ON THE ROGERS-RAMANUJAN IDENTITIES AND PARTIAL q-DIFFERENCE EQUATIONS

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1. Introduction

Perhaps the easiest proof of the Rogers-Ramanujan identities is the one expounded (in two different forms) by Rogers and Ramanujan [4]. The main idea is to show that two apparently different q-series both satisfy the q-difference equation

$$f(z) - f(zq) - zqf(zq^2) = 0. \tag{1.1}$$

It is an easy matter to show that if $f(z)$ is analytic at $z = 0$ and $f(0) = 1$, then $f(z)$ is uniquely determined by (1.1). This implies that the two q-series in question are actually identical, and the Rogers-Ramanujan identities follow by specializing z.

The object of this paper is to give a proof of the Rogers-Ramanujan identities which hinges almost entirely on showing that two systems of partial q-difference equations are compatible (i.e. any set of solutions for one system is a set of solutions for the other). In the final section of the paper, we discuss the extension of this technique to other problems in the theory of partitions and q-series identities.

2. Compatible q-difference equations

DEFINITION. Consider the systems of r equations

$$F_i(f_1(x, y), \cdots, f_n(x, y), f_1(xq, y), \cdots, f_n(xq, y), f_1(x, yq), \cdots,$$
$$f_n(x, yq), f_1(xq, yq), \cdots, f_n(xq, yq)) = 0,$$

and

$$G_j(f_1(x, y), \cdots, f_n(x, y), f_1(xq, y), \cdots, f_n(xq, y), f_1(x, yq), \cdots,$$
$$f_n(x, yq), f_1(xq, yq), \cdots, f_n(xq, yq)) = 0,$$

where $1 \leq i \leq s$, $1 \leq j \leq t$. These two systems are said to be *compatible* in case every solution set $\{f_1(x, y), \cdots, f_n(x, y)\}$ of analytic functions in x and y for one system is a solution set for the other system.

LEMMA 1. *Consider the partial q-difference equation*

$$\textstyle\sum_{j=0}^{r} \sum_{k=0}^{s} a_{j,k}(x, y) f(xq^j, yq^k) = b(x, y), \tag{2.1}$$

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where the $a_{j,k}(x, y)$ *and* $b(x, y)$ *are polynomials in* x, y, *and* q, $|q| < 1$. *Furthermore*

$$\text{(2.2)} \qquad \begin{aligned} a_{j,k}(0, 0) &= 1 \quad \text{if } j = k = 0 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then there exists at most one function $f(x, y)$ *which is analytic in* x *and* y *near* $(0, 0)$ *and satisfies* $f(0, y) = f(x, 0) = 1$.

Proof. We let

$$\text{(2.3)} \qquad a_{j,k}(x, y) = \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \alpha_{j,k}(h, i) x^h y^i;$$

$$\text{(2.4)} \qquad b(x, y) = \sum_{t=0}^{\infty} \sum_{u=0}^{\infty} \beta(t, u) x^t y^u;$$

$$\text{(2.5)} \qquad f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}\, x^m y^n,$$

where (2.3) and (2.4) are only formally infinite in that the $a_{j,k}(x, y)$ and $b(x, y)$ are polynomials.

Substituting these series into (2.1) and comparing coefficients of $x^t y^u$ on both sides we obtain

$$\text{(2.6)} \qquad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m+h=t,n+i=u} A_{m,n}\, q^{jm+kn} \alpha_{j,k}(h, i) = \beta(t, u).$$

By (2.2), we may rewrite (2.6) as

$$\text{(2.7)} \qquad A_{t,u} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m+h=t,n+i=u,(h,i)\neq(0,0)} A_{m,n}\, q^{jm+kn} \alpha_{j,k}(h, i) = \beta(t, u).$$

(2.7) shows that $A_{t,u}$ is defined in terms of $A_{m,n}$'s with at least one of m and n less than t and u respectively. Thus a simple double induction establishes the uniqueness of the $A_{t,u}$ given the initial condition

$$\text{(2.8)} \qquad \begin{aligned} A_{t,u} &= 1 \quad t = u = 0 \\ &= 0 \quad t = 0, u \neq 0 \\ &= 0 \quad t \neq 0, u = 0. \end{aligned}$$

LEMMA 2. *Suppose* $c(x, y, q)$ *and* $d(x, y, q)$ *are rational functions of* x, y, *and* q *without singularities at* $(x, y, q) = (0, 0, 0)$ *and* $c(0, 0, q) = 1$, $d(0, 0, q) = 0$. *Then for* $|q| < 1$, *there exists a unique function*, $f(x, y)$, *analytic in both* x *and* y *around* $(x, y) = (0, 0)$ *such that* $f(0, 0) = 1$, *and*

$$\text{(2.9)} \qquad f(x, y) = c(x, y, q) + qd(x, y, q) f(xq, yq).$$

Furthermore if $c(x, y, q)$ *and* $d(x, y, q)$ *have no singularities in* $|x| < W_1$, $|y| < W_2$, *then* $f(x, y)$ *is analytic in* x *and* y *in this region.*

