *Suppose that $n \geqq 2$. Then the following are equivalent.*

(1) $\psi_{h_1}(\sigma)^- = \psi_{h_2}(\sigma)^-$ *for all* $\sigma \in S$

(2) $\psi_{h_1}(\sigma) = \psi_{h_2}(\sigma)$ *for all* $\sigma \in S$

(3) $h_1 = h_2 r$ *for some* $r \in RL_n(V)$.

Proof. First we prove (1) $\Rightarrow$ (3). Put $g = h_2^{-1}h_1$. Then g is a semi-linear isomorphism of V onto V, and $\phi_g \circ \psi_{h_1} = \psi_{h_2}$.

Thus for $\sigma \in S$,

$$(\phi_{1_V}(\sigma))^- = \bar{\psi}_{h_2} \circ \bar{\psi}_{h_1}^{-1}(\bar{\sigma}) = \bar{\psi}_{h_2} \circ \psi_{h_1}^{-1}(\sigma)^- = \phi_g(\sigma)^-.$$

And so by 2.2, $g = r$, $r \in RL(V)$. Thus $h_1 = h_2 r$, $r \in RL(V)$. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. Q.E.D.

2.4. *Let S be any subgroup of $GL_n(V)$ that is full of transvections, $n \geqq 2$. Let $\bar{\phi}_g$ be an automorphism of $\bar{S}$. Suppose $\sigma \in DS$. Then $\phi_g(\sigma) \in DS$ and $\phi_{g^{-1}}(\sigma) \in DS$.*

Proof. Let σ_1 be any element of S. $\phi_g(\bar{\sigma}_1) = \bar{\tau}$, where we may assume $\tau \in S$. Thus $\phi_g(\sigma_1)^- = \bar{\tau}$, so $\phi_g(\sigma_1) = \alpha\tau$, $\alpha \in RL(V)$. Let

$$\sigma = \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1}, \quad \sigma_1, \sigma_2 \in S.$$

By the above, $\phi_g(\sigma_1) = \alpha\tau$, $\tau \in S$ and $\alpha \in RL(V)$. Also $\phi_g(\sigma_2) = \beta h$, $h \in S$ and $\beta \in RL(V)$. Thus

$$\begin{aligned}\phi_g(\sigma) = \phi_g(\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1}) &= \phi_g(\sigma_1)\phi_g(\sigma_2)[\phi_g(\sigma_1)]^{-1}[\phi_g(\sigma_2)]^{-1} \\ &= \alpha\tau\beta h \tau^{-1}\alpha^{-1}h^{-1}\beta^{-1} = \tau h \tau^{-1} h^{-1} \in DS.\end{aligned}$$

So we showed $\phi_g(\sigma) \in DS$ for all short commutators $\sigma \in DS$. Hence $\phi_g(\sigma) \in DS$ for all $\sigma \in DS$.

Now since $\bar{\phi}_g$ is an automorphism of $\bar{S}$, so is $\bar{\phi}_g^{-1} = \bar{\phi}_{g^{-1}}$. Thus what we have proved above shows $\phi_{g^{-1}}(\sigma) \in DS$ if $\sigma \in DS$. Q.E.D.

3. Determination of the automorphisms of any subgroup G full of projective transvections

3.1. *Let Λ be an isomorphism of G into $PGL_n(V)$ such that $\Lambda(\bar{T}(L)) \subseteq T(L)^-$ for all lines L in V. Suppose $n \geqq 2$. Then Λ equals the identity map on G.*

Proof. Take a typical $\sigma \in \Delta$ and a line L in V. Pick τ in $T(L) \cap \Delta$, $\tau \neq 1_V$. So $\bar{\tau}$ is a projective transvection with line L. Hence $\Lambda\bar{\tau}$ is a projective transvection with line L. But $\bar{\sigma}\bar{\tau}\bar{\sigma}^{-1}$ is a projective transvection with line σL. Let $\bar{\sigma}' = \Lambda\bar{\sigma}$. Then $\bar{\sigma}'\Lambda\bar{\tau}\bar{\sigma}'^{-1} = \Lambda(\bar{\sigma}\bar{\tau}\bar{\sigma}^{-1})$ is a projective transvection with line σL. But one also sees $\bar{\sigma}'\Lambda\bar{\tau}\bar{\sigma}'^{-1}$ is a projective transvection with line $\sigma' L$. Hence $\sigma L = \sigma' L$ for all lines L. So there is a scalar $\alpha \in RL(V)$ such that $\sigma = \alpha\sigma'$. Hence $\bar{\sigma} = \bar{\sigma}' = \Lambda\bar{\sigma}$ for all $\sigma \in \Delta$. Q.E.D.

Let $L_* =$ lines of V, $H_* =$ set of hyperplanes of V.

3.2. *Let V be an n-dimensional vector space over F, $n > 2$. Let G be a subgroup of $PGL_n(V)$ that is full of projective transvections, and let Λ be an automorphism of G. Then for all $X \in L_* \cup H_*$, there exists a unique $X' \in L_* \cup H_*$ such that $\Lambda\bar{T}(X) = \bar{T}(X')$.*

And the map $X \to X'$ is a bijection of $L_* \cup H_*$ onto $L_* \cup H_*$ such that:

(i) ($X \subseteq Y$ or $Y \subseteq X$) *iff* ($X' \subseteq Y'$ *or* $Y' \subseteq X'$)
(ii) $L'_* = L_*$ *and* $H'_* = H_*$; *or*, $L'_* = H_*$ *and* $H'_* = L_*$.

Proof. Let $X \in L_* \cup H_*$. By 1.14 every element of $\Lambda\bar{T}(X)$ is a transvection. Since $\bar{T}(X)$ is a maximal group of transvections in G, $\Lambda\bar{T}(X)$ is a maximal group of transvections in G. Since the groups $\bar{T}(X)$, where $X \in L_* \cup H_*$, are the only maximal groups of transvections in G, there will exist a unique $X' \in L_* \cup H_*$ such that $\Lambda\bar{T}(X) = \bar{T}(X')$. The same reasoning that showed X' is unique will show the map $X \to X'$ is 1-1. By considering Λ^{-1} one easily sees $X \to X'$ is surjective. So we have shown that $X \to X'$ is a bijection of $L_* \cup H_*$ onto $L_* \cup H_*$. The defining equation of the bijection $X \to X'$ is $\Lambda\bar{T}(X) = \bar{T}(X')$. Observe also $\bar{T}(X) \cap \bar{T}(Y) \neq \bar{I}_V$ iff $X \subseteq Y$ or $Y \subseteq X$, and therefore

$$(X \subseteq Y \text{ or } Y \subseteq X) \quad \text{iff} \quad (X' \subseteq Y' \text{ or } Y' \subseteq X').$$

So we have proved assertion (i).

Let us show either $L'_* = L_*$ and $H'_* = H_*$; or, $L'_* = H_*$ and $H'_* = L_*$. Suppose $L'_0 \in L_*$ for some line $L_0 \in L_*$. Let L be any line of V, $L \neq L_0$. Pick a hyperplane H containing L_0 and L. Then $L_0 \subset H$ and hence $L'_0 \subset H'$ by the above assertion (i); hence H' is a hyperplane, and hence $L' \subset H'$, again by (i). So L' is a line, and so $L'_* \subseteq L_*$. Now let H be a typical element of H^*. Fix a line L in H. Then since $L \subset H$ and L' is a line, the above assertion (i) shows $L' \subset H'$ and so H' is a hyperplane. Hence $L'_* \subseteq L_*$ and $H'_* \subseteq H_*$, and therefore $L'_* = L_*$ and $H'_* = H_*$. If $L' \in H_*$ for all $L \in L_*$, we show $L'_* = H_*$ and $H'_* = L_*$ in a similar way. Q.E.D.

Remark. Given an automorphism Λ of G, we shall refer to the bijection $X \to X'$ defined above in 3.2 as the bijection induced by Λ.

Now suppose Λ is an automorphism of G. By 3.2 Λ induces a bijection $X \to X'$ of $L_* \cup H_*$ onto $L_* \cup H_*$ such that $L'_* = L_*$ and $H'_* = H_*$; or, such that $L'_* = H_*$ and $H'_* = L_*$. We will use the same symbol $\bar{\Phi}_g$ for the map $\bar{\Phi}_g$ having domain all of $PGL_n(V)$ and also for the restriction of $\bar{\Phi}_g$ to G. We do the same thing with $\bar{\Psi}_h$. Of course we do not know a priori that $\bar{\Phi}_g(G) = G$ or that $\bar{\Psi}_h(G) = G$.

3.3. *Theorem. Let V be an n-dimensional vector space over the field F, $n > 2$. Let G be a subgroup of $PGL_n(V)$ full of projective transvections, and let Λ be an automorphism of G. Then Λ may be expressed in one and only one of the following two ways*:

(i) $\Lambda = \bar{\Phi}_g$, *for some semi-linear isomorphism g of V onto V*,
(ii) $\Lambda = \bar{\Psi}_h$, *for some semi-linear isomorphism h of V onto V'*.

Proof. By 2.1 it is clear that we cannot have both $\Lambda = \bar{\Phi}_g$ for some g, and also $\Lambda = \bar{\Psi}_h$, for some h. So the possibilities (i) and (ii) above are mutually exclusive.

To show that one of (i) and (ii) must hold, consider the bijection $X \to X'$ that Λ induces by 3.2. Let us suppose $L'_* = L_*$ and $H'_* = H_*$. Using 3.2 (i), we can deduce that $L \subset H$ implies $L' \subset H'$. Thus the Fundamental Theorem of Projective Geometry yields a semi-linear isomorphism g of V onto V'

such that $gL = L'$ for all lines $L \subset V$. Then

$$\bar{\Phi}_g^{-1} \circ \Lambda \bar{T}(L) \subseteq T(L)^-;$$

it follows from 3.1 that $\Lambda = \bar{\Phi}_g$, or $\Lambda^{-1} = \bar{\Phi}_{g^{-1}}$.

Now suppose $L'_* = H_*$ and $H'_* = L_*$. Using 3.2 (i) again, we can deduce $L \subset H$ implies $H' \subset L'$. Define a map $L \rightarrow (L')^0$, $(L')^0$ being the annihilator in V' of L'. The map $L \rightarrow (L')^0$ is a bijection of the lines of V onto those of V', which satisfies the hypotheses of the Fundamental Theorem of Projective Geometry. So we obtain a semi-linear isomorphism h of V onto V' such that $h(L) = (L')^0$. Then

$$\bar{\Psi}_h \circ \Lambda (\bar{T}(L) \subseteq T(L)^-;$$

so by 3.1 we have $\bar{\Psi}_h \circ \Lambda = \mathrm{id}\,(\bar{\Delta})$. So from this equation it follows that $\Lambda^{-1} = \bar{\Psi}_h$. We have proved the theorem for Λ^{-1}, so we have therefore proved it for any Λ. Q.E.D.

Remark. It follows from 2.2 and 2.3 that the semi-linear isomorphism g and h in the statement of 3.3 are uniquely determined by the automorphism Λ of G up to a scalar factor.

The following proposition is an immediate corollary to 3.3.

3.4. *Let $n > 2$ and let S be any subgroup of $GL_n(V)$ that is full of transvections. Let Λ be an automorphism of S. Then Λ may be expressed in one and only one of the following two ways*:

(i) $\Lambda\sigma = \chi(\sigma)\cdot\phi_g(\sigma)$ *for all* $\sigma \in S$,

(ii) $\Lambda\sigma = \chi(\sigma)\cdot\psi_h(\sigma)$ *for all* $\sigma \in S$,

where χ denotes a homomorphism of S into $RL(V)$.

4. The automorphism theory of $PGL_n(M)$, $PSL_n(M)$, $PTL_n(M)$, and their congruence subgroups

By a fractional ideal $\mathfrak{a}$ of an integral domain $\mathfrak{o}$ we mean a non-zero subset $\mathfrak{a}$ of $\mathfrak{o} \div \mathfrak{o}$ which is an $\mathfrak{o}$-module in the natural way and which has the property $\lambda\mathfrak{a} \subseteq \mathfrak{o}$ for some non-zero λ in $\mathfrak{o}$. We let F denote the quotient field, $\mathfrak{o} \div \mathfrak{o}$, of $\mathfrak{o}$. Recall any two bases of a free module over a commutative ring have the same cardinality and that this cardinality is called the dimension of the free module.

By a *lattice* M over the integral domain $\mathfrak{o}$ we mean an $\mathfrak{o}$-module which can be written $M = M_1 \oplus \cdots \oplus M_r$ in which each M_i is isomorphic as an $\mathfrak{o}$-module to some invertible fractional ideal $\mathfrak{a}_i$ of $\mathfrak{o}$. We say an $\mathfrak{o}$-module M is *bounded* if there is an $\mathfrak{o}$-linear isomorphism of M into a free module of finite dimension. So any submodule of a bounded module is bounded. For any non-zero module M it is easy to verify the following implications:

$$M \text{ is free of finite dimension} \Rightarrow M \text{ is a lattice}$$
$$\Rightarrow M \text{ is bounded.}$$

In all of Sections 4 and 5 we assume M is a bounded $\mathfrak{o}$-module contained in an n-dimensional vector space V over the quotient field F of $\mathfrak{o}$. And we also assume M spans V over F; i.e., $FM = V$ where FM is $\{\Sigma\alpha x \mid \alpha \in F,\ x \in M.\}$

We define

$$\begin{aligned} GL_n(M) &= \{\sigma \in GL_n(V) \mid \sigma M = M\}, \\ SL_n(M) &= GL_n(M) \cap SL_n(V), \\ TL_n(M) &= \text{the subgroup of } SL(M) \text{ generated by all} \\ &\quad\ \text{transvections } \tau \text{ in } SL(M), \\ RL_n(M) &= GL_n(M) \cap RL_n(V). \end{aligned}$$

Consider the bounded $\mathfrak{o}$-module M. For any non-zero vector x in FM we define the coefficient of x with respect to M to be the set

$$\mathfrak{c}_x = \{\alpha \in F \mid \alpha x \in M\};$$

$\mathfrak{c}_x$ is a fractional ideal of $\mathfrak{o}$. And for any non-zero linear functional ρ on V, $\rho(M)$ is a fractional ideal of $\mathfrak{o}$.

Define the linear congruence groups as follows for any non-zero ideal $\mathfrak{a}$ in $\mathfrak{o}$:

$$\begin{aligned} GL(M; \mathfrak{a}) &= \{\sigma \in GL(M) \mid (\sigma - 1_V)M \subseteq \mathfrak{a}M\}, \\ SL(M; \mathfrak{a}) &= GL(M; \mathfrak{a}) \cap SL(M) \end{aligned}$$

and $TL(M; \mathfrak{a})$ is defined to be the group generated by all the transvections in $GL(M; \mathfrak{a})$. We see that

$$TL(M; \mathfrak{a}) \subseteq SL(M; \mathfrak{a}) \subseteq GL(M; \mathfrak{a})$$

are normal subgroups of $GL(M)$. And

$$SL(M; \mathfrak{o}) = SL(M), \quad TL(M; \mathfrak{o}) = TL(M), \quad GL(M; \mathfrak{o}) = GL(M).$$

If we consider any non-trivial transvection $\tau_{a,\rho}$ in $SL(V)$, we easily see that

$$\rho(M) \subseteq \mathfrak{a}\cdot\mathfrak{c}_a \Rightarrow \rho(M)a \subseteq \mathfrak{a}\cdot M \Rightarrow \tau_{a,\rho} \in GL(M; \mathfrak{a}).$$

4.1. *For $n = \dim (FM) \geqq 2$ and M a bounded $\mathfrak{o}$ module, $TL_n(M; \mathfrak{a})$ is full of transvections.*

Proof. Let $L \subseteq H$ be given. Write $L = Fa$. Let ρ be a non-zero linear functional such that $\rho H = 0$. Now $\rho(M)$ is a fractional ideal so choose a non-zero λ in F such that $\lambda(\rho M) \subseteq \mathfrak{a}\cdot\mathfrak{c}_a$. Then the above remarks show $\tau_{a,\lambda\rho} \in TL_n(M; \mathfrak{a})$. Q.E.D.

Now let $PTL_n(M;\ \mathfrak{a})$, $PSL_n(M;\ \mathfrak{a})$, $PGL_n(M;\ \mathfrak{a})$ denote respectively $TL_n(M; \mathfrak{a})^-$, $SL_n(M; \mathfrak{a})^-$, $GL_n(M; \mathfrak{a})^-$, where $^-$ is the natural map of $GL_n(V)$ onto $PGL_n(V)$; the groups $PTL_n(M; \mathfrak{a})$, $PSL_n(M; \mathfrak{a})$, $PGL_n(M; \mathfrak{a})$ are the projective congruence groups.

4.2. *Let M be a bounded $\mathfrak{o}$-module with* $\dim V \geqq 3$, *where* $V = FM$. *Let* G *be one of the projective congruence groups*

$$PTL_n(M; \mathfrak{a}), \quad PSL_n(M; \mathfrak{a}), \quad PGL_n(M; \mathfrak{a}),$$

and let Λ *be an automorphism of* G. *Then* Λ *can be expressed in exactly one of the following two ways*:

(i) $\Lambda = \bar{\Phi}_g$ *for some semi-linear isomorphism* g *of* V *onto* V,
(ii) $\Lambda = \bar{\Psi}_h$ *for some semi-linear isomorphism* h *of* V *onto* V'.

Proof. 4.1 implies G is full of projective transvections. Now apply 3.3. Q.E.D.

Remark. 4.1 shows each of the groups $TL_n(M; \mathfrak{a})$, $SL_n(M; \mathfrak{a})$, $GL_n(M; \mathfrak{a})$ is full of transvections for M a non-zero bounded $\mathfrak{o}$-module. So all the automorphisms of the congruence groups $TL_n(M; \mathfrak{a})$, $SL_n(M; \mathfrak{a})$, $GL_n(M; \mathfrak{a})$ for $\dim FM \geqq 3$ and M any bounded $\mathfrak{o}$-module are given by 3.4.

5. The automorphisms $\bar{\Phi}_g$ and $\bar{\Psi}_h$ of $PGL(M, \mathfrak{o})$

By 4.2, if Λ is an automorphism of $PGL(M; \mathfrak{o})$ then Λ equals $\bar{\Phi}_g$ or $\bar{\Psi}_h$ for some g or h. In this section we give necessary and sufficient conditions for $\bar{\Phi}_g$ or $\bar{\Psi}_h$ to be automorphisms of $PGL(M; \mathfrak{o})$.

Let $\tau_{a,\rho}$ be a transvection in $GL_n(V)$. The following proposition is clear.

5.1. $\tau_{a,\rho}(M) = M$ *iff* $\tau_{a,\rho}(M) \subseteq M$ *iff* $(\rho M)\cdot a \subseteq M$.

Definition. Let M be a bounded $\mathfrak{o}$-module. Let

$$M^{\#} = \{Q \in V' \mid Q(M) \subseteq \mathfrak{o}\}.$$

If $\sigma M = M$, $\sigma \in GL_n(V)$, then $\sigma^{\vee}(M^{\#}) = M^{\#}$. For any $\sigma \in GL_n(V)$, we say σ is *on* M iff $\sigma M = M$.

Remark. Let M be a *free* $\mathfrak{o}$-module, and $x_1 \cdots x_n$ a base for M. So $M = \oplus_{i=1}^{n} x_i \mathfrak{o}$. Let $\{Q_i\}$ be the dual base to $\{x_i\}$. Let $\tau_{\lambda x_i, Q_j}$, $i \neq j$, be an elementary transvection with respect to the base $\{x_i\}$ of $FM = V$. Then it is easy to see that $\tau_{\lambda x_i, Q_j}$ is on M iff $\lambda \in \mathfrak{o}$.

5.2. *Let M be a free $\mathfrak{o}$ module, let $V = FM$, and let S be one of the groups* $GL(M)$, $SL(M)$, $TL(M)$. *Suppose* $n = \dim V \geqq 3$, $\bar{\Phi}_g$ *is an automorphism of* $\bar{S}$. *Then* $\mathfrak{o}^u = \mathfrak{o}$ *where u is the field automorphism of g.*

Proof. Write $M = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$ with $\{x_i\}$ a base for V and let $\{Q_i\}$ denote the dual base of $\{x_i\}$. To begin, notice that if $\tau_{a,Q}$ is any transvection on M then $(QM)a \subseteq M$, hence $(QM)\lambda a \subseteq M$ for any $\lambda \in \mathfrak{o}$; and so $\tau_{a,Q} \in S$ implies $\tau_{\lambda a, Q} \in S$ for all $\lambda \in \mathfrak{o}$.

Put $y_i = gx_i$ for $1 \leqq i \leqq n$, and $\phi_j = \mu Q_j g^{-1}$ for $1 \leqq j \leqq n$. Since g is a semi-linear isomorphism, it follows $y_1, \cdots, y_n$ are a basis for V, and that $\phi_1, \cdots, \phi_n$ are linear functionals which constitute the dual basis of the y_i's.

Now by 5.1, $\tau_{x_i,Q_j} \in S$ for $i \neq j$, and hence

$$\tau_{x_i,Q_j} = [\tau_{x_i,Q_k}\,;\,\tau_{x_k,Q_j}] \in DS \quad k \neq i,\ k \neq j.$$

Hence $\tau_{y_i,\phi_j} = g\tau_{x_i,Q_j}g^{-1} \in S$ by 2.4 so

$$\tau_{\lambda y_i,\phi_j} = [\tau_{\lambda y_i,\phi_k}\,;\,\tau_{y_k,\phi_j}] \in DS \quad \text{if } \lambda \in \mathfrak{o}.$$

Hence $g^{-1}(\tau_{\lambda y_i,\phi_j})g$ is on M by 2.4. So $\tau_{\lambda y_1,\phi_2}$ is on gM whenever $\lambda \in \mathfrak{o}$. Now

$$gM = \mathfrak{o}^{\mu}y_1 + \cdots + \mathfrak{o}^{\mu}y_n.$$

So by the remark immediately before the statement of 5.2, since $\mathfrak{o}^{\mu} \div \mathfrak{o}^{\mu} = F$ and gM is a free $\mathfrak{o}^{\mu}$ module, we have $\lambda \in \mathfrak{o}^{\mu}$. Thus $\mathfrak{o} \subseteq \mathfrak{o}^{\mu}$. By considering $\bar{\phi}_g^{-1}$ instead of $\bar{\phi}_g$ we get $\mathfrak{o} \subseteq \mathfrak{o}^{\mu^{-1}}$. Hence $\mathfrak{o} = \mathfrak{o}^{\mu}$. Q.E.D.

Recall $t_{ij}(\lambda)$ denotes the matrix with λ in (i, j) position, $i \neq j$, 1's on the main diagonal, and zeroes everywhere else. $EL_n(\mathfrak{o})$ equals the group generated by all the $t_{ij}(\lambda)$, $\lambda \in \mathfrak{o}$. Note $EL_n(\mathfrak{o}) \subseteq SL_n(\mathfrak{o})$.

5.3. *Let ε be in $\mathfrak{o}$; $\varepsilon^n = 1$. Then the n by n matrix* diag $(\varepsilon, \cdots, \varepsilon)$ *is in* $EL_n(\mathfrak{o})$.

Proof. A computation using elementary row and column operations. Q.E.D.

5.4. *Let M be a free module over the integral domain $\mathfrak{o}$, where $\mathfrak{o} \div \mathfrak{o} = F$, $FM = V$. Let h be a linear isomorphism of V onto V' such that $hM = M^{\#}$. Then ψ_h is an automorphism of S where S equals one of the groups $TL_n(M)$, $SL_n(M)$ or $GL_n(M)$.*

Proof. One need only show $\psi_h(S) = S$. Clearly $\psi_h(S) \subseteq S$ since $hM = M^{\#}$. Since $h^tM^{\#} = (M^{\#})^{\#} = M$ we have similarly

$$\psi_h^{-1}(S) = \psi_{h^t}(S) \subseteq S. \qquad \text{Q.E.D.}$$

5.5. *Let $A \in GL_n(\mathfrak{o})$ and $\alpha \in F$. Suppose* trace $(\alpha B_1AB_2) \in \mathfrak{o}$ *whenever B_1 and B_2 are in $EL_n(\mathfrak{o})$. Then $\alpha \in \mathfrak{o}$.*

Proof. Apply 5.1 of [8]. Q.E.D.

5.6. *Let M be a free $\mathfrak{o}$ module, let $V = FM$, $n = \dim V \geqq 3$ and let S be one of the groups $TL(M)$, $SL(M)$, $GL(M)$. Let $\bar{\phi}_g$ be an automorphism of $\bar{S}$; then ϕ_g is an automorphism of S.*

Proof. To show ϕ_g is an automorphism of S, it is clearly enough to show $\phi_g(S) \subseteq S$. For $\bar{\phi}_g^{-1} = \bar{\phi}_{g^{-1}}$ is an automorphism of $\bar{S}$, and if we have shown $\phi_g(S) \subseteq S$, we can show $\phi_{g^{-1}}(S) \subseteq S$. So $\phi_g(S) = S$ and ϕ_g is an automorphism of S.

So let us show $\phi_g(S) \subseteq S$. Take $\sigma \in S$. Since $\bar{\phi}_g(\bar{\sigma}) \in \bar{S}$ there is a (fixed) $\tau_\sigma \in S$ such that $\bar{\phi}_g(\bar{\sigma}) = \bar{\tau}_\sigma$. So $\phi_g(\sigma)^- = \bar{\tau}_\sigma$, or $\chi(\sigma)\cdot\phi_g(\sigma) = \tau_\sigma$ for a scalar $\chi(\sigma) \in RL(V)$.

To show $\phi_g(S) \subseteq S$ it is enough to show $\phi_g(\sigma) \in S$, and this will be true if we can show $\chi(\sigma) \in S$.

So let us show $\chi(\sigma) \in S$. It is enough to show $\chi(\sigma) \in RL_n(M)$, for if $S = GL(M)$, surely $\chi(\sigma) \in RL(M)$ will imply $\chi(\sigma) \in S$. If $S = SL(M)$ or $TL(M)$ then $\det \sigma = 1$ and $\det \tau_\sigma = 1$. This implies $\det \chi(\sigma) = 1$. Let $\chi(\sigma) = \varepsilon \cdot 1_V$, ε in F. Then $\varepsilon^n = 1$, and hence by 5.4 the matrix $\operatorname{diag}(\varepsilon, \cdots, \varepsilon)$ is in $EL_n(\mathfrak{o})$. Taking a basis $x_1, \cdots, x_n$ for the free module M then shows

$$\chi(\sigma) = \varepsilon \cdot 1_V \in TL(M) \subseteq S.$$

So $\chi(\sigma) \in RL(M)$ implies $\chi(\sigma) \in S$ in all cases.

It follows immediately from 2.4 that $\sigma \in DS$ implies $\chi(\sigma) \in S$ and $\phi_g(\sigma) \in S$. Now consider a typical σ in S. We wish to show $\chi(\sigma) \in RL_n(M)$. Express $\chi(\sigma)$ in the form $\chi(\sigma) = \alpha \cdot 1_V$ for some $\alpha \in \dot{F}$. It is enough to prove this α is in $\mathfrak{o}$. For then the equation $\chi(\sigma) \cdot \phi_g(\sigma) = \tau_\sigma$ shows that $\det \chi(\sigma)$ is a unit. (To see this, use the fact $g\sigma g^{-1}$ has matrix $PS^\mu P^{-1}$ in the base X of V where S is the matrix of σ, P that of g.) But we are assuming $\alpha \in \mathfrak{o}$. Hence α is a unit, and so $\chi(\sigma) \in RL_n(M)$ as required. So it is indeed enough to show $\alpha \in \mathfrak{o}$.

Write $\alpha = \beta^\mu$. We will show $\beta \in \mathfrak{o}$. Let $\sigma \sim A$ in X; so $A \in GL_n(\mathfrak{o})$. Consider matrices B_1 and B_2 in $EL_n(\mathfrak{o})$. If $\operatorname{tr}(\beta B_1 A B_2) \in \mathfrak{o}$ we are done by 5.5. Take τ_1 and τ_2 in S with $\tau_1 \sim B_1$ and $\tau_2 \sim B_2$ in the base X. Now it follows $\operatorname{tr} \phi_g(\Sigma) = (\operatorname{tr} \Sigma)^\mu$ for all Σ in $GL_n(V)$. Hence it is enough to show $\operatorname{tr} \phi_g(\beta\tau_1\sigma\tau_2) \in \mathfrak{o}$. But this follows from the fact

$$\phi_g(\beta\tau_1\sigma\tau_2) = \alpha \cdot \phi_g(\tau_1) \cdot \phi_g(\sigma) \cdot \phi_g(\tau_2) = \phi_g(\tau_1) \cdot \tau_\sigma \cdot \phi_g(\tau_2) \in S. \qquad \text{Q.E.D.}$$

5.7. *Let the hypotheses on n, S and M be as in* 5.6. *Suppose the mapping $\bar{\psi}_h$ is an automorphism of $\bar{S}$. Then $\psi_h(S) = S$.*

Proof. Take a fixed linear isomorphism k of V onto V' such that $kM = M^\#$. By 5.4, ψ_k is an automorphism of S, and so $\bar{\psi}_k$ is an automorphism of $\bar{S}$. Put $g = k^{-1}h$. So g is a semi-linear isomorphism of V onto V. We have $\phi_g \circ \psi_h = \psi_k$, so ψ_h is on S iff ϕ_g is. But $\bar{\phi}_g \circ \bar{\psi}_h = (\phi_g \circ \psi_h)^{-1} = \psi_{hg^{-1}} = \bar{\psi}_k$. Therefore $\bar{\phi}$ is an automorphism of $\bar{S}$. By 5.6, $\phi_g(S) = S$. Hence $\psi_h(S) = S$. Q.E.D.

Assume M to be a free $\mathfrak{o}$-module of dim $\geqq 3$ and let S be as in 5.6. By 5.6 and 5.7 we see that if $\bar{\phi}_g$ (respectively $\bar{\psi}_h$) is an automorphism of $\bar{S}$, then ϕ_g (respectively ψ_h) is an automorphism of S. But surely if ϕ_g (respectively ψ_n) is an automorphism of S, then $\bar{\phi}_g$ (respectively $\bar{\psi}_h$) is an automorphism of $\bar{S}$.

Hence we conclude that

$\bar{\phi}_g$ (respectively $\bar{\psi}_h$) is an automorphism of $\bar{S}$ iff ϕ_g (respectively ψ_h) is an automorphism of S.

Now in 5.5 and 5.6 of [8], O'Meara has given the following necessary and sufficient conditions for ϕ_g (respectively ψ_h) to be an automorphism of S:

(i) $\mathfrak{o}^u = \mathfrak{o}$ for u the field automorphism of g (respectively u the field automorphism of h), and (ii) $gM = \mathfrak{a}M$ for some invertible fractional ideal $\mathfrak{a}$ of $\mathfrak{o}$ (respectively $hM = \mathfrak{a}M^{\#}$ for some invertible fractional ideal $\mathfrak{a}$ of $\mathfrak{o}$). Hence we conclude the above two conditions are also necessary and sufficient conditions for $\bar{\phi}_g$ (respectively $\bar{\psi}_h$) to be an automorphism of $\bar{S}$ when M is free.

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Indiana University
Bloomington, Indiana