ON DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE $n \equiv 2 \bmod 4$

BY

MICHAEL ASCHBACHER

Let G be a doubly transitive permutation group on a set Ω of degree $n \equiv 2 \bmod 4$, $n > 2$. Let α and β be distinguished and distinct points in Ω, and let $H = G_\alpha$, $D = G_{\alpha\beta}$, and T be a Sylow 2-subgroup of D. We prove the following two results:

THEOREM 1. *G contains a unique minimal normal subgroup $M(G)$. $M(G)$ is simple and doubly transitive on Ω, with $G \leq$ Aut $M(G)$.*

THEOREM 2. (*i*) *if u is an involution in T such that the fixed point set of every element of $T^{\#}$ is contained in that of u, then $M(G)^\Omega$ is A_6, $L_2(q)$, or $U_3(q)$ in their natural doubly transitive representations.*

(*ii*) *if T is abelian then $M[G]^\Omega$ is an in* (i).

Theorem 1 reduces the problem of determining all groups mentioned in the title, or any subclass thereof, to the problem of finding all such simple groups. A natural question that arises is which doubly transitive groups satisfy theorem 1. The only groups known to the author that do *not* are those with regular normal subgroups, and the Ree group, $R(3)$.

To date most characterizations of doubly transitive groups seem to be in terms of the structure of the stabilizer of two points, and particular its involutions. Theorem 2 is a result of this kind. Lemmas 3 and 4 are useful in general problems of the sort discussed in this paragraph.

1. Throughout this paper G, H, T, etc. are as above. For $X \subseteq G$ we write $F(X)$ for the set of ω in Ω with $\omega x = \omega$ for every x in X. More generally our permutation group theoretic notation is as in [5].

A_n is the alternating group on n letters. $L_2(q)$, $U_3(q)$, are the simple normal subgroups of the two, three, dimensional projective special linear, unitary, group over $GF(q)$, respectively.

We require the following two results, which we quote without proof.

LEMMA 1 (Witt, [6]). *Let A be a t-transitive permutation group on $\Omega = \{1, 2, \cdots, n\}$ and let B be the subgroup of A fixing each i, $1 \leq i \leq t$. Let $U \subseteq B$. Then $(N_G U)^{F(U)}$ is t-transitive if and only if for a in A, $U^a \subseteq B$ implies there exists b in B with $U^a = U^b$.*

LEMMA 2 (Suzuki, [4]). *Let U be a 2-group and u an involution in U such that $C_U(u)$ is the four group. Then U is dihedral or semidihedral*

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We also require the classification theorems for groups with dihedral, semidihedral, and wreathed Sylow 2-subgroups [1], [3].

2. The proof of Theorem 1 depends on the next lemma.

LEMMA 3. *Let A be a nontrivial normal subgroup of G, and let R be a Sylow 2-subgroup of $B = A \cap D$. Then $F(R) = \{\alpha, \beta.\}$*

Proof. Suppose $R^g \leq D$ for some g in G. Then $R^g \leq B$ and as a Sylow group of B, $R^g = R^b$ for some b in B. It follows from Lemma 1 that $N = N_G R$ is doubly transitive on $F(R)$.

N contains a Sylow 2-subgroup T of D, and T has index two in a Sylow group of G, so $|F(R)| = f \equiv 2 \bmod 4$.

Let $E = N \cap A$. $E^{F(R)}$ is 3/2-transitive, so a Sylow 2-subgroup of $E^{F(R)}$ has order two and thus $E^{F(R)}$ is solvable. So $N^{F(R)}$ contains a solvable normal subgroup. Therefore $N^{F(R)}$ has a regular normal subgroup, and f is a prime power. Thus $f = 2$.

We now prove Theorem 1. Let $1 < A \trianglelefteq G$. Claim that A^Ω is doubly transitive. A is 3/2-transitive, so it suffices to show that $H = (A \cap H)D$. But as A is 3/2-transitive, a Sylow 2-subgroup R of $A \cap D$ is also Sylow in $A \cap H$. Thus $H = (A \cap H)N_H R$, while by Lemma 3, $N_H R \leq D$.

Next if $A, B \leq G$ with A^Ω and B^Ω doubly transitive, then $[A, B] \neq 1$. For if not, A acts on $F(B_\alpha) = \{\alpha\}$, contradicting the transitivity of A^Ω. Finally we recall that a minimal normal subgroup M is semisimple. Theorem 1 follows from the last three remarks.

Note that we have also shown that $F(T) = \{\alpha, \beta\}$.

3. We now prove two lemmas to be used in the proof of Theorem 2.

LEMMA 4. *Let $U \leq D$ with $F(U) \neq \{\alpha, \beta\}$. Assume there exists a prime p such that for g in G, if $U \leq D^g$ then a Sylow p-subgroup of $N_{D^g}(U)$ fixes exactly two points. Then $(N_G(U))^{F(U)}$ is doubly transitive.*

Proof. Let h be in H with $U^h \leq D$. Let P be a Sylow p-subgroup of $N_{D^{h^{-1}}}(U)$. Then by hypothesis $|F(P)| = 2$. Thus $N_H P \leq D^{h^{-1}}$, so P is Sylow in $N_H U$. Let Q be a Sylow group of $N_D U$. Then $Q = P^x$ for some x in $N_H U$. Thus $\{\alpha, \beta\} = F(Q) = F(P)x = \{\alpha, \beta\}h^{-1}x$, so that $h^{-1}x = d^{-1}$ is in D, and $U^d = U^h$.

Lemma 1 now implies that $N_H U$ is transitive on $F(U) - \{\alpha\}$. Similarly $N_{G_\beta}(U)$ is transitive on $F(U) - \{\beta\}$. Since $|F(U)| > 2$, the desired result now follows.

