THE REDUCED SYMMETRIC PRODUCT OF PROJECTIVE SPACES AND THE GENERALIZED WHITNEY THEOREM

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1. Statement of results

We give a simple way of computing the cohomology structure of the reduced symmetric product of a projective space (real and complex). This can then be used in the study of the embedding problem for these spaces. A comparison of the reduced product with the projective tangent bundle of the manifold yields relations between embeddings and immersions. In particular we get:

THEOREM 5.2. *The regular homotopy classes of embeddings* $RP^k \subset R^{2k}$ *for* k *even and greater than* 2 *coincide with the isotopy classes of such embeddings and these in turn are in one-to-one correspondence with the integers.*

There we must note that the condition $k > 2$ is essential. Massey [7] has recently proved a conjecture of Whitney which claims that there are only two regular homotopy classes of embeddings $P^2 \subset R^4$.

Previously the author and D. Handel [4] have independently found that $RP_n \subset R^{2n-2}$ for $n = 2^s + 2$. More recently F. Nussbaum, using the results of this paper and obstruction theory for nonorientable bundles, has shown that if $n = 2^s + 2$ then $RP_n \subset R^{2n-3}$.

2. Preliminaries

Let X be a topological space and Δ the diagonal of $X \times X$. A map $F : X \times X - \Delta \rightarrow S^{n-1}$ is called *equivariant* if $F(x, y) = -F(y, x)$ for all $(x, y) \in X \times X - \Delta$. Any topological embedding $f : X \rightarrow R^n$ gives rise to such an equivariant map, namely define

$$F(x, y) = (f(x) - f(y))/ \| f(x) - f(y) \| .$$

Two isotopic embeddings give rise to equivariantly homotopic maps from $X \times X - \Delta$ to S^{n-1}.

For compact manifolds the following is a corollary to Haefliger [5]:

THEOREM. *A manifold* M^m *embeds in* R^n *if there exists an equivariant map* $M \times M - \Delta \rightarrow S^{n-1}$ *and* $n \geq 3(m + 1)/2$. *Moreover if* $n \geq 3(n + 1)/2$ *then the isotopy classes of such embeddings are in one-to-one correspondence with the equivariant homotopy classes of such maps.*

The equivariant homotopy classes of maps from $M \times M - \Delta$ into S^{n-1} are further in one-to-one correspondence with homotopy classes of maps

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$f: M^* = M \times M - \Delta/Z_2 \to P^{n-1}$ for which $f^*(x) = u$ where x is the generator of $H^*(P^{n-1}, Z_2)$ and u is the class of the double covering $M \times M - \Delta \to M^*$. These in turn are in one-to-one correspondence with non-zero sections of the bundle $n\zeta \to M^*$ where ζ is the line bundle associated to the double covering $M \times M - \Delta \to M^*$.

There is a similar theorem concerning immersions due to Haefliger and Hirsch [6] which can be restated as follows:

THEOREM. *A manifold M^m immerses in R^n if and only if there exists an equivariant map $\mathbf{S}(M^m) \to S^{n-1}$ and $n \geq (3m + 1)/2$. Moreover, if $n > (3m + 1)/2$ then the regular homotopy classes of such immersions are in one-to-one correspondence with equivariant homotopy classes of such maps.*

Here $\mathbf{S}(M)$ denotes the tangent sphere bundle of M. Let $\mathbf{P}(M)$ be the projective tangent bundle of M and let η be the canonical line bundle over $\mathbf{P}(M)$. The problem can again be reduced to a question about section of $n\eta$.

The canonical embedding $\mathbf{P}(M) \subset M^*$ may be used to compare embedding and immersion results about M.

In [3] the author has determined the cohomology of the reduced symmetric product of real projective spaces. Using the basic idea of [3] we shall determine the structure of the reduced symmetric product of real and complex projective spaces which will be denoted by RP_n and CP_n respectively. To give a unified treatment we shall use the symbol FP_n where F will be the field of either real or complex numbers. The reduced symmetric product is $FP_n{}^* = FP_n \times FP_n - \Delta/Z_2$ where Z_2 acts on $FP_n \times FP_n - \Delta$ by interchanging the two coordinates.

