A CONTINUOUS LOWER ESTIMATE FOR THE DISTANCE BETWEEN AN OPERATOR AND COMPACT PERTURBATIONS OF NORMAL OPERATORS

BY

Takashi Ito[1] and Tin Kin Wong

1. Introduction and preliminaries

Let H be a complex infinite dimensional Hilbert space, and let $\mathfrak{N}$ and $\mathcal{K}$ be the sets of normal operators and compact operators on H, respectively. By an operator we mean a bounded linear operator on H. It is well known that the distance between the unilateral shift U and the set $\mathfrak{N} + \mathcal{K}$ is exactly 1. This is shown in [10] by J. G. Stampfli.

In this paper we introduce a continuous lower estimate $\rho_0(T)$ for the distance between an arbitrary operator T and the set $\mathfrak{N} + \mathcal{K}$. We prove that for a certain class of operators this lower estimate $\rho_0(T)$ gives the exact distance. On the other hand, if T_f is a Toeplitz operator, and f in $H^\infty + C$, then $\rho_0(T_f)$ can be replaced by a more natural and computable lower estimate. Hence our main result (Theorem 2) is

$$\text{dist}\,(T_f,\ \mathfrak{N} + \mathcal{K}) \geqq \max_{\lambda \in \sigma(T_f)} \text{dist}\,(\lambda,\ R(f)) \quad \text{for } f \in H^\infty + C,$$

where C is the space of all continuous complex-valued functions on the unit circle, $R(f)$ is the essential range of f, and $\sigma(T_f)$ is the spectrum of T_f. One can easily see that Stampfli's theorem dist $(U, \mathfrak{N} + \mathcal{K}) = 1$ follows from our result (see also Remark 2). The techniques that we use here are based on the Fredholm Index Theory (cf. [6]).

We recall that an operator T is said to be *left-Fredholm* if the range $R(T)$ is closed and the null space $N(T)$ is finite dimensional, or, equivalently, if T is left-invertible modulo the compact operators [13]. An operator T is *right-Fredholm* if its adjoint T^* is left-Fredholm, *semi-Fredholm* if it is either left-Fredholm or right-Fredholm, and *Fredholm* if it is both left-Fredholm and right-Fredholm. The set $\mathfrak{F}_s$ ($\mathfrak{F}$) of all semi-Fredholm (Fredholm) operators is open in the algebra $\mathcal{B}(H)$ of all operators on H. For a semi-Fredholm operator T, $\dim N(T) - \dim N(T^*)$ is called the Fredholm *index* of T and is denoted by ind T.

In relation to semi-Fredholm operators, several kinds of essential spectra of an operator can be introduced in a natural way. For an operator T, the Fredholm spectrum is the set $\sigma_F(T) = \{\lambda;\ T - \lambda I \text{ is not Fredholm}\}$, the *left-Fredholm spectrum* is the set $\sigma_F^l(T) = \{\lambda;\ T - \lambda I \text{ is not left-Fredholm}\}$ and the right-Fredholm spectrum $\sigma_F^r(T)$ is defined similarly. We call the

Received July 12, 1971.

[1] The research of the first author was partially supported by a National Science Foundation grant.

set $\sigma_F^l(T) \cap \sigma_F^r(T)$ the *semi-Fredholm spectrum* of T and we denote it by $\sigma_s(T)$. The *Weyl essential spectrum* of T is, by definition, the set $\bigcap_{K \in \mathcal{K}} \sigma(T + K)$ and it is denoted by $\omega(T)$. All these essential spectra of T are non-empty compact subsets of the plane and are invariant under compact perturbation of T. The following inclusions are clear:

$$\sigma_s(T) \subset \sigma_F(T) \subset \omega(T) \subset \sigma(T).$$

It is known that the larger essential spectrum is obtained from the smaller one by filling in the holes. Moreover, the set valued mapping $T \to \omega(T)$ is upper semi-continuous. These facts follow easily from the continuity of the Fredholm index.