LEMMA 5. *Let G be simple, and u an involution in T such that $F(x) \subseteq F(u)$ for all x in T. Then G^Ω is A_6, $L_2(q)$, or $U_3(q)$.*

Proof. By double transitivity of G there exists an involution $x = (\alpha, \beta) \cdots$ with x in u^G and $T^x = T$. Let $C = C_T(x)$. Let $I = \langle u^G \cap T \rangle$. Clearly I satisfies the hypothesis of Lemma 1, so $(N_G(I))^{F(I)}$ is doubly transitive. As T is contained in this normalizer, $|F(I)| \equiv 2 \bmod 4$. Also $F(I) = F(u)$, so as x is in u^G, a 2-group acting semiregularly on $F(x)$ has order at most two.

Claim that C acts semiregularly on $F(x)$. Suppose c in $C^{\#}$ fixes a point of $F(x)$. Then c fixes at least two points of $F(x)$ and we can assume c is an involution. But then $\langle x, c \rangle$ is contained in some conjugate of T, contradicting $F(x) \not\supseteq F(c)$.

It follows from Lemma 2, that a Sylow group $T^* = \langle T, x \rangle$ of G is dihedral or semidihedral. As such groups have been classified, G is a known simple group. Consideration of possible doubly transitive representations, of the given sort, of these groups, leads to the desired result.

Note that Lemma 5 classifies all G with T cyclic or generalized quaternion.

4. We now assume Theorem 2 to be false and let G be a counter example of minimal order. Lemma 5 implies that T is abelian but not cyclic. Theorem 1 implies that G is simple. Lemmas 3 and 4 imply that if $U \leq T$ with $F(U) \neq \{\alpha, \beta\}$, then $(N_G U)^{F(U)}$ is doubly transitive. $N_D U$ contains T, so $|F(U)| \equiv 2 \bmod 4$. Therefore by minimality of G we have one of the following.

(1) $T^{F(U)}$ is cyclic and $(T^*)^{F(U)}$ is dihedral or semidihedral where T^* is a Sylow group of G containing T.

(2) $N^{F(U)} = A_6$.

(3) $T^{F(U)}$ is the direct product of two cyclic groups of order 2 and 2^i, $i \geq 2$, and $N^{F(U)} \leq \operatorname{Aut} L$ with $|N^{F(U)} : L|$ odd, where L is generated by $L_2(q^2)$ and the transformation $v : a \to a^q$ on $GF(q^2) \cup \{\infty\}$, $q \equiv 1 \bmod 4$.

Let t be an involution in T with $F(t)$ maximal. Lemma 5 implies $F(t) \neq \{\alpha, \beta\}$, so $N = C_G(t)$ acts as one of the groups discussed above on $F(t)$. Let $B = T_{F(t)}$. There exist u and s in $T^* - T$ with u fixing points of $F(B)$ and s in t^G. Thus $\langle u, B \rangle$ is abelian, so B is in the center of T^*, and s is in the center of a Sylow group S^* containing B. By maximality of $|F(t)|$, $B \cap S_{F(s)} = 1$, so B is isomorphic to a subgroup of $S^{F(s)}$. Since further $F(b) = F(t)$ for b in $B^{\#}$, either B is cyclic or $N^{F(t)} = A_6$.

Assume first $N^{F(t)} = A_6$. Then B is elementary and every coset of T/B contains an involution so T is elementary. N contains an element x, normalized by s, inducing an automorphism of order three on T. So $T = B \times [T, x]$ with $\langle s, [T, x] \rangle$ dihedral of order eight. Therefore a Sylow 2-subgroup of G is the direct product of a dihedral group of order eight with an elementary group of order two or four. With a theorem of Fong [2], this contradicts the fact that G is simple.

So B is cyclic. Assume next $(T^*)^{F(t)}$ is dihedral or semidihedral. Then

$$T^* = \langle s, y, B : y^s = y^{r-1}b, b \in B \rangle.$$

Let $2k$ be the order of y mod B; then r is 0 or k. As $T = \langle y, B\rangle$ is not cyclic, we can assume $y^{2k} = 1$. Let s be conjugate to y^k or $y^k t$ in N. As $s \in t^G$, Lemma 1 implies that t is conjugate to y^k or $y^k t$ in $N_D(T)$. Thus T^* is wreathed and the classification of groups with wreathed Sylow 2-subgroups [1] yields a contradiction.

So s is not conjugate to y^k or $y^k t$. Thus $N^{F(t)} = PGL_2(q)$ and s acts fixed point free on $F(t)$. Thus $B = C_B(s) = \langle t\rangle$. As ys is conjugate to y^k or $y^k t$, ys is an involution and so s inverts y. Thus T^* is the product of a nonabelian dihedral group with a cyclic group of order 2. But this is impossible as G is simple.

Finally assume $N^{F(t)} \leq \operatorname{Aut} L$. Let v be an element of T acting on $F(t)$ as the transformation $a \to a^q$. v fixes common points with some element in $T^* - T$ so v is in the center of T^*. As B is cyclic either $B \leq \langle v\rangle$ or we can choose v to be an involution. In the former case a contradiction can be derived as above; so v is an involution.

N contains a subgroup M of index two with $v \in M - N$.

$$R = T \cap M = \langle B, y\rangle.$$

As G is simple, v is conjugate to an element of R; as above this conjugation takes place in $N_D(T)$. v is not rooted in T so v is not conjugate to the involution in $\langle y\rangle$. Thus $v \in t^G$ and again T^* is the direct product of a nonabelian dihedral group with the four group. So Fong's result yields a contradiction.

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UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS