ZERO SETS IN $H^p(U^n)$

BY

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1. Introduction

In this paper we are concerned with zero sets of functions in the spaces $H^p(U^n)$ and in the more general classes $H_\varphi(U^n)$. Here U is the unit disk and U^n is the n-dimensional polydisk. In the one dimensional case it is well known that a sequence $\{a_n\}$ in U is the zero set of a member of $H^p(U)$, $0 < p \leq \infty$, if and only if $\{a_n\}$ satisfies the Blaschke condition

$$\sum (1 - |a_n|) < \infty.$$

The necessity of this condition is a consequence of Jensen's theorem and the sufficiency is a result of the fact that every such sequence is in fact the zero set of a Blaschke product.

Rudin [1], [2] has studied zero sets of members of $H^p(U^n)$ and $H_\varphi(U^n)$ for $n \geq 2$. He showed the situation in higher dimensions is quite different from one dimension by proving in particular that for $0 < p < \infty$ and $n \geq 2$ there exists $f \in H^p(U^n)$ such that if $g \in H^\infty(U^n)$ and g vanishes at every zero of f, then $g \equiv 0$. Rudin [2, p. 63] posed the problem of comparing the zero sets of members of $H^p(U^n)$ and $H^q(U^n)$ for different finite values of p and q. In this paper we provide a solution to this problem by showing that for $0 < p < \infty$ and $n \geq 2$ there exists $f \in H^p(U^n)$ such that if $g \in H^q(U^n)$ for some $q > p$ and the zero set of g contains the zero set of f, then $g \equiv 0$. The technique in [1] is based heavily on the fact that if $f \in H^\infty(U^n)$ then every "slice function" of f is in $H^\infty(U)$. This method is, however, unavailable for our problem because the corresponding property is false for $H^q(U^n)$, $0 < q < \infty$. Our approach is to modify Rudin's techniques through a use of Jensen's theorem so as to avoid this difficulty.

2. Notation

We let C be the complex numbers, U the open unit disk, T the unit circle, U^n and T^n the Cartesian products of n copies of U and T respectively, and m_n Lebesgue measure on T^n normalized so that $m_n(T^n) = 1$. If

$$\varphi : (-\infty, \infty) \to [0, \infty)$$

is convex, non-decreasing, and $\varphi(t)/t \to \infty$ as $t \to \infty$, φ is said to be strongly convex. If φ is strongly convex, $H_\varphi(U^n)$ is the class of all holomorphic functions on U^n such that

$$\sup_{0<r<1} \int_{T^n} \varphi(\log |f(rw)|)\, dm_n(w) < \infty. \tag{2.1}$$

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For the special case $\varphi(t) = e^{pt}$ we denote $H_\varphi(U^n)$ by $H^p(U^n)$. The class of all bounded holomorphic functions on U^n is $H^\infty(U^n)$. If f is a complex-valued function, $f^{-1}\{0\}$ is called the zero set of f and is denoted by $Z(f)$. If $w \in T^n$ and $f : U^n \to C$, we define $f_w : U \to C$ by $f_w(\lambda) = f(w\lambda)$. The function f_w is called a slice function associated with f. It is an easy exercise that if f is holomorphic on U^n then f_w is holomorphic on U.

3. Statement and proof of theorem

We choose to state our result in somewhat greater generality than indicated in the introduction.

THEOREM. *Suppose φ_1 is a strongly convex function. For $n \geq 2$ there exists a function $f \in H_{\varphi_1}(U^n)$ with the property that if φ_2 is a strongly convex function satisfying, for each $a > 0$,*

$$\varphi_2(t - a) > \varphi_1(t) \ (\log \varphi_1(t))t^{1/2}$$

for all $t > t_0(a)$, then $g \in H_{\varphi_2}(U^n)$ and $Z(g) \supset Z(f)$ imply $g \equiv 0$.

We observe that if $0 < p < q < \infty$ then $\varphi_1(t) = e^{pt}$ and $\varphi_2(t) = e^{qt}$ satisfy the hypothesis of the theorem and thus there exists $f \in H^p(U^n)$ such that if $g \in H^q(U^n)$ and $Z(g) \supset Z(f)$ then $g \equiv 0$. The hypothesis

$$\varphi_2(t - a) > \varphi_1(t) \ (\log \varphi_1(t))t^{1/2}$$

for $t > t_0(a)$ is chosen not because it is the weakest hypothesis which implies the desired conclusion but because of its convenience in the statement and proof and because it enables us to demonstrate the differences in the zero sets of the various spaces $H^p(U^n)$. An examination of the proof in fact shows the exponent 1 on $\log \varphi_1(t)$ could be replaced by any constant greater than $\frac{1}{2}$ and independent of φ_2. However we do not know that even this would be sharp.