2. The general case

In this section we introduce a continuous lower estimate for the distance between an arbitrary operator and all the compact perturbations of normal operators. We need the following definitions.

DEFINITION 1. For an operator T, we define $\gamma(T)$, $\gamma_0(T)$, $\rho(T)$ and $\rho_0(T)$ as follows:

(i) $\gamma(T) = \inf\{\|Tx\| : x \in H \text{ and } \|x\| = 1\}$ and
$\gamma_0(T) = \inf\{\|T(x)\| : x \in N(T)^{\perp}, \|x\| = 1\}$,

(ii) $\rho(T) = \sup\{\gamma(T - \lambda I + K) : \lambda \in \omega(T), K \in \mathcal{K}\}$,

(iii) $\rho_0(T) = \max\{\rho(T), \rho(T^*)\}$.

Now we are ready to state and prove the main result of this section.

THEOREM 1. *$\rho_0(T)$ has the following properties:*

(1) *$\rho_0(T) = 0$ if and only if $\omega(T) = \sigma_s(T)$,*

(2) *$\rho_0(T)$ is continuous on $\mathcal{B}(H)$,*

(3) dist $(T, \mathfrak{N} + \mathcal{K}) \geqq \rho_0(T)$.

Proof. (1) If there is some λ_0 in $\omega(T) \backslash \sigma_s(T)$, then $T - \lambda_0 I$ is semi-Fredholm. In case the index of $T - \lambda_0 I$ is negative, it is easy to construct a finite rank operator K_0 such that $T - \lambda_0 I + K_0$ is left-invertible. So $\gamma(T - \lambda_0 I + K_0) > 0$, hence $\rho_0(T) \geqq \rho(T) > 0$ in this case. If the index of $T - \lambda_0 I$ is positive, by considering the adjoint of $T - \lambda_0 I$ we can conclude that $\gamma(T^* - \bar{\lambda}_0 I + K_0) > 0$ for some finite rank operator K_0. So $\rho_0(T) \geqq \rho(T^*) > 0$. Therefore, in either case we have $\rho_0(T) > 0$ if $\sigma_s(T)$ is a proper subset of $\omega(T)$. Now suppose $\sigma_s(T) = \omega(T)$. For every λ in $\omega(T)$, $T - \lambda I$ is not semi-Fredholm, so $T - \lambda I + K$ is not semi-Fredholm for any compact K as well. In particular, $T - \lambda I + K$ is not left-invertible for all λ in $\omega(T)$ and all compact K. Hence $\gamma(T - \lambda I + K) = 0$. Thus $\rho(T) = 0$. Similarly $\rho(T^*) = 0$ if $\sigma_s(T) = \omega(T)$. Therefore $\rho_0(T) = 0$ when $\sigma_s(T) = \omega(T)$. This completes the proof of (1).

(2) To prove (2), it is enough to prove that $\rho(T)$ is continuous on $\mathcal{B}(H)$. We first show the upper semi-continuity. Let T_0 be an operator, and let

$T_n \to T_0$ in $\mathcal{B}(H)$. For any given ε-neighborhood V of $\omega(T_0)$, we see that $\omega(T_n) \subset V$ for sufficiently large n (by upper semi-continuity). Then for any λ_n in $\omega(T_n)$, there is λ_n' in $\omega(T_0)$ such that $|\lambda_n - \lambda_n'| < \varepsilon$ for large n. For any unit vector x in H and any compact operator K, we have

$$\begin{aligned} \|(T_n - \lambda_n I + K)x\| & \\ &\leqq \|(T_0 - \lambda_n' I + K)x\| + \|(T_0 - T_n)x\| + |\lambda_n - \lambda_n'| \\ &\leqq \|(T_0 - \lambda_n' I + K)x\| + 2\varepsilon \quad \text{for large } n. \end{aligned}$$

Taking the infimum over all the unit vector x on both sides of the above inequality, we have

$$\gamma(T_n - \lambda_n I + K) \leqq \gamma(T_0 - \lambda_n' I + K) + 2\varepsilon \leqq \rho(T_0) + 2\varepsilon.$$

