AMENABLE GROUPS AND VARIETIES OF GROUPS[1]

BY

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1. Introduction

It has been conjectured on the basis of current knowledge that the class of amenable groups consists precisely of all groups G which do not possess a subgroup isomophic to the free group on two generators.

In this paper we will not settle this conjecture. However this work is motivated by a desire to give an algebraic description of the class of all amenable groups. If the conjecture mentioned were true then certainly any group satisfying a nontrivial law would be amenable.

In an attempt to determine which varieties of groups consist entirely of amenable groups the notion of a uniformly amenable group is introduced.

In Section 4 uniformly amenable groups are defined and elementary properties are derived. The motivation behind the definition of uniformly amenable groups lies within the framework of nonstandard analysis and many of the proofs are nonstandard.

In Section 5 we derive results indicating the relationships between varieties and uniformly amenable groups. We prove that the variety generated by a group G is amenable if and only if the direct product of a countably infinite number of copies of G is uniformly amenable.

Section 6 contains a necessary and sufficient condition that a group G satisfy a nontrivial law. This condition appears to have been unknown previously and poses some interesting algebraic questions.

Section 2 contains a brief summary of elementary information about varieties of groups together with a few lemmas about varieties necessary in the sequel.

Section 3 contains a brief summary of information about amenable groups.

No attempt is made to explain the methods of nonstandard analysis used here. The reader is referred to [1] and [2] for further information.

2. Varieties of groups

We present here some of the ideas and results necessary in the sequel. Further information can be obtained by consulting [3].

Let $X = \{x_i \mid i \text{ is a positive integer}\}$ be an alphabet. Let X_m be the free group of rank n generated by $\{x_i \mid 1 \leq i \leq n\}$, and let X_∞ be the free group generated by X. If $w \in X_n$ we will often write $w(x_1, \cdots, x_n)$ to denote w.

If G is any group and $a_1, a_2, \cdots, a_n \in G$, then by

$$w(a_1, a_2, \cdots, a_n) \quad \text{for } w \in X_n$$

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we mean $\alpha(w)$ where α is the unique homomorphism from X_n into G with $\alpha(x_i) = a_i$ for $1 \leq i \leq n$.

A word $w(x_1, x_2, \cdots, x_n)$ is called an identical relation or a law for a group G if and only if $w(a_1, a_2, \cdots, a_n) = 1$ for every $a_1, a_2, \cdots, a_n$ in G. A law is called trivial if it is satisfied by all groups and this happens if and only if $w = 1$.

A variety of groups is the class of all groups satisfying each law in a given set of laws. If $L \subseteq X_\infty$ then we denote by $V(L)$ the variety of all groups for which the words in L are laws.

We mention a few examples of varieties. The class of all abelian groups is obviously $V(x_1^{-1}x_2^{-1}x_1x_2)$. For any positive integer t the class of all solvable groups of derived length $\leq t$ form a variety. The variety $V(x^m)$ is called the Burnside variety of exponent m. The class of all groups is a variety we will denote by $\mathfrak{O}$.

A reduced free group in a variety V is a group with a set of generators S such that any map from S into a group in V can be extended to a homomorphism. For any cardinal number h, V contains a reduced free group with h generators. For every natural number n, $F_n(V)$ will denote a reduced free group with n generators. A reduced free group with a countably infinite set of generators will be denoted by $F_\infty(V)$.

The Cartesian product of $\{A_\lambda \mid \lambda \in \Lambda\}$ of groups will be denoted by $C_\lambda \prod_{\lambda\in\Lambda} A_\lambda$, and the direct product will be denoted by $\prod_{\lambda\in\Lambda} A_\lambda$. The direct product of a countably infinite number of groups isomorphic to a group G will be denoted by $\prod G$.

Let V be a variety of groups. A set of groups $\mathfrak{D} \subseteq V$ is said to discriminate V if and only if for every finite set of words W in X_∞ which are not laws for V there exists $G \in \mathfrak{D}$ and elements $g_1, g_2, \cdots, g_n \in G$ such that

$$w(g_1, g_2, \cdots, g_n) \neq 1$$

for any $w \in W$. (n is some integer for which $W \subseteq X_n$.) Put another way any finite set of nonlaws for V can be simultaneously falsified.

The set of all varieties of groups is partially ordered under inclusion. It is a complete lattice where $U \wedge V = U \cap V$. The variety generated by a class of groups $\mathfrak{C}$ is denoted by $V(\mathfrak{C})$ and defined by $V(\mathfrak{C}) = \bigcap_{V \supseteq \mathfrak{C}} V$.