We first remark it is sufficient to prove the theorem in two dimensions. Suppose this is already accomplished and consider a value of n greater than 2. If $f(z_1, z_2)$ is the function in $H_{\varphi_1}(U^2)$ whose existence is asserted by the theorem, let

$$f_1(z_1, z_2, \cdots, z_n) = f(z_1, z_2).$$

Certainly $f_1 \in H_{\varphi_1}(U^n)$. We show in fact f_1 is the required function. Suppose g is holomorphic on U^n, $g \in H_{\varphi_2}(U^n)$, and $Z(g) \supset Z(f_1)$. We claim for each $\tilde{z} \in U^{n-2}$ the function $\tilde{g}(z_1, z_2) = g(z_1, z_2, \tilde{z})$ is in $H_{\varphi_2}(U^2)$. Because $g \in H_{\varphi_2}(U^n)$ it follows [2, p. 41] that $\varphi_2(\log | g |)$ has an n-harmonic majorant u. Let $\tilde{u}(z_1, z_2) = u(z_1, z_2, \tilde{z})$. Thus $\tilde{u}$ is a 2-harmonic majorant of $\varphi_2(\log | \tilde{g} |)$, and consequently $\tilde{g} \in H_{\varphi_2}(U^2)$. For each $\tilde{z} \in U^{n-2}$, the function $\tilde{g}$ certainly vanishes on $Z(f)$. We conclude $\tilde{g} \equiv 0$ on U^2. Since this is true for each $\tilde{z} \in U^{n-2}$, we have $g \equiv 0$ on U^n.

We now prove the result for $n = 2$. We first define sequences R_k, n_k, r_k, and β_k which will play a central role in the entire discussion. For each integer $k > \max(2, \varphi_1(2))$ we define R_k to be a real number satisfying

$$\varphi_1(1 + \log R_k) = k. \tag{3.1}$$

We then let

$$(3.2) \qquad n_k = [k^4 \log R_k (\log k)^4]$$

and

$$(3.3) \qquad r_k = 1 - 1/k^4 (\log k)^3$$

where [ ] denotes the greatest integer function.

For large values of k, we have

$$(3.4) \qquad \log R_k r_k^{n_k} < \log R_k - n_k/k^4 (\log k)^3 < -2 \log k.$$

Hence

$$(3.5) \qquad \sum_k R_k r_k^{n_k} < \infty.$$

We also have

$$(3.6) \qquad \sum_k \varphi_1(1 + \log R_k) \sqrt{(1 - r_k)} = \sum_k 1/k(\log k)^{3/2} < \infty.$$

We define an increasing bounded sequence by

$$(3.7) \qquad \alpha_k = \sum_{q=k_0}^{k} 1/q (\log q)^2$$

where k_0 is chosen so that

$$(3.8) \qquad \sum_{k=k_0}^{\infty} R_k r_k^{n_k} < 1 - \log 2,$$

$$(3.9) \qquad 0 < \alpha_k < \pi/2, \quad k \geq k_0,$$

$$(3.10) \qquad e^{-1/n_k} < 1 - 1/8n_k, \quad k \geq k_0,$$

$$(3.11) \qquad \sqrt{(1 - r_{k_0})} < \pi/16,$$

$$(3.12) \qquad 8(k + 1)(\log (k + 1))^2 < k^2 (\log k)^{3/2}, \quad k \geq k_0,$$

and

$$(3.13) \qquad \text{if } \theta \in (-\pi, \pi) \text{ and } r_{k_0} < \cos\theta, \text{ then } \cos\theta < 1 - \theta^2/4.$$

From (3.7) we see that the integers greater than or equal to k_0 can be partitioned into sets $I_0, I_1, I_2, \cdots$ such that

$$(3.14) \qquad \sum_{k \in I_j} 1/k \log k = \infty, \quad j = 0, 1, 2, \cdots,$$

and

$$(3.15) \qquad | \alpha_m - \alpha_k | < \pi/2^{j+1} \quad \text{if} \quad m \in I_j \quad \text{and} \quad k \in I_j.$$

If $k \in I_j$ we define

$$(3.16) \qquad \beta_k = \alpha_k + \pi(1 - 2^{-j}).$$

For any function φ_2 satisfying the hypothesis of the theorem and for each $b > 0$ we have for $k > k_0(b)$,

$$(3.17) \qquad \begin{aligned} &\frac{\varphi_2(\log R_k - b)}{\sqrt{n_k}} \\ &\geq \frac{\varphi_1(1 + \log R_k)(\log \varphi_1 (1 + \log R_k))(1 + \log R_k)^{1/2}}{k^2(\log R_k)^{1/2}(\log k)^2} \\ &> \frac{1}{k \log k}. \end{aligned}$$

Consequently, for each $b > 0$,

$$\sum_{k\epsilon I_j} \varphi_2(\log R_k - b)/\sqrt{n_k} = \infty, \quad j = 0, 1, 2, \cdots . \tag{3.18}$$

The desired holomorphic function is

$$f(z_1, z_2) = \prod_{k=k_0}^{\infty} (1 - R_k((z_1 + e^{i\beta_k} z_2)/2)^{n_k}). \tag{3.19}$$

The product converges to a holomorphic function on U^2 by (3.5). We remark that f depends only on φ_1, noting in particular that the partition $I_0, I_1, I_2, \cdots$ is independent of φ_2.