Therefore $\rho(T_n) \leqq \rho(T_0) + 2\varepsilon$ for large n. This proves the upper semi-continuity of $\rho(T_0)$ at T_0. To prove the lower semi-continuity at T_0, we need only prove this for $\rho(T_0) > 0$. Let $T_n \to T_0$ in $\mathcal{B}(H)$ and $\rho(T_0) > 0$. For any $\varepsilon > 0$ ($\rho(T_0) > \varepsilon > 0$) there is some λ_0 in $\omega(T_0)\backslash\sigma_s(T_0)$ and some compact operator K_0 such that

$$0 < \rho(T_0) - \varepsilon < \gamma(T_0 - \lambda_0 I + K_0).$$

Therefore $T_0 - \lambda_0 I + K_0$ is left-invertible but not invertible, because λ_0 is in $\omega(T_0)$. Hence ind $(T_0 - \lambda_0 I + K_0) < 0$. By the Fredholm Index Theorem, we see that for large n, $T_n - \lambda_0 I + K_0$ is left-Fredholm and

$$\text{ind}\,(T_n - \lambda_0 I) = \text{ind}\,(T_n - \lambda_0 I + K_0) = \text{ind}\,(T_0 - \lambda_0 I + K_0) < 0.$$

Hence λ_0 belongs to $\omega(T_n)$ for large n. Moreover, if x is any unit vector and if n is large then

$$\begin{aligned} \|(T_n - \lambda_n I + K_0)x\| &\geqq \|(T_0 - \lambda_0 I + K_0)x\| - \|T_n - T_0\| \\ &\geqq \gamma(T_0 - \lambda_0 I + K_0) - \varepsilon \\ &\geqq \rho(T_0) - 2\varepsilon. \end{aligned}$$

It follows that

$$\rho(T_n) \geqq \gamma(T_n - \lambda_0 I + K_0) \geqq \rho(T_0) - 2\varepsilon \quad \text{for large } n.$$

This completes the proof of part (2).

(3) We note that (3) holds trivially if $\rho_0(T) = 0$. So we may and do assume that $\rho_0(T) > 0$, say $\rho_0(T) = \rho(T)$. For any positive $\delta < \rho(T)$ there is a λ_δ in $\omega(T)$ and a compact K_δ such that $\gamma(T - \lambda_\delta I + K_\delta) > \delta > 0$. It follows that $T - \lambda_\delta I + K_\delta$ is left-invertible and ind $(T - \lambda_\delta I + K_\delta) < 0$. This implies that

$$\|T - (N + K)\| \geqq \gamma(T - \lambda_\delta I + K_\delta) > \delta,$$

for any normal operator N and any compact operator K. Indeed, if there were some normal N and compact K such that $\|T - (N + K)\| < \gamma(T - \lambda_\delta I + K_\delta)$.

Since

$$\gamma(T - \lambda_\delta I + K_\delta) = \gamma_0(T - \lambda_\delta I + K_\delta),$$

then it would follow from the Fredholm Index Theorem that

$$0 = \operatorname{ind}(N - \lambda_\delta I) = \operatorname{ind}(T - \lambda_\delta I + K_\delta + N + K - T)$$
$$= \operatorname{ind}(T - \lambda_\delta I + K_\delta) < 0,$$

contradiction. Hence $\| T - (N + K) \| > \delta$ for any $\delta < \rho(T)$, where N is normal and K is compact. Consequently, the inequality

$$\operatorname{dist}(T, \mathfrak{N} + \mathfrak{K}) \geqq \rho_0(T)$$

holds in case $\rho_0(T) = \rho(T)$. To complete the proof, we must also consider the case $\rho_0(T) = \rho(T^*)$. However, by considering the adjoint, the same argument will suffice. This concludes the proof of Theorem 1.

For a certain class of operators T, the lower estimate $\rho_0(T)$ gives the exact distance between T and the set $\mathfrak{N} + \mathfrak{K}$. This is shown in the following corollary.

We consider a very special class of operators T such that either $T^*T - I$ or $TT^* - I$ is compact. It is observed in [5] that this is equivalent to $T = V + K$ where V is either an isometry or a co-isometry and K is a compact operator. From this we deduce the following.

COROLLARY 1. *Let T be an operator such that either $T^*T - I$ or $TT^* - I$ is compact. Then we have the following alternative:*

(1) *Conditions* (i) *through* (v) *are equivalent:*

(i) $0 \notin \omega(T)$,

(ii) $\rho_0(T) = 0$,

(iii) $\operatorname{dist}(T, \mathfrak{N} + \mathfrak{K}) = 0$,

(iv) $T = V + K$ *for some unitary V and compact K,*

(v) $\sigma_s(T) = \omega(T) \subset$ the unit circle.

(2) *Conditions* (i′) *through* (iv′) *are equivalent.*

(i′) $0 \in \omega(T)$,

(ii′) $\rho_0(T) = 1$,

(iii′) $\operatorname{dist}(T, \mathfrak{N} + \mathfrak{K}) = 1$,

(iv′) $\omega(T)$ *is the closed unit disk, and $\sigma_s(T)$ is the unit circle.*

Observe that the above alternative holds for T if and only if it holds for T^*. Therefore we may assume that $T = V + K$ where V is an isometry and K is compact. Moreover, since $\omega(T) = \omega(V)$, $\sigma_s(T) = \sigma_s(V) \subset$ the unit circle,

$$\operatorname{dist}(T, \mathfrak{N} + \mathfrak{K}) = \operatorname{dist}(V, \mathfrak{N} + \mathfrak{K}),$$

and $\rho_0(T) = \rho_0(V)$, if $T = V + K$, we may further assume that T is an isometry V in proving (1) and (2).

To prove (1) we show that (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (v) $\Rightarrow$ (i). The implications (iv) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (i) are trivial. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (v) follows from Theorem 1. To see (i) $\Rightarrow$ (iv) notice that since $0 \notin \omega(T)$, V is Fredholm and ind $V = 0$. So V must be unitary.

To prove (2) we show that (i′) $\Rightarrow$ (ii′) $\Rightarrow$ (iii′) $\Rightarrow$ (iv′) $\Rightarrow$ (i′). Assuming (i′), then V is left-Fredholm and ind $V < 0$ hence $1 \leqq \rho_0(V) \leqq \| V \| = 1$, and (ii′) follows. Suppose (ii′) holds. Then (iii′) follows from $1 = \rho_0(V) \leqq \operatorname{dist}(V, \mathfrak{N} + \mathfrak{K}) \leqq \| V \| = 1$. If (iii′) holds, then we must have $\omega(V) \neq \sigma_s(V)$ by (1). Therefore $\omega(V)$ is the union of $\sigma_s(V)$ and some "holes" of $\sigma_s(V)$. It follows that $\sigma_s(V)$ = the unit circle and $\omega(V)$ = the closed unit disk. Finally, the implication (iv′) $\Rightarrow$ (i′) is trivial.

Remark 1. Theorem 1 immediately suggests the following question. Does the set $\{T : \sigma_s(T) = \omega(T)\}$ have a non-empty interior in $\mathcal{B}(H)$? During the preparation of this paper, Professor J. G. Stampfli communicated to us that the answer to this question is negative [12]. He also points out to us that there are operators T such that $\sigma_s(T) = \omega(T)$ while $\operatorname{dist}(T, \mathfrak{N} + \mathfrak{K}) > 0$.

3. Toeplitz Operators in $H^\infty + C$

In this section we wish to consider Toeplitz operators T_f for f in $H^\infty + C$. We aim to replace the lower estimate $\rho_0(T_f)$ defined in §2 by a more natural and computable one (see Definition 2 below).

Let L^p $(1 \leqq p \leqq \infty)$ denote the Lebesgue space for the normalized measure defined on the unit circle, let H^p denote the corresponding Hardy space of functions in L^p, and let C denote the space of continuous complex-valued functions on the unit circle. For f in L^∞ we define the Toeplitz operator T_f on H^2 by $T_f\,\varphi = P(f\varphi)$ for φ in H^2, where P is the orthogonal projection of L^2 onto H^2. Let $H^\infty + C$ denote the linear span of H^∞ and C. D. Sarason shows in [9] that $H^\infty + C$ is a closed subalgebra of L^∞. We need some known results about Toeplitz operators.

Lemma A (Coburn [2]). *$\sigma(T_f) = \omega(T_f)$ for any Toeplitz operator T_f.*

For f in L^∞, let $R(f)$ denote the essential range of f. Then $R(f)$ is a compact set in the plane. Let $\bar{H}^\infty$ denote the set of complex conjugate of functions in H^∞. The following lemma is contained implicitly in the work of R. G. Douglas and D. Sarason [4].

Lemma B. *$R(f) = \sigma_s(T_f)$ for f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$*

It follows from Lemma A and Lemma B that for f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$ the spectrum of T_f is obtained from $R(f)$ by filling in the holes. (See [11] for a different proof of this fact.) We need the following definition.

Definition 2. For f in L^∞, we define $\delta(f)$ as

$$\delta(f) = \sup\{\operatorname{dist}(\lambda, R(f)) : \lambda \in \sigma(T_f)\}.$$

It is immediate from Lemma A and Lemma B and the definition of $\delta(f)$, (for f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$) that the sets determining $\delta(f)$ are the same as those appearing in the definition of $\rho_0(T_f)$. This makes it possible to replace ρ_0 by δ for the special class of Toeplitz operators T_f with f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$ as shown in the following theorem. We need one more lemma which is also known (cf. [3] and [11]).

LEMMA C. *$T_g T_f - T_{gf}$ is compact for g in L^∞ and f in $H^\infty + C$.*

We are now ready to prove our principal result.

THEOREM 2. *For f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$, $\delta(f)$ has the following properties:*

(1) *$\delta(f) = 0$ if and only if $R(f) = \sigma(T_f)$,*

(2) *$\delta(\cdot)$ is continuous on $(H^\infty + C) \cup (\bar{H}^\infty + C)$,*

(3) *$\rho_0(T_f) \geqq \delta(f)$, and hence $\operatorname{dist}(T_f, \mathfrak{N} + \mathfrak{K}) \geqq \delta(f)$.*

Proof. (1) holds trivially for all f in L^∞.

To prove (2) it is enough to prove that $\delta(\cdot)$ is continuous on $H^\infty + C$. The idea of the proof is the same as that of part (2) of Theorem 1. We first show that $\delta(\cdot)$ is upper semi-continuous on $H^\infty + C$ (in fact, on L^∞). Suppose $f_n \to f_0$ in L^∞. For any ε-neighborhood V of $\sigma(T_{f_0})$, we have $\sigma(T_{f_n}) \subset V$ and $\|f_n - f_0\|_\infty < \varepsilon$ for large n. Then for any λ_n in $\sigma(T_{f_n})$, there is λ_n' in $\sigma(T_{f_0})$, such that $|\lambda_n - \lambda_n'| < \varepsilon$ provided n is large. Hence

$$\begin{aligned}\operatorname{dist}(\lambda_n, R(f_n)) &\leqq |\lambda_n - \lambda_n'| + \operatorname{dist}(\lambda_n', R(f_0)) + \|f_n - f_0\|_\infty \\ &\leqq 2\varepsilon + \operatorname{dist}(\lambda_n', R(f_0)) \\ &\leqq 2\varepsilon + \delta(f_0),\end{aligned}$$

for all λ_n in $\sigma(T_{f_n})$. Consequently,

$$\delta(f_n) \leqq 2\varepsilon + \delta(f_0) \quad \text{for large } n.$$

Now we proceed to show the lower semi-continuity of $\delta(\cdot)$ on $H^\infty + C$. We may assume that $\delta(f_0) > 0$. Let $f_n \to f_0$ in $H^\infty + C$. Choose some λ_0 in $\sigma(T_{f_0})$ such that

$$0 < \delta(f_0) = \operatorname{dist}(\lambda_0, R(f_0)).$$

Hence $(f_0 - \lambda_0)^{-1}$ is in L^∞. By Lemma C, we see that

$$T_{(f_0-\lambda_0)^{-1}} T_{(f_0-\lambda_0)} = I + K$$

for some compact K. Therefore $T_{f_0-\lambda_0}$ is left-Fredholm, and $\operatorname{ind}(T_{f_n-\lambda_0}) \neq 0$ by Lemma A. It follows from the Fredholm Index Theorem that $T_{f_n-\lambda_0}$ is left-Fredholm and $\operatorname{ind}(T_{f_n-\lambda_0}) \neq 0$ for large n. Consequently λ_0 is in $\sigma(T_{f_n})$ for large n. Therefore,

$$\delta(f_n) \geqq \operatorname{dist}(\lambda_0, R(f_n)) \geqq \operatorname{dist}(\lambda_0, R(f_0)) - \|f_n - f_0\|_\infty \geqq \delta(f_0) - \varepsilon$$

for large n. This concludes the proof of (2).

For the proof of part (3), we first note that $\rho_0(T_{\bar{f}}) = \rho_0(T_f^*) = \rho_0(T_f)$ and $\delta(f) = \delta(\bar{f})$. Hence it is enough to prove (3) for f in $H^\infty + C$. Observe that $\rho_0(T_f) = 0$ if and only if $\delta(f) = 0$. So we may and do assume that $\delta(f_0) > 0$ for a given f_0 in $H^\infty + C$. By the continuity of $\rho_0(\cdot)$ and $\delta(\cdot)$, it is enough to show that $\rho_0(T_{f_0}) \geqq \delta(f_0)$ for functions f_0 of the form $\bar{z}^n h_0$, where h_0 is in H^∞, because functions of the form $\bar{z}^n h_0$ are dense in $H^\infty + C$. Let $f_0 = \bar{z}^n h_0$, with h_0 in H^∞ and $\delta(f_0) > 0$. Choose λ_0 in $\sigma(T_{f_0})$ such that $\delta(f_0) = \operatorname{dist}(\lambda_0, R(f_0))$. Then $T_{f_0 - \lambda_0}$ is left-Fredholm and $\operatorname{ind}(T_{f_0-\lambda_0}) \neq 0$ by Lemma C and Lemma A respectively. Once again by Lemma C,

$$\begin{aligned} T_{f_0} - \lambda_0 I = T_{f_0 - \lambda_0} &= T_{\bar{z}^n h_0 - \lambda_0} = T_{\bar{z}^n (h_0 - \lambda_0 z^n)} \\ &= T_{\bar{z}^n} T_{h_0 - \lambda_0 z^n} \\ &= T_{(h_0 - \lambda_0 z^n)} \cdot T_{\bar{z}^n} + K_0 \quad \text{for some compact } K_0. \end{aligned}$$

Let $U = T_z$; then we have

$$T_{f_0} - \lambda_0 I - K_0 = T_{(h_0 - \lambda_0 z^n)} U^{*n}.$$

We recall that $\gamma_0(T) = \inf\{\| Tx \| : x \in N(T)^\perp, \| x \| = 1\}$ for any operator T. We shall show that

$$\gamma_0(T_{f_0} - \lambda_0 I - K_0) \geqq \delta(f_0).$$

It is clear that $N(T_{f_0} - \lambda_0 I - K_0) \supset N(U^{*n})$, so for any φ in

$$N(T_{f_0} - \lambda_0 I - K_0)^\perp$$

we have

$$\begin{aligned} \| (T_{f_0} - \lambda_0 I - K_0)\varphi \| &= \| T_{(h_0 - \lambda_0 z^n)} U^{*n} \varphi \| \\ &\geqq \operatorname{ess\,inf}_{|z|=1} | h_0(z) - \lambda_0 z^n | \, \| U^{*n} \varphi \| \\ &= \operatorname{ess\,inf}_{|z|=1} | \bar{z}^n h_0(z) - \lambda_0 | \, \| \varphi \| \\ &= \operatorname{dist}(\lambda_0, R(f_0)) \, \| \varphi \| \\ &= \delta(f_0) \, \| \varphi \|. \end{aligned}$$

It follows that $\gamma(T_{f_0} - \lambda_0 I - K_0) \geqq \delta(f_0)$. We wish to show that either

$$\gamma(T_{f_0} - \lambda_0 I - K_0 + K_1) \geqq \delta(f_0) \quad \text{or} \quad \gamma(T_{f_0}^* - \bar{\lambda}_0 I - K_0^* + K_1) \geqq \delta(f_0)$$

for some finite-rank operator K_1. (For the definition of $\gamma(\cdot)$, see Definition 1(i).) We have already seen that $T_{f_0} - \lambda_0 I$ is left-Fredholm and

$$\operatorname{ind}(T_{f_0} - \lambda_0 I) \neq 0.$$

So is $T_{f_0} - \lambda_0 I - K_0$ as well. If $\operatorname{ind}(T_{f_0} - \lambda_0 I - K_0) < 0$, then one can easily construct a finite rank operator K_1 such that $T_{f_0} - \lambda_0 I - K_0 + K_1$ is left-invertible and

$$\gamma(T_{f_0} - \lambda_0 I - K_0 + K_1) = \gamma_0(T_{f_0} - \lambda_0 I - K_0) \geqq \delta(f_0).$$

In case $\operatorname{ind}(T_{f_0} - \lambda_0 I - K_0) > 0$, on taking the adjoint of $T_{f_0} - \lambda_0 I - K_0$ we see that

$$\gamma(T_{f_0}^* - \bar{\lambda}_0 I - K_0^* + K_1)$$
$$= \gamma_0(T_f^* - \bar{\lambda}_0 I - K_0^*) = \gamma_0(T_{f_0} - \lambda_0 I - K_0) \geqq \delta(f_0)$$

for a suitable finite-rank operator K_1. Note that $\gamma_0(T^*) = \gamma_0(T)$ for any operator T. In conclusion, we have $\rho_0(T_{f_0}) \geqq \delta(f_0)$ for $f_0 = \bar{z}^n h_0$ with h_0 in H^∞. Therefore $\rho_0(T_f) \geqq \delta(f)$ for all f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$. Hence

$$\operatorname{dist}(T_f, \mathfrak{N} + \mathfrak{K}) \geqq \rho_0(T_f) \geqq \delta(f)$$

for f in $(H^\infty + C) \cup (\bar{H}^\infty + C)$. This completes the proof of Theorem 2.

Note. One could prove the inequality $\operatorname{dist}(T_f, \mathfrak{N} + \mathfrak{K}) \geqq \delta(f)$ for f in $H^\infty + C$ without appealing to Theorem 1. However, this inequality does not hold for all f in L^∞. Any non-constant characteristic function of a measurable set will provide us with a counterexample.

Let f be a unimodular function in $(H^\infty + C) \cup (\bar{H}^\infty + C)$. That is $|f| = 1$ a.e. By Lemma C, we see that either $T_f T_f^* - I$ or $T_f^* T_f - I$ is compact. Therefore Corollary 1 is applicable in view of Lemmas A and B. Hence we have the following result (see also [11]):

Corollary 2. *Let f be a unimodular function in $(H^\infty + C) \cup (\bar{H}^\infty + C)$. Then we have the following alternative:*

(1) *Conditions* (i) *through* (v) *are equivalent.*
- (i) T_f *is invertible,*
- (ii) $\delta(f) = 0$,
- (iii) $\operatorname{dist}(T_f, \mathfrak{N} + \mathfrak{K}) = 0$,
- (iv) $T = V + K$ *for some unitary V and compact K, and*
- (v) $\sigma(T_f) = R(f) \subset$ *the unit circle.*

(2) *Conditions* (i′) *through* (iv′) *are equivalent.*
- (i′) T_f *is not invertible,*
- (ii′) $\delta(f) = 1$,
- (iii′) $\operatorname{dist}(T_f, \mathfrak{N} + \mathfrak{K}) = 1$,
- (iv′) $\sigma(T_f)$ *is the closed unit disk and $R(f)$ is the unit circle.*

Remark 2. Theorem 2 gives no information on $\operatorname{dist}(T_f, \mathfrak{N} + \mathfrak{K})$ for f in the class $\{f \in H^\infty + C : R(f) = \sigma(T_f)\}$. In view of Theorem 2, it is clear that this class is a closed subset of $H^\infty + C$. It is also not hard to see that there are non-constant continuous functions in this class. We want to point out that *there even exist functions f in* $H^\infty \cap C$—*the disk algebra—such that* $\sigma(T_f) = R(f) =$ *the closed unit disk.* The construction of such f goes as follows. Take the Cantor set S with zero measure on the unit circle. Then there is a continuous function φ on S whose range is the closed unit disk. For example, the Lebesgue singular function followed by a Peano curve which maps the unit interval onto the closed unit disk provide such a function φ. Since S has measure zero, by Rudin's extension theorem [1, Theorem 2.4.10], there is a function f in the disk algebra such that the restriction of f to S is φ and $\|f\|_\infty = 1$. This f is our desired function, because $\sigma(T_f)$ = the closed unit disk $= R(f)$. (See [8] for a similar kind of function constructed by a different method.) Notice that the equalities $\sigma(T_f)$ = the closed unit disk $= R(f)$ follow from a result of Hartman and Wintner [7]. We thank our colleagues Leon Brown, Franz Schnitzer, and Bertram Schreiber for discussions on the construction of such a function.

Added in proof. Recently L. G. Brown, R. G. Douglas and P. A. Fillmore (see Notices Amer. Math. Soc., vol. 20 (1973), no. 1, and their preprint *Extensions of C^*-algebras, operators with compact self-commutators, and K-homology*) have proved that the class $\mathfrak{N} + \mathfrak{K}$ consists of those operators T whose self-commutator $T^*T - TT^*$ is compact and whose Fredholm spectrum and Weyl spectrum are identical, i.e. $\sigma_F(T) = \omega(T)$.

References

1. A. Browder, *Introduction to function algebra*, W. A. Benjamin, New York, 1969.
2. L. A. Coburn, *Weyl's theorem for nonnormal operators*, Mich. Math. J., vol. 13 (1966), pp. 285–288.
3. R. G. Douglas, *Toeplitz and Wiener-Hopf operators in $H^\infty + C$*, Bull. Amer. Math. Soc., vol. 74 (1968), pp. 895–899.
4. R. G. Douglas and D. Sarason, *Fredholm Toeplitz operators*, Proc. Amer. Math. Soc., vol. 26 (1970), pp. 117–120.
5. P. A. Fillmore, J. G. Stampfli and J. P. Williams, *On the essential numerical range, the essential spectrum and a problem of Halmos*, Acta Sci. Math., to appear.
6. S. Goldberg, *Unbounded linear operators: theory and applications*, McGraw-Hill, New York, 1966.
7. P. Hartman and A. Wintner, *The spectra of Toeplitz's matrices*, Amer. J. Math., vol. 76 (1954), pp. 867–882.
8. G. Piranian, C. J. Titus and G. S. Young, *Conformal mappings and Peano curves*, Mich. Math. J., vol. 1 (1952), pp. 69–72.
9. D. Sarason, *Generalized interpolation on H^∞*, Trans. Amer. Math. Soc., vol. 127 (1967), pp. 179–203.
10. J. G. Stampfli, *Perturbations of the shift*, J. London Math. Soc., vol. 40 (1965), pp. 345–347.
11. ———, *On hyponormal and Toeplitz operators*, Math. Ann., Vol. 183 (1969), pp. 328–336.
12. ———, unpublished note.
13. F. Wolf, *On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators*, Indag. Math., vol. 21 (1959), pp. 142–147.

Wayne State University
Detroit, Michigan