Multiplication can be defined on varieties where UV is the variety of all extensions of groups in U by groups in V.

LEMMA 2.1. *A set $\mathfrak{D}$ of groups generates $\mathfrak{O}$, the variety of all groups, if and only if $\mathfrak{D}$ discriminates $\mathfrak{O}$.*

Proof. If $\mathfrak{D}$ discriminates $\mathfrak{O}$ then $\mathfrak{D}$ obviously generates $\mathfrak{O}$. Suppose $\mathfrak{D}$ generates $\mathfrak{O}$. We must show that any finite set of words in X_∞ not containing the null word can be simultaneously falsified.

Suppose W is a finite subset of X_∞ with $1 \notin W$ and let n be chosen such that $W \subseteq X_n$. It suffices to show that there exists a word $t \in X_n$, $t \neq 1$ such that falsifying t will falsify every word in W. Then the result follows, for t can be falsified in $\mathfrak{D}$ since we assumed $\mathfrak{D}$ generates $\mathfrak{O}$.

We prove that such a word t exists by induction on the number of words in W. Clearly we need only consider the case where W contains two words w_1 and w_2. If $w_1 w_2 \neq w_2 w_1$ let $t = w_1 w_2 w_1^{-1} w_2^{-1}$. If $w_1 w_2 = w_2 w_1$ then there exists a word $w \in X_n$ and integers p and q such that $w^p = w_1$ and $w^q = w_2$. In this case let $t = w^{pq}$.

The result follows.

LEMMA 2.2. *Let V be a variety of groups and let $\mathfrak{D}$ be a set of groups contained in V. Let M be a full structure whose individuals include all elements of all groups in $\mathfrak{D}$ all elements of X_∞ and all natural numbers. Let $^*M$ be an enlargement of M. Then $\mathfrak{D}$ discriminates V if and only if $F_\infty(V)$ is isomorphic to a subgroup of a group $D \in {}^*\mathfrak{D}$.*

Proof. Suppose $\mathfrak{D}$ discriminates V.

If W is a finite set of nonlaws for V and $W \subseteq X_n$ then there exists $D \in \mathfrak{D}$ and $d_1, d_2, \cdots, d_n \in D$ such that $w(d_1, d_2, \cdots, d_n) \neq 1$ for any $w \in W$.

Now let $Y \subseteq X_\infty$ consist of all nonlaws for V. Let E be a *-finite subset of $^*Y$ such that $Y \subseteq E$ and let $\omega \in {}^*N - N$ be chosen so that $E \subseteq X_\omega$.

Since every element of Y is a nonlaw for V so is every element of E. We get: There exists $D \in {}^*\mathfrak{D}$ and $d_1, d_2, \cdots, d_\omega \in D$ such that

$$e(d_1, d_2, \cdots, d_\omega) \neq 1$$

for every $e \in E$. In particular the only relations on the elements d_i are identities for V and so the group they generate is free reduced on an infinite set of generators and so $F_\infty(V)$ is certainly isomorphic to a subgroup of D.

On the other hand suppose $\mathfrak{D}$ does not discriminate V. Then there is a finite set W of nonlaws for V which cannot be simultaneously falsified in any $D \in \mathfrak{D}$. Suppose $W \subseteq X_n$. We have:

If $D \in \mathfrak{D}$ and $d_1, \cdots, d_n \in D$ then there exists $w \in W$ with

$$w(d_1, \cdots, d_n) = 1.$$

So, if $D \in {}^*\mathfrak{D}$ and $d_1, \cdots, d_n \in D$ then there exists $w \in W$ with

$$w(d_1, \cdots, d_n) = 1.$$

That is, every n elements in any group in $^*\mathfrak{D}$ satisfy a relation which is not a law for V. So $F_\infty(V)$ is not isomorphic to a subgroup of D for any $D \in {}^*\mathfrak{D}$ and the proof is complete.

THEOREM 2.3. *If $\mathfrak{C}$ is a nonempty class of groups which contains all homomorphic images, subgroups, direct products and nonstandard models of groups in $\mathfrak{C}$ then $\mathfrak{C}$ is a variety of groups.*

Proof. Let Q be the set of all words in X_∞ which are not laws in $\mathfrak{C}$. We may assume Q is not empty. For every $q \in Q$ let $G_q \in \mathfrak{C}$ be chosen with $G_q \notin V(q)$.

