SKLAR-STIELTJES INTEGRALS

BY

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Many definitions of Stieltjes-type integrals have been proposed since the original one by T. J. Stieltjes in 1894 [18]. A particularly simple one was put forward by A. Sklar in 1964 [16]; he called it the "uniform Stieltjes integral". The simplicity of the uniform Stieltjes integral makes it an appealing object for investigation, and this paper presents some of the results of such an investigation. The principal results obtained are the existence of the integral as a real linear functional on $Q_0 \times BV$, $Q \times BV_1$, $BV_0 \times Q$ and $BV \times Q_1$ (see Table 1 for explanation of symbols), compatibility with the Lebesgue-Stieltjes integral, integration-by-parts and integration-by-substitution theorems, and the establishment of the Sklar-Stieltjes integral in Q_0 as the analog of the ordinary Stieltjes integral in C.

In the first of the three sections of this paper a brief historical survey is presented along with the definition of the Sklar-Stieltjes integral and the preliminaries needed in the remaining sections. Existence of the Sklar-Stieltjes integral and its compatibility with the Lebesgue-Stieltjes integral over $Q_0 \times BV$, $Q \times BV_1$, $BV_0 \times Q$ and $BV \times Q_1$ are established in the second section. The third section contains the development of additional properties of the Sklar-Stieltjes integral.

1. Historical survey

T. J. Stieltjes generalized the Riemann integral by replacing the "variable of integration" by a "function" of this "variable". Stieltjes defined a normed type of integral in which he restricted the integrand to be continuous and the integrator to be of bounded variation. These conditions guaranteed the existence of the integral. (As currently used, the expression "Stieltjes integral" refers to a Stieltjes limit in which the integrand and integrator are not restricted.)

S. Pollard [13] modified the normed Stieltjes integral to obtain a refinement type integral. (The distinction between normed and refinement types of integrals can be found in [7].) The integrals of Pollard and Stieltjes have many common properties and are referred to as the "ordinary Stieltjes integrals".

H. L. Smith [17] averaged the integrand over subintervals and modified the ordinary Stieltjes integrals to obtain normed and refined "mean Stieltjes integrals". Mean Stieltjes integrals of order p [4] are further generalizations of the mean Stieltjes integrals. T. H. Hildebrandt [7] has extensively surveyed other modifications leading to Riemann-type Stieltjes integrals.

Fundamental changes in the process of integration resulted in other gen-

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eralizations of the Stieltjes integral. The integration process of Lebesgue is generalized in the Lebesgue-Stieltjes integral [8], [10]. W. H. Young [19] added still another variant to the development of the Lebesgue-Stieltjes integral. J. Radon [14] constructed two generalizations in the same direction. P. J. Daniell [3] devised an integral that includes as special cases the Lebesgue-Stieltjes, Radon and ordinary Stieltjes integrals.

J. Kurzweil [9] introduced a Riemann-type integral that was investigated by R. Henstock [5], [6]. The Kurzweil integral was shown to be more general than the Lebesgue integral. E. J. McShane [11] further generalized the concept of integral to obtain as special cases the Lebesgue-Stieltjes integral and the Bochner integral over locally compact domains and a generalization of the Ito stochastic integral as well as various Stieltjes-type integrals.

W. F. Osgood [12] initiated another change in the process of integration by employing in the definition a sequence of partitions, P_n, of the interval $[a, b]$ in which P_n consists of all integral multiples of 2^{-n} in the closed interval $[a, b]$, together with the endpoints a and b. A. Sklar [15] has generalized the notion of sequences of regular sets of points in his definition of the uniform integral. The uniform integral of Sklar is defined as follows: Let f be a real finite-valued function on $[a, b]$; and, for $c > 0$, let

$$\sum_{c=a}^{b} f = \sum_{a/c \leq n < b/c} f(nc)$$

where n is an integer. If there exists a number I such that

$$\lim_{c \to 0+} \sum_{c=a}^{b} f = I,$$

then Sklar calls the number I the uniform integral of f and denotes it by $U \int_a^b f$. It has been shown [15] that for bounded functions the uniform integral is not only an extension of the Riemann integral but it is also compatible with the Lebesgue integral, i.e., if $U \int_a^b f$ and $L \int_a^b f$ exist, then $U \int_a^b f = L \int_a^b f$. Compatibility of uniform and Lebesgue integrals depends crucially on a lemma of J. C. van der Corput [15] which is itself equivalent to a result of P. T. Bateman [1] in the geometry of numbers. This indicates a connection between the geometry of numbers and the uniform integral.

In a further generalization of the uniform integral, A. Sklar [16] has defined a uniform Stieltjes integral as follows: Given a function f defined on the closed interval $[a, b]$, we can extend f to the entire real line by the convention that $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. Sklar approximative sums are defined as

$$\sum (f, g; a, b; c) = \sum_{n=-\infty}^{+\infty} f(nc)[g(nc + c) - g(nc)]$$

where f and g satisfy the extension convention.

Finally, we set

$$U \int_a^b f\, dg = \lim_{c \to 0+} \sum (f, g; a, b; c)$$

whenever the limit exists.

This definition differs slightly, and inessentially, from that of Sklar in that this definition always takes into account the left endpoint of the interval $[a, b]$. This modification facilitates comparison with the Lebesgue-Stieltjes integral.

It follows directly from the definition that the Sklar-Stieltjes integral is a bilinear function of f and g while the interval additive identity

$$U\int_a^b f\,dg = U\int_a^{b'} f\,dg + U\int_{b'}^b f\,dg, \qquad a < b' < b,$$

is valid if either $f(b') = f(b'-)$ or $g(b') = g(b'+)$.

Subspaces of the Banach space, B, of bounded real functions on $[a, b]$ with the norm of an element, f, in B defined by

$$\|f\| = \text{l.u.b.}_{x\in[a,b]} |f(x)|$$

are identified in Table 1.

TABLE 1. Function spaces on the closed interval $[a, b]$

B	bounded functions
Q	quasi-continuous functions, i.e., Q is the set of functions f such that $f(x^-)$ and $f(x^+)$ exist for all x $[a, b]$ (with $f(a^-) \equiv_{\text{def}} f(a)$ and $f(b^+) \equiv_{\text{def}} f(b)$)
Q_λ	for fixed $\lambda \in [0, 1]$, the set of all functions f in Q such that $f(x) = (1-\lambda)f(x^-) + \lambda f(x^+)$ for all $x \in [a, b]$
BV	functions of bounded variation
BV_λ	$BV \cap Q_\lambda$
K	step functions, i.e., functions which are constant on each of a finite number of disjoint intervals whose union is $[a, b]$
K_λ	$K \cap Q_\lambda$
C	continuous functions

The following lattice diagram shows inclusion relations among the various spaces:

$$\begin{tikzcd}
 & B \arrow[d, dash] & \\
 & Q \arrow[dl, dash] \arrow[dr, dash] & \\
BV \arrow[d, dash] \arrow[dr, dash] & & Q_\lambda \arrow[dl, dash] \arrow[d, dash] \\
K \arrow[dr, dash] & BV_\lambda \arrow[d, dash] & C \\
 & K_\lambda &
\end{tikzcd}$$

2. Existence of the Sklar-Stieltjes Integral

Compatibility of the Sklar-Stieltjes and Lebesgue-Stieltjes integrals on $K_0 \times K$ and $K \times K_1$ is established in Theorem 1. For (f, g) in $K \times K$, the Lebesgue-Stieltjes integral is defined as

$$L[f, g] \equiv \textstyle\sum_{a\le x\le b} f(x)\{g(x^+) - g(x^-)\}.$$

Under suitable topologies [4] defined on $Q \times BV$ and $BV \times Q$ the Lebesgue-Stieltjes integral has a unique continuous extension from $K \times K$ to $Q \times BV$ and $BV \times Q$. Because of its compatibility with the Lebesgue-Stieltjes integral on $K_0 \times K$ and $K \times K_1$, the Sklar-Stieltjes integral inherits a unique continuous extension from $K_0 \times K$ to $Q_0 \times BV$ and $BV_0 \times Q$ (Theorem 2) and from $K \times K_1$ to $Q \times BV_1$ and $BV \times Q_1$ (Theorem 3).

THEOREM 1. *Let $k_0 \in K_0$, $k_1 \in K_1$ and $k \in K$. then*

$$\text{(i)} \qquad U\int_a^b k_0\, dk = L[k_0, k]$$

and

$$\text{(ii)} \qquad U\int_a^b k\, dk_1 = L[k, k_1].$$

Proof. Let $c_j^n \in [a, b], j = 1, 2, \cdots, (n+1), c_1^n = a, c_{n+1}^n = b, n \in I^+$, and $C^n = \{c_j^n\}_{j=1}^{n+1}$. $\{C^n\}_{n=1}^{\infty}$ is called a sequence of uniform sets in $[a, b]$ provided that

(i) $c_{j+1}^n - c_j^n = \alpha_n > 0, j = 2, 3, 4, \cdots, (n-1)$,
(ii) for each $n \in I^+$ there is k such that $c_2^n = k \cdot \alpha_n$, $k \in I^+$,
(iii) $\lim_n \alpha_n = 0$.

The collection of all such sets is denoted by Γ and is defined by

$$\Gamma = \{\{C^n\}_{n=1}^{\infty} \mid \{C^n\}_{n=1}^{\infty} \text{ is a sequence of uniform sets in } [a, b]\}.$$

Part (i). Since $k \in K$ then k has a finite set of discontinuities. Let $D = \{x_i \mid k \text{ is discontinuous at } x_i \text{ in } [a, b]\}$. Moreover, require that $a \leq x_i \leq x_{i+1} \leq b$ for $x_i \in D$. Let $\min (x_{i+1} - x_i) = B > 0$ and $\{C^n\}_{n=1}^{\infty} \in \Gamma$. There is $M > 0$ such that $n > M$ implies $0 < \alpha_n < B/_2$. For $n > M$, each $[c_j^n, c_{j+1}^n], j = 1, 2, \cdots, n+1$, cannot contain more than one $x_i \in D$. It is possible, however, for

$$x_i \in [c_r^n, c_{r+1}^n] \quad \text{and} \quad x_i \in [c_{r+1}^n, c_{r+2}^n]$$

for some $1 \leq r \leq (n-1)$, i.e., some points of discontinuity can belong to two consecutive intervals. For $n > M$,

$$\sum_{j+1}^{n} k_0(c_j^n) \cdot k(c_{j+1}^n) - k(c_j^n)$$
$$= \sum_{x_j \in D_n} k_0(x_j -)[k(x_j +) - k(x_j -)] + \sum_{x_p \in D_n'} k_0(x_p)[k(x_p +) - k(x_p -)]$$

where $D_n \cup D_n' = D$ and

$$x_j \in D_n \Leftrightarrow x_j \in [c_r, c_{r+1}] \text{ for some r} \quad \text{and} \qquad x_j \notin [c_i^n, c_{i+1}^n], \quad i \neq r.$$

Since $k_0(x_j -) = k_0(x_j)$ then for $n > M$,

$$\sum_{j=1}^{n} k_0(c_j^n) \cdot [k(c_{j+1}^n) - k(c_j^n)] = \sum_{x_i \in D} k_0(x_i)[k(x_i +) - k(x_i -)].$$

Then

$$\lim_n \sum_{j=1}^{n} k_0(c_j^n) \cdot [k(c_{j+1}^n) - k(c_j^n)] = \sum_{x_i \in D} k_0(x_i) \cdot [k(x_i +) - k(x_i -)]$$

for $\{C^n\}_{n=1}^{\infty} \in \Gamma$. Therefore, $U\int_a^b k_0\, dk = L[k_0, k]$ for $(k_0, k) \in K_0 \times K$.

Part (ii). Argument similar to that used in Part (i). Topologies for $Q \times BV$ and $BV \times Q$ are prescribed by the following conditions. Let f be in Q and $\{v_n\}_{n=1}^{\infty}$ be a sequence of functions in K such that $\lim_n \| v_n - f \| = 0$. The sequence, $\{v_n\}_{n=1}^{\infty}$ is said to satisfy condition A relative to f. Moreover, if f is in BV, $\{v_n\}_{n=1}^{\infty}$ satisfies condition A relative to f, and there exists an $M > 0$ such that $V_a^b(v_n) \leq M$ for $n = 1, 2, 3, \cdots$, then $\{v_n\}_{n=1}^{\infty}$ is said to satisfy condition B relative to f.

THEOREM 2. *Let $f_0 \in Q_0$, $f \in Q$, $g_1 \in BV_1$ and $g \in BV$. Then*
(i) *$U \int_a^b f_0\, dg$ exists and equals $L[f_0, g]$,*
and
(ii) *$U \int_a^b f dg_1$ exists and equals $L[f, g_1]$.*

Proof. Part (i). From Theorem 1 (i) and condition A relative to f_0 it follows [4] that there is a unique continuous extension of the Lebesgue-Stieltjes and Sklar-Stieltjes integrals to $Q_0 \times K$. Moreover, for $f_0 \in Q_0$, $h \in BV$, and $k \in K$

$$\text{(i)} \qquad \left| U \int_a^b f_0\, dk \right| \leq \| f_0 \| \cdot V_a^b(k)$$

and

$$\text{(ii)} \qquad \left| U \int_a^b h\, dk \right| \leq |h(a)k(a)| + |h(b)k(b)| + \| k \| \cdot V_a^b(h).$$

From inequalties (i) and (ii) and condition B relative to g, a unique continuous extension of the Sklar-Stieltjes integral from $Q_0 \times K$ to $Q_0 \times BV$ is obtained [4].

Part (ii). Argument similar to that used in Part (i).

THEOREM 3. *Let $f_1 \in Q_1$, $f \in Q$, $g_0 \in BV_0$, and $g \in BV$. Then*
(i) *$U \int_a^b g_0\, df$ exists and equals $L[g_0, f]$,*
and
(ii) *$U \int_a^b g\, df_1$ exists and equals $L[g, f_1]$.*

Proof. Part (i). From Theorem 1 (ii) and condition B relative to g_0 the unique continuous extension from $K_0 \times K$ to $BV_0 \times Q$ as in Theorem 2 [4].

Part (ii). Argument similar to that used in Part (i).

3. Additional properties of the Sklar-Stieltjes integral

Existence of $U \int_a^b fdg$ for all $g \in BV$ implies that $f \in Q_0$. This result establishes the Sklar-Stieltjes integral as an analog of the ordinary Stieltjes integral which has a similar property over the space of continuous functions.

THEOREM 4 *Let $U \int_a^b fdg$ exist for all $g \in BV$. Then $f \in Q_0$.*

Proof. Let $d \in [a, b]$, $k_d(x) = 0$, if $a \leq x \leq d$ and $k_d(x) = 1$, if $x > d$. Obviously, $k_d \in BV$. Let $\{x_k\}_{k=1}^{\infty} \in [a, b]$ such that $\lim_n x_k = d$ and $x_k < d$ for

$k \in I^+$. There is $\{C^n\}_{n=1}^{\infty} \in \Gamma$ such that $\{x_k\}_{k=1}^{w}$ is imbedded in $\{C^n\}_{n=1}^{\infty}$. Then

$$\lim_n \sum_{j=1}^{n} f(x_j^n)\cdot[k_d(x_{j+1}^n) - k_d(x_j^n)] = \lim_k f(x_k)\cdot 1 = U\int_a^b f\,dk_d.$$

Therefore $f(d-)$ exists and

$$f(d-) = V\int_a^b f\,dk_d.$$

Let $\{Y_k\}_{k=1}^{\infty} \in [a, b]$ such that $Y_k = d$ for $k \in I^+$. There is $\{C^n\}_{n=1}^{\infty} \in \Gamma$ such that $\{d\}_{k=1}^{\infty}$ is imbedded in $\{C^n\}_{n=1}^{\infty}$. Again it follows that $f(d) = U\int_a^b f dk$, and $f(d-) = f(d)$ for $d \in [a, b]$. Consequently, f is in Q_0.

THEOREM 5 (Integration-by-parts). *Let $(f_0, g_0) \in Q_0 \times BV_0$. Then*

$$U\int_a^b f_0\,dg_0 = f_0\cdot g_0]_a^b - U\int_a^b g_0\,df_0 + 2M[f_0, g_0]$$

where M denotes the ordinary mean-Stieltjes integral [17].

Proof.

$$U\int_a^b f_0\,dg_0 = L[f_0, g_0] = f_0\cdot g_0]_a^b - L[g_0, f_0] + 2M[f_0, g_0]$$
$$= f_0\cdot g_0]_a^b - U\int_a^b g_0\,df_0 + 2M[f_0, g_0].$$

The following theorem provides an additional application of the compatibility obtained in Theorem 1.

THEOREM 6. *Let $(f_0, g) \in Q_0 \times BV$. Then*

$$U\int_a^b f_0\cdot h_0\,dg = U\int_a^b h_0\,ds \quad \textit{where } s(x) = U\int_a^x f_0\,dg, \qquad a < x \leq b,$$

and

$$s(a) = 0.$$

Proof. This property is possessed by the Lebesgue-Stieltjes integral [4]. Theorem 1 implies that the Sklar-Stieltjes integral inherits the property.

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