Proof. Clearly if $f(x, y)$ does exist, then setting $x = y = 0$ in (2.9), we obtain

$$f(0, 0) = 1.$$

Iterating (2.9) n times we obtain

$$f(x, y) = \sum_{j=0}^{n-1} c(xq^j, yq^j, q)q^j \prod_{r=0}^{j-1} d(xq^r, yq^r, q) + q^n f(xq^n, yq^n) \prod_{r=0}^{n-1} d(xq^r, yq^r, q). \tag{2.10}$$

This suggests that

$$f(x, y) = \sum_{j=0}^{\infty} c(xq^j, yq^j, q)q^j \prod_{r=0}^{j-1} d(xq^r, yq^r, q). \tag{2.11}$$

Indeed if $f(x, y)$ is defined by (2.11) then the ratio test guarantees that f is analytic in x and y in the neighborhood of $(0, 0)$ for $|q| < 1$ $|x| < W_1$, $|y| < W_2$, and a simple shift of the summation index shows that (2.9) is satisfied.

Finally if $\varphi(x, y)$ is also a solution of the prescribed type, then $\varphi(x, y)$ satisfies (2.10)

Letting $n \to \infty$ in (2.10), we note that

$$q^n \to 0, \quad \prod_{r=0}^{n-1} d(xq^r, yq^r, q) \to 0, \quad \varphi(xq^n, yq^n) \to 1,$$

and consequently $\varphi(x, y)$ satisfies (2.11). Hence the solution is unique.

THEOREM 1. *If* $|q| < 1$,

(A)
$$g(x, y) - yh(x, y) = 1 - y + y^2xq(1 - xq)h(xq, y) \tag{2.12}$$
$$h(x, y) = 1 - y + y(1 - xq)g(xq, y) \tag{2.13}$$

and

(B)
$$\gamma(x, y) = 1 - x^2y^2q^2 - x^2y^3q^3 \frac{(1 - xq)}{(1 - yq)} \gamma(xq, yq) \tag{2.14}$$
$$\eta(x, y) = 1 - xyq - x^2y^3q^4 \frac{(1 - xq)}{(1 - yq)} \eta(xq, yq) \tag{2.15}$$

then (A) *and* (B) *are compatible systems of equations with a unique analytic solution set. Furthermore the solutions are analytic in* x *and* y *for all* x *and* $|y| < |q|^{-1}$.

Proof. First we note from system (B) that

$$\gamma(x, 0) = \gamma(0, y) = \eta(x, 0) = \eta(0, y) = 1.$$

From system (A) we have clearly $g(x, 0) = h(x, 0) = 1$. Setting $x = 0$ in (A), we obtain a system of two equations in the two unknowns $g(0, y)$, $h(0, y)$, and the unique solution set is $g(0, y) = h(0, y) = 1$ provided $y \neq \pm 1$. Thus if analytic $g(x, y)$ and $h(x, y)$ exist, $g(0, y) = h(0, y) = 1$ for all y.

Thus by Lemma 2, system (B) has a unique solution set and the solutions are analytic for all x and $|y| < |q|^{-1}$. Substituting (2.13) into (2.12), we find by Lemma 1 that at most one $g(x, y)$ exists, and thus by (2.13) at most one $h(x, y)$ exists. Consequently if we can show that $\gamma(x, y)$ and $\eta(x, y)$ (the unique solution set of system (B))) satisfy (2.12) and (2.13)), then Theorem 1 will be proved.

Let

$$L(x, y) = \gamma(x, y) - y\eta(x, y). \tag{2.16}$$

If we multiply equation (2.15) by y and subtract from equation (2.14), we obtain

$$L(x, y) = 1 - y + xy^2q(1 - xq) - x^2y^3q^3 \frac{(1 - xq)}{(1 - yq)} L(xq, yq). \tag{2.17}$$

Define $H(x, y)$ by the following equation.

$$L(x, y) = 1 - y + y^2xq(1 - xq)H(xq, y). \tag{2.18}$$

Substituting (2.18) into (2.17) we obtain

$$H(xq, y) = 1 - xyq^2 - x^2y^3q^6 \frac{(1 - xq^2)}{(1 - yq)} H(xq^2, yq). \tag{2.19}$$

Replacing x by xq^{-1} in (2.18), we obtain

$$H(x, y) = 1 - xyq - x^2y^3q^4 \frac{(1 - xq)}{(1 - yq)} H(xq, yq). \tag{2.20}$$

Thus $H(x, y)$ satisfies (2.15), and hence $H(x, y) = \eta(x, y)$ by Lemma 2. Consequently

$$\begin{aligned} \gamma(x, y) - y\eta(x, y) &= L(x, y) \\ &= 1 - y + y^2xq(1 - xq)H(xq, y) \\ &= 1 - y + y^2xq(1 - xq)\eta(xq, y). \end{aligned} \tag{2.21}$$

Thus $\gamma(x, y)$ and $\eta(x, y)$ satisfy (2.12).