FP_n^* can be viewed as the set of unordered pairs of distinct points in FP_n or the set of pairs of distinct lines through the origin in F^{n+1}. This gives us the fibration

$$FP_1^* \to FP_n^* \to FG_{n+1,2}$$

where $FG_{n+1,2}$ is the Grassmanian of (unoriented in the real case) 2-planes in F^{n+1}.

The fiber is an open Moebius band for $F = R$ and is $S^2 \times S^2 - \Delta/Z_2$ in the case $F = C$. In either case the fiber has a real projective space $RP_d (d = 1$ if $F = R$, $d = 2$ if $F = C)$ as a deformation retract. Using this deformation we can deform the total space onto a subspace, which we shall also denote by FP_n^*. We thus obtain a bundle η_F:

$$RP_d \to FP_n^* \to FG_{n+1,2}\,. \tag{1}$$

The deformation can be interpreted in the following way: each pair of distinct lines in F^{n+1} defines a 2-plane in F^{n+1}; we move the two lines within this plane until they become mutually orthogonal. This gives an interpretation of the bundle (1) in terms of the canonical 2-plane bundle γ_F over

$FG_{n+1,2}$: we take the sphere bundle associated to γ_F and identify points which lie on pairs of mutually orthogonal lines. In the case when $F = C$ this amounts to taking the projectification of γ_C with fiber $CP_1 = S^2$, which, as shall be seen later, has an associated R^3-bundle and then the projectification of this bundle (with respect to R) yields the bundle η_C.

This interpretation of FP_n^* gives a particularly simple way of computing its cohomology.

Since η_F is a projectification of real vector bundle (of dimension 2 when $F = R$ and dimension 3 when $F = C$) a standard spectral sequence argument yields the cohomology of the total space. Namely, we have the following proposition and corollary (e.g. Bott [2]):

PROPOSITION 3.1. *Let ξ be a real vector bundle over B. Then $H^*(\mathbf{P}(\xi), Z_2)$ is a free module over $H^*(B, Z_2)$ generated by $1, X_\xi, \cdots, X_\xi^{k-1}$, $k = \dim \xi$, where $X_\xi \in H^1(\mathbf{P}(\xi), Z_2)$ is equal to $w_1(S_\xi)$.*

S_ξ is the canonical line bundle over $\mathbf{P}(\xi)$. We also have the following:

COROLLARY. *There are unique classes $w_i(\xi) \in H^i(B, Z_2)$ $i = 0, \cdots, \dim \xi = k$, $w_0 = 1$, such that the equation*

$$\sum_{i=0}^{k} x^{k-1} w_i(\xi) = 0$$

holds in $H^*(\mathbf{P}(\xi), Z_2)$. This is the defining relation of $\mathbf{P}(\xi)$ and $w_i(\xi)$ are the Stiefel-Whitney classes of the bundle ξ.

To complete our computations it suffices to find the cohomology of the Grassmanians (Borel [1]) and the stiefel-Whitney classes which correspond to η_F.

4. The bundles η_F and the cohomology of FP_n^*

Whenever a homomorphism $h : G \to H$ between two Lie groups is given we can associate a principal H-bundle to a principal G-bundle using this homomorphism. More precisely if $E \to X$ is a principal G-bundle it is induced by a map

$$X \xrightarrow{E} B_G,$$

composing this map with the map

$$B_G \xrightarrow{h_*} B_H$$

we obtain the map

$$X \xrightarrow{h_* \circ E} B_H$$

which induces the desired principal H-bundle.

The bundle η_F is the projectification of an O_2 (when $F = R$) or an O_3 (when $F = C$) bundle ξ_F. In order to get at the cohomology of FP_n^* we must calculate the Stiefel-Whitney classes of this bundle ξ_F.

From the description of η_F it is clear that ξ_R is associated to the canonical 2-plane bundle ξ_R over $RG_{n+1,2}$ via the homomorphism

$$h^R : O(2) \to O(2)$$

with kernel

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}$$

(i.e. the center of $O(2)$). This map induces the identity homomorphism on $H^0(O(2), Z_2)$ and the trivial map on $H^1(O(2), Z_2)$. It follows from the spectral sequence for $H^*(BO(2), Z_2)$ that $w_1(\xi_R) = w_1(\gamma_R)$ and $w_2(\xi_R) = 0$.