Now $\prod_{q \in Q} G_q = H$ is in $\mathcal{C}$ and discriminates $V(\mathcal{C})$. So $\mathcal{C} \subseteq V(H) = V(\mathcal{C})$. Since $\mathcal{C}$ contains all nonstandard models of groups in $\mathcal{C}$ and since H discriminates $V(\mathcal{C})$, Lemma 2.2 implies there is a group in $\mathcal{C}$ containing an isomorphic copy of $F_\infty(V(\mathcal{C}))$ and hence $\mathcal{C}$ contains $F_\infty(V(\mathcal{C}))$. Since $\mathcal{C}$ is closed under homomorphic images $\mathcal{C}$ contains every finitely generated group in $V(\mathcal{C})$. Now every group is the directed union of its finitely generated subgroups and any directed union of groups is a homomorphic image of a direct product of the groups in the directed set. So $\mathcal{C} \supseteq V(\mathcal{C})$ and the proof is complete.

COROLLARY 2.4 (Birkoff). *If $\mathcal{C}$ is a nonempty class of groups which contains all homomorphic images, subgroups, and cartesian products of groups in $\mathcal{C}$ then $\mathcal{C}$ is a variety of groups.*

Proof. In the proof of the theorem the only nonstandard model required is an enlargement of an arbitrary group in $\mathcal{C}$. But such a model can always be realized by an ultrapower which is a quotient of a cartesian power. The result follows.

3. Amenable groups

In this section we give a brief summary of some pertinent information about amenable groups. Further information is available in such sources as [4], [5] and [6].

Let S be a discrete semigroup. Let $m(S)$ be the space of bounded real-valued functions on S endowed with the sup norm (i.e. l_∞).

A linear functional μ on $m(S)$ is called a mean if

$$\inf_{x \in S} f(x) \leq \mu(f) \leq \sup_{s \in S} f(x)$$

for every $f \in m(S)$.

For a fixed element $a \in S$ let ${}_af$ $[f_a]$ be the function on S such that ${}_af(x) = f(ax)$ $[f_a(x) = f(xa)]$ for all $x \in S$.

A left [right] invariant mean is a mean such that $\mu({}_xf) = \mu(f)$ $[\mu(f_x) = \mu(f)]$ for all $f \in m(S)$, $x \in S$. If a mean μ satisfies $\mu({}_xf) = \mu(f_x) = \mu(f)$ for $f \in m(S)$ and $x \in S$, then μ is called a two-sided invariant mean or simply an invariant mean.

If a group has a left or a right invariant mean it has a two-sided invariant mean and is called amenable. Several equivalent formulations of this concept are known and may be found in the references mentioned. The proposition most suitable for our purposes is due to Følner [7]. If B is a finite set let $|B|$ denote the number of elements in B.

THEOREM (Følner). *A group G is amenable if and only if for every finite subset A of G and for every k with $0 < k < 1$ there exists a finite subset E of G with*

$$|E \cap Ea| \geq k|E|$$

for every $a \in A$.

Homomorphic images and subgroups of amenable groups are amenable, as well as extensions of amenable groups by amenable groups.

Cartesian products of amenable groups are not necessarily amenable. However direct products and directed unions of amenable groups are amenable.

A group is amenable if and only if every finitely generated subgroup is amenable.

Finite groups are amenable. Abelian groups are amenable. All other known amenable groups can be formed from these classes using the properties we have just mentioned.

On the other hand, X_2 is not amenable, and every known group which is not amenable contains X_2 as a subgroup.

4. Uniformly amenable groups

If G is an amenable group and $^*G$ is a nonstandard model for G then $^*G$ may not be amenable. Examples will become evident later. The concept of a uniformly amenable group is designed to force $^*G$ to be amenable. In fact this constitutes the major result of this section.

In the sequel let N be the natural numbers and let $I = (0, 1)$, the open unit interval.

DEFINITION 4.1. A group G is called uniformly amenable (or u.a.) if and only if there exists a function $F : N \times I \to N$ satisfying: Given any $n \in N$ and $A \subseteq G$ with $|A| \leq n$ and given any $k \in I$ there exists $E \subseteq G$ with $|E| \leq F(n, k)$ such that

$$|E \cap Ea| \geq k|E|$$

for every $a \in A$.

Obviously a uniformly amenable group is amenable since Følner's condition is satisfied.

THEOREM 4.2. *A group G is uniformly amenable if and only if every nonstandard model for G is amenable.*

Let M be a full structure whose individuals include the elements of G and all real numbers. Let $^*M$ be a nonstandard model for M.

Suppose G is uniformly amenable.

Given any internal subset A of $^*G$ with $^*|A| \leq n \in {}^*N$ and given any $k \in {}^*I$ there exists an internal subset $E \subseteq {}^*G$ with $^*|E| \leq {}^*F(n, k)$ such that $^*|E \cap Ea| \geq k^*|E|$ for every $a \in A$.

Since every finite subset of $^*G$ is internal we get immediately that $^*G$ is uniformly amenable where F satisfies the necessary conditions since the restriction of $^*F$ to $N \times I$ is F.

Now suppose $^*G$ is amenable. To define a function $F : N \times I \to N$ which will guarantee that G is uniformly amenable we pick $n \in N$ and $k \in I$ arbitrarily and show a suitable choice of $F(n, k)$ can be made.

Let $n \in N$ and $k \in I$ be chosen. Since $^*G$ is amenable we claim:

There exists $m \in {}^*N$ such that for every internal subset A of $^*G$ with $^*|A| \leq n$ there exists an internal subset E of $^*G$ with $^*|E| \leq m$ such that $^*|E \cap Ea| > k\ {}^*|E|$ for every $a \in A$.

This follows by taking $m \in {}^*N - N$ and remembering Følner's condition and the fact that finite sets are always internal.

Now there exists $m \in N$ such that for every subset A of G with $|A| \leq n$ there exists a subset E of G with $|E| \leq m$ such that $|E \cap Ea| > k\,|E|$ for every $a \in A$. Choosing such an m for $F(n, k)$ the proof is complete.

Note that G is a u.a. group if and only if $^*G$ is a u.a. group. Also, $^*G$ is amenable if and only if $^*G$ is uniformly amenable.

We now establish some properties of u.a. groups.

THEOREM 4.5. *Homomorphic images and subgroups of u.a. groups are uniformly amenable. Also extensions of u.a. groups by u.a. groups are uniformly amenable.*

Proof. Let H, K be arbitrary groups, and let G be an arbitrary extension of H by K. Let M be a full structure whose individuals include the elements of H, G, K and all real numbers. Let $^*M$ be a nonstandard model for M.

Now if H is u.a., and L is a subgroup of H we have $^*L$ is a subgroup of $^*H$ which is amenable by Theorem 4.2. So $^*L$ is amenable and L is u.a., by Theorem 4.2.

Now consider the exact sequence $1 \to H \to G \to K \to 1$. This yields an exact sequence $1 \to {}^*H \to {}^*G \to {}^*K \to 1$. If two of the three groups H, G, and K are u.a. then two of the three groups $^*H$, $^*G$, and $^*K$ are amenable by Theorem 4.2. Hence all three are amenable and by Theorem 4.2, H, G and K are uniformly amenable.

COROLLARY 4.6. *Finite products of u.a. groups are uniformly amenable.*

Proof. The result is obvious since extensions of u.a. groups by u.a. groups are uniformly amenable.

5. Amenable varieties

It is not known whether all groups which satisfy a nontrivial identity are amenable. We approach this problem by describing necessary and sufficient conditions for a variety to contain only amenable groups.

DEFINITION 5.1. A class $\mathcal{C}$ of groups if called amenable if and only if all groups in $\mathcal{C}$ are amenable.

DEFINITION 5.2. A class $\mathcal{C}$ of groups is called uniformly amenable if and only if there exists a function $F : N \times I \to N$ such that each group in $\mathcal{C}$ is uniformly amenable with F satisfying the conditions of Definition 4.1.

LEMMA 5.3. *Let $\mathcal{S}$ be a set of groups. Let M be a full structure whose indivi-*

duals include the elements of all groups in $\mathcal{S}$ and all real numbers. Let $^*M$ be a nonstandard model for M. Then $\mathcal{S}$ is uniformly amenable if and only if $^*\mathcal{S}$ is amenable.

Proof. The function F satisfying the conditions of Definition 4.1 for every group in $\mathcal{S}$ also satisfies the conditions of Definition 4.1 for every group in $^*\mathcal{S}$. To see this consider this statement in M:

Given any $G \in \mathcal{S}$, any $n \in N$, any $A \subseteq G$ with $|A| \leq n$, and any $k \in I$, there exists $E \subseteq G$ with $|E| \leq F(n, k)$ such that $|E \cap Ea| \geq k|E|$ for every $a \in A$.

In $^*M$ we get the desired result by observing that every finite subset of a group in $^*\mathcal{S}$ is internal and that the restriction of $^*F$ to $N \times I$ is F.

Now suppose $^*\mathcal{S}$ is amenable. Let $n \in N$, $k \in I$ be chosen. We claim:

There exists $m \in {}^*N$ such that if $S \in {}^*\mathcal{S}$ and A is an internal subset of S with $^*|A| \leq n$ then there exists an internal subset E of S with $^*|E| \leq m$ such that $^*|E \cap Ea| \geq k^*|E|$ for every $a \in A$.

This follows by taking $m \in {}^*N - N$ and Følner's condition.

Hence there exists $m \in N$ such that if $S \in \mathcal{S}$ and A is a subset of S with $|A| \leq n$ then there exists a subset E of S with $|E| \leq m$ such that $|E \cap Ea| \geq k|E|$ for every $a \in A$. Choosing such an m for $F(n, k)$ the proof is complete.

THEOREM 5.5. *Let $\mathfrak{D}$ be a set of groups which discriminates a variety V. Then V is amenable if and only if $\mathfrak{D}$ is uniformly amenable.*

Proof. Let M be a full structure whose individuals include all elements of all groups in $\mathfrak{D}$ and all real numbers. Let $^*M$ be an enlargement for M.

If $\mathfrak{D}$ is uniformly amenable then every group in $^*\mathfrak{D}$ is uniformly amenable by Lemma 5.3. Now by Lemma 2.2, $F_\infty(V)$ is isomorphic to a subgroup of a group in $^*\mathfrak{D}$, and is not only amenable but uniformly amenable by Theorem 4.5. Now V is amenable since every group in V is the directed union of groups which are homomorohic images of $F_\infty(V)$.

If $\mathfrak{D}$ is not uniformly amenable there exists a nonamenable group $G \in {}^*\mathfrak{D}$ by Lemma 5.3. But $^*\mathfrak{D} \subseteq V$ since $\mathfrak{D} \subseteq V$. Therefore V is not amenable.

COROLLARY 5.6. *If D is a group which discriminates a variety V then V is amenable if and only if D is uniformly amenable.*

Proof. This is simply a special case of the theorem.

COROLLARY 5.7. *Let G be any group. Then $V(G)$ is amenable if and only if $\prod G$ is uniformly amenable. ($\prod G$ is the direct product of a countably infinite number of copies of G.)*

Proof. This follows from Corollary 5.6 and the fact that $\prod G$ discriminates $V(G)$ for any group G.

It is not known if $\prod G$ can be replaced by G in Corollary 5.7. In particular is $\prod G$ u.a. if G is u.a.?

COROLLARY 5.8. *V is amenable if and only if V is uniformly amenable.*

Proof. If V is u.a. it is obviously amenable. If V is amenable let $\mathfrak{D}$ consist of one group from each isomorphism class of finitely generated groups in V. $\mathfrak{D}$ discriminates V and by the theorem is u.a. Now since $\mathfrak{D}$ contains a copy of every finitely generated group in V a function F satisfying Definition 5.2 for $\mathfrak{D}$ also satisfies Definition 5.2 for V, and V is u.a.

COROLLARY 5.9. *If $\mathcal{G}$ is a uniformly amenable set of groups then $V(\mathcal{G}) \neq \mathfrak{O}$. In particular if G is a u.a. group then G satisfies a nontrivial law.*

Proof. Suppose $V(\mathcal{G}) = \mathfrak{O}$. Then $\mathcal{G}$ discriminates $\mathfrak{O}$ by Lemma 2.1, which forces $\mathfrak{O}$ to be amenable by the theorem. But $\mathfrak{O}$ is obviously not amenable and so $V(\mathcal{G}) \neq \mathfrak{O}$. The remainder of the corollary is now obvious.

It is not known whether a group satisfying a nontrivial identity is amenable. In fact, suppose G satisfies a nontrivial law and is amenable. It is not known whether G must then be uniformly amenable.

LEMMA 5.10. *All solvable groups are uniformly amenable. If G is a finite group then $V(G)$ is amenable.*

Proof. If G is solvable any nonstandard model for G is clearly solvable of the same derived length. Hence all nonstandard models for G are amenable and G is u.a.

If G is finite, $V(G)$ is locally finite and so is amenable. (See [3, 15.71].)

LEMMA 5.11. *If U and V are amenable varieties then $U \wedge V$, $U \vee V$, and UV are all amenable. If Y is any subvariety of U then Y is amenable.*

The last conclusion and the result that $U \wedge V$ is amenable are obvious. Now UV is amenable since extensions of amenable groups by amenable groups are amenable, and $U \vee V$ is amenable since $U \vee V \subseteq UV$.

6. Laws in groups

The fact that u.a. groups satisfy nontrivial laws motivated a search for similar conditions which would be necessary and sufficient for a group to satisfy a law. The object of this section is to establish such conditions. The results appear to be new.