Define $M(x, y)$ by the following equation.

$$M(xq^{-1}, y) = 1 - y + y(1 - x)\gamma(x, y). \tag{2.22}$$

Substituting (2.22) into (2.14), we obtain

$$M(xq^{-1}, y) = 1 - xy - x^2y^3q^2 \frac{(1 - x)}{(1 - yq)} M(x, yq). \tag{2.23}$$

Replacing x by xq in (2.23), we find

$$M(x, y) = 1 - xyq - x^2y^3q^4 \frac{(1 - xq)}{(1 - yq)} M(xq, yq). \tag{2.24}$$

Thus $M(x, y)$ satisfies (2.15), and hence $M(x, y) = \eta(x, y)$ by Lemma 2. Therefore

$$1 - y + y(1 - xq)\gamma(xq, y) = M(x, y) = \eta(x, y). \tag{2.25}$$

Thus $\gamma(x, y)$ and $\eta(x, y)$ satisfy (2.13), and so (A) and (B) are compatible systems.

COROLLARY (The Rogers-Ramanujan identities).

$$(2.26)\quad 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}=\prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1};$$

$$(2.27)\quad 1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}=\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})^{-1}.$$

Proof. By $\gamma(x, y)$ and $\eta(x, y)$ we denote the unique solution set for the compatible systems (A) and (B) of Theorem 1. If we set $y = x$ in (2.14) we obtain

$$(2.28)\qquad \gamma(x, x) = 1 - x^4q^2 - x^5q^3\gamma(xq, xq).$$

Iteration of this equation yields

$$(2.29)\qquad \gamma(x, x) = \sum_{n=0}^{\infty} (-1)^n x^{5n} q^{(n/2)(5n+1)}(1 - x^4q^{4n+2}).$$

Therefore by the Jacobi identity [3, p. 282],

$$(2.30)\qquad \gamma(1, 1) = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+2})(1 - q^{5n+3})$$

Setting $y = x$ in (2.15), we obtain

$$(2.31)\qquad \eta(x, x) = 1 - x^2q - x^5q^4\eta(xq, xq).$$

Iteration yields in this case

$$(2.32)\qquad \eta(x, x) = \sum_{n=0}^{\infty} (-1)^n x^{5n} q^{(n/2)(5n+3)}(1 - x^2q^{2n+1}).$$

Thus by the Jacobi identity [3, p. 282],

$$(2.33)\qquad \eta(1, 1) = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+1})(1 - q^{5n+4}).$$

If we set $y = 1$ in system (A) and solve for $g(x, 1) = \gamma(x, 1)$, we obtain

$$(2.34)\quad \gamma(x, 1) = (1 - xq)\gamma(xq, 1) + xq(1 - xq)(1 - xq^2)\gamma(xq^2, 1).$$

Thus if $G(x) = \gamma(x, 1) \prod_{n=1}^{\infty} (1 - xq^n)^{-1}$, then

$$(2.35)\qquad G(x) = G(xq) + xqG(xq^2).$$

We now proceed as in [3, p. 293] and obtain

$$(2.36)\quad \gamma(x,1)\prod_{n=1}^{\infty}(1-xq^n)^{-1} = G(x) = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}x^n}{(1-q)\cdots(1-q^n)}.$$

From (2.13), we find

$$(2.37)\quad \eta(x,1)\prod_{n=1}^{\infty}(1-xq^n)^{-1} = G(xq) = 1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}x^n}{(1-q)\cdots(1-q^n)}.$$

Thus setting $x = 1$ in (2.36) and combining with (2.30) we obtain

$$(2.38)\quad \begin{aligned} 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)} &= \gamma(1,1)\prod_{n=1}^{\infty}(1-q^n)^{-1} \\ &= \prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}. \end{aligned}$$

Setting $x = 1$ in (2.37) and combining with (2.33), we obtain

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \eta(1,1)\prod_{n=1}^{\infty}(1-q^n)^{-1} \tag{2.39}$$
$$= \prod_{n=0}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}.$$

Hence the corollary is proved.

3. Extended results

The preceding technique can easily be extended to give a full proof of the Rogers-Ramanujan-Gordon identities utilizing the analytic-combinatorial approach of [1]. In this case there are two systems of $(k + 1)$-equations. Namely

(A′)

$$(3.1)\quad C_{k,i}(x, y) - yC_{k,i-1}(x, y) = 1 - y + y^i(xq)^{i-1}(1 - xq)C_{k,k-i+1}(xq; y), \qquad 1 \leqq i \leqq k;$$

$$(3.2)\quad C_{k,0}(x, y) = 0.$$

(B′)

$$(3.3)\quad C_{k,i}(x, y) = 1 - x^iy^iq^i - x^ky^{k+1}q^{2k-i+1}\frac{(1 - xq)}{(1 - yq)}C_{k,i}(xq; yq), \qquad 0 \leqq i \leqq k.$$

The technique may also be extended to cover the results considered in [2]. It is to be hoped that general theorems on compatible systems of partial q-difference equations could be found. Such results would surely have interesting ramifications in the theory of basic hypergeometric series and partitions.

References

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