In the case of η_C we have a similar phenomenon. The bundle ξ_C is obtained from γ_C via the homomorphism

$$h^C : U(2) \to SO(3) \to O(3)$$

whose kernel is the center of $U(2)$. This homomorphism leads to the fibration

$$K(Z, 2) = BSO(2) \to BU(2) \to BSO(3)$$

which yields $w_1(\xi_C) = 0$, $w_2(\xi_C) = \rho c_1(\gamma_C)$ and $w_3(\xi_C) = 0$ where ρ denotes reduction mod 2.

The cohomology of the Grassmanian has been determined by Borel [1] and is given by

$$H^*(FG_{n+1,2}, Z_2) = S(x_1, \cdots x_{n-1}) \otimes S(x_n, x_{n+1})/S^+(x_1, \cdots, x_{n+1})$$

where $S(x_1, \cdots, x_r)$ is the algebra of symmetric polynomials over $x_1, \cdots, x_r$ (all the generators are of dimension 1 if $F = R$ and of dimension 2 if $F = C$) and $S^+(x_1, \cdots, x_r)$ is the ideal of elements of positive degree.

One can easily obtain a description of this ring which is more suitable for computations. Namely we have (cf. [3])

PROPOSITION 4.1. *$H^*(FG_{n+1,2}, Z_2)$ is a ring on two generators x, y (dim $x = d$, dim $y = 2d$) with the only relations:*

$$a_n = \sum_{i=0} \binom{n-i}{i} x^{n-2i} y^i = 0 \quad \textit{and} \quad a_{n+1} = \sum_{i=0} \binom{n+1-i}{i} x^{n+1-2i} y^i = 0$$

The Steenrod algebra structure is given by

$$Sq^1 y = xy \text{ if } F = R \quad \text{and} \quad Sq^2 y = xy \text{ if } F = C.$$

The elements x and y are the characteristic classes.

The proposition easily yields the following corollaries which are useful for calculations:

COROLLARY 4.1. *$x^{2i} y^{n-1-i} \neq 0$ if and only if $i = 2^s - 1$*

Proof. We prove this assertion by induction on n. It is clearly true for $n = 2$ (the smallest possible value for n). Suppose that the statement is true for $n = k$. That means that the system $a_k = 0$, $a_{k+1} = 0$ yields our proposition. The system for $n = k + 1$ can be taken to be $a_{k+1} = 0$, $ya_k = 0$ (cf. (3)) so in the top dimension we get all the previous equations multiplied by y and the equation $x^{k-1}a_{k+1} = 0$. This last equation is

$$(*) \qquad 0 = \binom{k+1}{0}x^{2k} + \binom{k}{1}x^{2k-2}y + \cdots + \binom{k+1-j}{j}x^{2k-2j}y^{d} + \cdots.$$

By the induction hypothesis the other equations yield

$$x^{2k-2j}y^{j-1}y = 0$$

unless $k - j = 2^s - 1$ for some s. The coefficients of the remaining terms in (*) are

$$\binom{k+1-j}{j}$$

where $k - j = 2^s - 1$
or

$$\binom{2^s}{j}$$

and $2^s + j = k + 1$. This can be non-zero only when $j = 2^s$ or $k + 1 = 2^{s+1}$ in which case the added equation reads

$$x^{2k} + y^{i} = 0 \quad \text{and} \quad k = 2^{s+1} - 1,$$

which concludes the induction.

It follows from Corollary 4.1 that the height of y is maximal.

COROLLARY 4.2. *The height of x does not change for $2^{r-1} \leq n \leq 2^r - 1$ and is equal to $2^r - 2$.*

Proof. Since $a_n = 0$, $a_{n+1} = 0 \Rightarrow a_{n+1} = 0$, $a_{n+2} = 0$ the height of a increases with n. For $n = 2^r - 1$

$$a_n = \sum \binom{n-i}{i}x^{n-2i}y^{i} = x^{n} = 0.$$

Since this is the first relation we conclude the height of x is $n - 1 = 2^r - 2$. On the other hand for $n = 2^{r-1}$, Corollary 4.1 states that $a^{2n-2} = x^{2(2^{r-1}-1)} \neq 0$ thus the height of x is again $2^r - 2$ as was claimed.