By the length of a word in X_∞ we will always mean the number of letters in a reduced word. If φ is a map from X_n into a group, φ_m will denote the restriction of φ to the set of all words in X_n of length $\leq m$.

DEFINITION 6.1. *Let $\mathfrak{C}$ be a class of groups, and let $u \in N$. We say $B(u)$ holds in $\mathfrak{C}$ if and only if for every group $G \in \mathfrak{C}$ and every homomorphism φ from X_2 into G, there exists a finite set $E \subseteq G$ with*

$$|\varphi_u^{-1}[E \cap E\varphi(x_i)]| \geq \tfrac{1}{2}|E|$$

for $i = 1, 2$.

THEOREM 6.2. *Let $\mathcal{S}$ be a set of groups. There exists a nontrivial law w such that $\mathcal{S} \subset V(w)$ if and only if there exists a positive integer u such that $B(u)$ holds in $\mathcal{S}$.*

In order to prove this theorem we will need several lemmas.

LEMMA 6.3. *Let $\mathcal{G}$ be a set of groups. Let M be a full structure whose individuals include all elements of all groups in $\mathcal{G}$, the elements of X_2 and all real numbers. Let ${}^*M$ be an enlargement of M. Suppose $B(u)$ holds in $\mathcal{G}$ for some positive integer u. Then $B(u)$ holds in ${}^*\mathcal{G}$.*

Proof. Let Y be a group in ${}^*\mathcal{G}$ and let θ be a homomorphism from X_2 into Y with $\theta(x_i) = y_i$ for $i = 1, 2$.

For every group $G \in \mathcal{G}$ and for every $g_1, g_2 \in G$ there is a homomorphism $\varphi : X_2 \to G$ with $\varphi(x_i) = g_i$ for $i = 1, 2$.
Therefore there is an internal homomorphism $\psi : {}^*H_2 \to Y$ with $\psi(x_i) = y_i$ for $i = 1, 2$. Clearly θ is the restriction of ψ to X_2.
Since $B(u)$ holds in $\mathcal{G}$ we get a *-finite set $E \subseteq Y$ such that

$$(6.4) \qquad {}^*|\psi_u^{-1}[E \cap E\psi(x_i)]| \geq \tfrac{1}{2}{}^*|E|$$

for $i = 1, 2$. Now $\psi_u = \theta_u$ since $\psi = \theta$ on X_2. Also, the left side of (6.4) is clearly finite implying that E must be finite. This gives

$$(6.5) \qquad |\theta_u^{-1}[E \cap E\theta(x_i)]| \geq \tfrac{1}{2}|E|$$

and we have shown $B(u)$ holds in ${}^*\mathcal{G}$ establishing the lemma.

LEMMA 6.5. *If $W(n) = \{w \in X_2 \mid l(w) \leq n\}$ then $|W(n)| = 2 \cdot 3^n - 1$ and*

$$(6.6) \qquad \frac{|W(n) \cap W(n)x_i|}{|W(n)|} = \frac{3^n - 1}{2 \cdot 3^n - 1}$$

for $i = 1, 2$.

Since $|W(0)| = 1$, $|W(1)| = 5$ and the number of words of length k is triple the number of words of length $k - 1$ for $k \geq 2$ then $|W(n)| = 2 \cdot 3^n - 1$ by induction. Clearly the number of words ending in one letter is the same as for any other. This gives $(3^n - 1)/2$ words in $W(n)$ ending in any fixed letter.
Now any word in $W(n)$ ending in x_i is clearly in $W(n) \cap W(n)x_i$. Also, if w is any word in $W(n)$ ending in x_i^{-1} then wx_i is in $W(n) \cap W(n)x_i$. It is easily seen that these sets of words are disjoint and account for all words in $W(n) \cap W(n)x_i$. This gives $|W(n) \cap W(n)x_i| = 3^n - 1$ and (6.6) follows.

LEMMA 6.7. *If a group G satisfies a nontrivial law w then $B(u)$ holds in G for some positive integer u which is dependent only on the law w.*

Proof. The law w implies a nontrivial law h in ≤ 2 variables. Let u be the length of h.

Now we show $B(u)$ holds in G by setting $E = \varphi(W(u))$ where φ is an arbitrary homomorphism from X_2 into G. We have

$$\varphi_u^{-1}[\varphi(W(u)) \cap \varphi(W(u))\varphi(x_i)] \supseteq W(u) \cap W(u)x_i$$

for $i = 1, 2$. Also, since $\varphi(h) = 1$, $|\varphi(W(u))| \leq |W(u)| - 1 = 2\cdot(3^u - 1)$ by Lemma 6.5. This gives

$$|\varphi_u^{-1}[\varphi(W(u)) \cap \varphi(W(u))\varphi(x_i)]|$$
$$\geq |W(u) \cap W(u)x_i| = 3^u - 1 \geq \tfrac{1}{2}|\varphi(W(u))|.$$

Therefore $B(u)$ holds in G and the proof is complete.

LEMMA 6.8. *There is no finite subset E of X_2 such that*

$$(6.9) \qquad |E \cap Ex_i| \geq \tfrac{1}{2}|E| \quad \textit{for } i = 1 \textit{ and } i = 2.$$

Proof. We proceed by contradiction. Suppose a finite set $E \subseteq X_2$ and satisfying (6.9) does exist. Let E_i be the set of words in E whose final letters are x_i or x_i^{-1}.

Now if $w \in E \cap Ex_i$ then $w = ux_i$ for some u in E. This implies that either $w \in E_i$ and u does not, or w does not belong to E_i and u does. This gives

$$(6.10) \qquad |E \cap Ex_i| \leq |E_i| \quad \text{for } i = 1, 2.$$

Combining (6.9) and (6.10) we get $|E_1| + |E_2| \geq |E|$. But E_1 and E_2 are disjoint subsets of E. Therefore

$$(6.1) \qquad |E| = |E_1| + |E_2|$$

and $E = E_1 \cup E_2$. Hence the null word, $1 \notin E$. However, equality in (6.11) implies equality in (6.9) and (6.10). This means that if $w \in E$ then the word formed by removing the final letter from w is in E. By taking w to be of minimum length in E the contradiction is obtained.

Proof of Theorem 6.2. Suppose $\mathcal{S} \subseteq V(w)$ for some law $w \neq 1$. Then by Lemma 6.7 $B(u)$ holds in $\mathcal{S}$ for some positive integer u.

Now suppose $B(u)$ holds in $\mathcal{S}$ for some positive integer u. Let M be a full structure whose individuals include all elements of all groups in $\mathcal{S}$, the elements of X_2, and all real numbers. Let $^*M$ be an enlargement of M. By Lemma 6.3, $B(u)$ holds in $^*\mathcal{S}$. By Lemma 6.8, X_2 is not isomorphic to a subgroup of G for any $G \in {}^*\mathcal{S}$. By Lemmas 2.1 and 2.2, $V(\mathcal{G}) \neq \emptyset$ and so $\mathcal{G} \subseteq V(w)$ for some $w \neq 1$.

7. Two generator groups

As we have mentioned previously it has been conjectured that a group G is amenable if and only if G does not contain a copy of X_2. If this were true then a group G would be amenable provided all two generator subgroups of G are amenable. Even this is not known.

In this section we give a few elementary results that show something of the role that could be played by two generator groups in an attempt to describe all amenable groups.

LEMMA 7.1. *Every finitely generated amenable group H is a homomorphic image of a subgroup of a two generator amenable group G. If H is u.a., then G can be chosen from the class of u.a. groups.*

Proof. We construct a subgroup of an appropriate wreath product. However we will give a matrix description of the construction as it is easily presented in this form.

If H is generated by $h_1, h_2, \cdots, h_n$ consider the $n \times n$ matrices $A = (a_{ij})$ where $a_{ij} = \delta_{ij} h_i$ and $B = (b_{ij})$ where $b_{ij} = \delta_{i,j+1}$ for $j \leq n$ and $b_{in} = \delta_{i1}$. With the obvious multiplication A and B generate a group G in which diagonal matrices form a normal subgroup D with cyclic factor group. D is a subgroup of a product of n copies of H and the mapping $(t_{ij}) \to t_{11}$ from D into H is a homomorphism onto H. The lemma follows from the elementary properties of amenable groups and u.a. groups.

Clearly this can be considered as a reduction of the problem of classifying all amenable groups to classifying all two generator amenable groups.

However, that problem seems very difficult. It would have been nice if all two generator amenable groups had to satisfy a nontrivial law but there are simple examples to show this is not the case.

For instance let S be the group of all permutations on Z with finite support. This group is amenable since it is locally finite and satisfies only trivial laws since it contains an isomorphic copy of every finite group. Now let T be the permutation on Z defined by $T(i) = i + 1$ for all i in Z. Let G be generated by T and S. G is amenable since it is an extension of S by a cyclic group. Clearly G satisfies only trivial laws and is generated by two elements namely T and any transposition in S.

As we have indicated we do not know whether groups which satisfy nontrivial laws are amenable or equivalently whether u.a. groups are precisely the groups satisfying nontrivial laws.

We now show that this also only depends on two generator groups.

LEMMA 7.2. *Every variety of groups distinct from the variety of all groups is amenable iff every two generator group satisfying a law is amenable.*

Let V be a variety distinct from the variety of all groups. Consider $F_\infty(V)$. Let σ be a permutation on the free generators of $F_\infty(V)$ with just one orbit. Build a semidirect product of $F_\infty(V)$ with an infinite cyclic group using the automorphism σ. The result is clearly a two generator group which satisfies a nontrivial law and the lemma follows.

Now let G be a two generator group which satisfies a law. It is well known that $V(G)$ is the variety of groups satisfying all laws of G in ≤ 2 variables.

(See [3, 16.1].) Thus the problem of determining which laws yield amenable varieties is reduced to the situation where the laws are in ≤ 2 variables.

8. A property of u.a. groups

Many of the properties of u.a. groups which have been obtained in this paper have been derived from the nonstandard characterization of u.a. groups. The definition does not seem to be easily adaptable to standard proofs.

While searching for a more versatile standard condition on a group which would be equivalent to being uniformly amenable we discovered a necessary condition. It is not known whether it is sufficient.

Before we present the condition we remind the reader that whenever we say a group G is amenable we mean that G is amenable as a discrete group.

THEOREM 8.1. *Suppose G is a u.a. group, T is a Hausdorff group and φ is an isomorphism (algebraic) from G into T. Then* Cl $\varphi(G)$, *the closure of $\varphi(G)$ in T, is amenable. In fact,* Cl $\varphi(G)$ *is a u.a. group.*

It suffices to prove the theorem when $T = \mathrm{Cl}\,\varphi(G)$ and so we will assume that $\varphi(G)$ is dense in T.

Let M be a full structure whose individuals include the elements of G, T and all real numbers, and let ${}^*M$ be an enlargement of M.

Let τ be the topology on T. Let $\mu_\tau(x)$ be the τ-monad of any point x in T and let $K = \{t \in {}^*T \mid t \in \mu_\tau(x) \text{ for some } x \in T\}$. These elements are called the near-standard elements of ${}^*T$. (See [2, p. 61 ff.] for a detailed discussion.) Since T is a Hausdorff group K is a subgroup of ${}^*T$ and the map st_τ which maps each near-standard element a to the unique standard element x with $a \in \mu_\tau(x)$ is a homomorphism with kernel $\mu_\tau(1)$ and image T.

Let $H = ({}^*\varphi)^{-1}(K)$ and consider the map $\psi : H \to T$ defined by $\psi(h) = st_\tau^*\varphi(h)$ for $h \in H$. Now ψ is obviously a homomorphism and $\psi(H) = T$ since $\varphi(G)$ is dense in T. So T is a homomorphic image of a subgroup of ${}^*G$. Since G is a u.a. group ${}^*G$ is a u.a. group by Theorem 4.2 and T is a u.a. group by Theorem 4.5.

We formulate the conclusion of Theorem 8.1 as a group property.

DEFINITION 8.2. A group G is said to satisfy condition (w) if for every Hausdorff group T and every monomorphism φ from G into T, Cl $\varphi(G)$ is amenable.

Of course Theorem 8.1 then says that any u.a. group satisfies condition (w). The converse is unknown in general. However the following special case is of interest because it provides an alternate proof that $\prod G$ uniformly amenable is sufficient to guarantee $V(G)$ is amenable.

THEOREM 8.3. *Let H be any infinite direct product of copies of a group G and suppose that H satisfies (w). Then $V(H) = V(G)$ is amenable. In particular, H is a u.a. group.*

Proof. Let H be any infinite direct product of copies of a group G and let K be the corresponding cartesian product of copies of G. With the discrete topology on G and the product topology on K we have H dense in K and K amenable since H satisfies (w). Since K is an infinite cartesian product $F_n(V(G))$ is amenable for every n and therefore, $V(G)$ is amenable. Since all groups in an amenable variety are u.a. groups, H is a u.a. group and the proof is complete.

Obviously Theorems 8.1 and 8.3 give an alternate proof that if $\prod G$ is u.a., $V(G)$ is amenable. In Theorem 5.4 we showed that if G discriminates $V(G)$ and is u.a., then $V(G)$ is amenable. We haven't been able to show this when G is only assumed to satisfy (w). In other words we don't know if (w) implies u.a. even when G discriminates $V(G)$.

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