To obtain a complete description of $H^*(FP_n^*, Z_2)$ we now apply Proposition 3.1 and get

THEOREM 4.3. *$H^*(FP_n^*, Z_2)$ as a module over $H^*(FG_{n+1,2}, Z_2)$ is generated by $1, u, \cdots, u^d$, $\dim u = 1$ ($d = 1$ if $F = R$, $d = 2$ if $F = C$). The ring structure is given by the relation:*

$$u^2 = ux \text{ if } F = R \quad \text{and} \quad u^3 = ux \text{ when } F = C.$$

5. Applications

We can apply the results of the previous section to obtain the classification (first up to isotopy) of embeddings of RP_k in R^{2k} when k is even *and greater than* 2.

Such embeddings are classified by the homotopy classes of non-zero sections of the bundle

$$\begin{array}{c} 2k\zeta \\ \downarrow \\ RP_k^* \end{array}$$

where ζ is the line bundle associated to the double covering

$$P^k \times P^k - \Delta \to P^k \times P^k - \Delta/Z_2 .$$

Since $G_{k+1,2}$ is non-orientable and the first Stiefel-Whitney class of its tangent bundle is the same as $w_1(\eta_R) = w_1(\xi_R) = x$ the manifold RP_k^* is a $(2k-1)$-dimensional orientable manifold. Moreover $2k\zeta$ is an orientable bundle, we have thus

$$H^{2k-1}(RP_k^*, \pi_{2k-1}(S^{2k-1})) = H^{2k-1}(RP_k^*; Z) \cong Z.$$

Since each element of $H^*(RP_k^*; Z)$ can be realized as an obstruction to a homotopy between two different cross-sections of $2k\zeta$, we have

PROPOSITION 5.1. *The isotopy classes of embeddings $RP^k \subset R^{2k}$ for k even $(k > 2)$ are in one-to-one correspondence with the integers.*

To compare these isotopy classes of embeddings with regular homotopy classes we must study the inclusion map $\mathbf{P}(\tau) \subset RP_k^*$. Viewing $\tau(RP_k)$ as pairs of lines in R^{k+1} which are close to each other we see that $\mathbf{P}(\tau)$ is orientable so

$$\iota^* : H^{2k-1}(RP_k^*, Z) \to H^{2k-1}(\mathbf{P}(\tau), Z) \cong Z$$

is just multiplication by 2. This means that if two embeddings are regularly homotopic they are already isotopic. This together with Proposition 5.1 yields:

THEOREM 5.2. *The regular homotopy classes of embeddings $RP^k \subset R^{2k}$ for k even and greater than 2 coincide with the isotopy classes of such embeddings and these in turn are in one-to-one correspondence with the integers.*

D. Handel [4] has computed the cohomology of RP_k^* and proved the following embedding theorem:

If $n = 2^s + 2$ $(s > 2)$ then $RP^n \subset R^{2n-2}$.
The following holds for complex projective spaces:

PROPOSITION 5.3.[1] *If $n \neq 2^i$ and $n > 3$ then CP_n embeds in R^{4n-2}.*

Indeed, there is only one obstruction to a non-zero section of

[1] This proposition has been known; we include it here to give an example which may serve for other computations.

$$\begin{array}{c}(4n-2)\zeta\\ \downarrow\\ CP_n^*\end{array}$$

Since dim $CP_n^* = 4n - 2$, the obstruction is $\chi(4n - 2)\zeta)$—the Euler class.

The Euler class of an even multiple of a line bundle ζ has the property that $2\chi(m\zeta) = \chi((m-2)\zeta)\chi(2\zeta)$ and if the mod 2 reduction of $\chi((m-2)\zeta)$ is zero then $\chi((m-2)\zeta) = 2\cdot c$ and

$$\chi(m\zeta) = 2\cdot\chi(2\zeta)\cdot c = 0.$$

Thus if $u^{m-2} = 0$ then $\chi(m\zeta) = 0$.

In CP_n^* we have $u^k = 0$ if $k \geq 2^{r+1} - 1$ where $2^{r-1} \leq n \leq 2^r - 1$ (this follows from the fact that $x^{2r-1} = 0$ and $u^{2s+1} = ux^s$) so $\chi((4n-2)\zeta) = 0$ whenever $n \neq 2^i$.

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