We first show $f \epsilon H_{\varphi_1}(U^2)$. Our approach is to show the partial products f_p for f are uniformly bounded on T^2 except on a small set, namely on $\bigcup_{k=k_0}^{\infty} A_k$, and that

$$\int_{A_k} \varphi_1(\log |f_p(w)|)\, dm_2(w)$$

is dominated for all p by the k^{th} term of a convergent series. This fact, combined with the 2-subharmonicity of $\varphi_1(\log |f_p|)$ and the uniform convergence of the f_p to f on compact sets, is sufficient to imply $f \epsilon H_{\varphi_1}(U^2)$.

We define $A_k \subset T^2$ by

$$A_k = \{(w_1, w_2) : |w_1 + e^{i\beta_k} w_2| > 2r_k\}. \tag{3.20}$$

If $(w_1, w_2) \epsilon A_k$ then for appropriate choices of real α and β we have $w_1 = e^{i\alpha}$, $w_2 = e^{i\beta}$ and

$$\begin{aligned} r_k &< \tfrac{1}{2} |1 + e^{i(\beta_k + \beta - \alpha)}| = \cos \tfrac{1}{2}(\beta_k + \beta - \alpha) \\ &< 1 - (\beta_k + \beta - \alpha)^2/16, \end{aligned} \tag{3.21}$$

where the last inequality follows from (3.13).

We let $B_k \subset T^2$ be

$$B_k = \{(e^{i\alpha}, e^{i\beta}) : |\beta_k + \beta - \alpha| \leq 4\sqrt{(1 - r_k)}\} \tag{3.22}$$

and note from (3.21) that $A_k \subset B_k$.

For each real α,

$$m_1\{e^{i\beta} : (e^{i\alpha}, e^{i\beta}) \epsilon B_k\} = 4(\sqrt{(1 - r_k)})/\pi. \tag{3.23}$$

Hence

$$m_2(B_k) = 4(\sqrt{(1 - r_k)})/\pi. \tag{3.24}$$

We next show the sets B_k are disjoint. We note this implies the sets A_k are also disjoint. Suppose $(w_1, w_2) \epsilon B_k \cap B_m$ and $k < m$. From (3.11) and the fact that β_k and β_m are in $(0, 3\pi/2)$ we see there exist real α and β such that $w_1 = e^{i\alpha}$, $w_2 = e^{i\beta}$,

$$|\beta_k + \beta - \alpha| \leq 4\sqrt{(1 - r_k)} \tag{3.25}$$

and

$$|\beta_m + \beta - \alpha| \leq 4\sqrt{(1 - r_m)}. \tag{3.26}$$

First suppose k and m belong to the same I_j. From (3.7) and (3.16),

$$\beta_m - \beta_k = \alpha_m - \alpha_k \geq 1/(k+1)(\log(k+1))^2. \tag{3.27}$$

Now suppose $k \in I_j$ and $m \notin I_j$. It follows from (3.15) and (3.16) that if $p \in I_j$ and $p > k$ then $|\beta_k - \beta_p| < |\beta_k - \beta_m|$. Hence

$$|\beta_m - \beta_k| > \textstyle\sum_{q=k+1}^{\infty} 1/q(\log q)^2 > 1/(k+1)(\log(k+1))^2. \tag{3.28}$$

However from (3.12), (3.25), and (3.26) we conclude

$$|\beta_m - \beta_k| \leq 8\sqrt{(1-r_k)} < 1/(k+1)(\log(k+1))^2, \tag{3.29}$$

which is the desired contradiction.

Let

$$\textstyle\prod_{k=k_0}^{\infty}(1 + R_k r_k^{n_k}) = c \tag{3.30}$$

where $c < e/2$ by (3.8). For $p \geq k_0$, define

$$f_p(z_1, z_2) = \textstyle\prod_{k=k_0}^{p}(1 - R_k((z_1 + e^{i\beta_k}z_2)/2)^{n_k}). \tag{3.31}$$

If $(w_1, w_2) \in T^2 - \bigcup_{k=k_0}^{p} A_k$, then by (3.20),

$$|f_p(w_1, w_2)| \leq \textstyle\prod_{k=k_0}^{p}(1 + R_k r_k^{n_k}) < c. \tag{3.32}$$

If $(w_1, w_2) \in A_q$ for some q, $k_0 \leq q \leq p$, then by the disjointness of the sets A_k,

$$|f_p(w_1, w_2)| \leq c(1 + R_q) < 2cR_q < eR_q. \tag{3.33}$$

Thus from (3.24), (3.32), and (3.33) we have

$$\begin{aligned} &\int_{T^2} \varphi_1(\log|f_p(w)|)\, dm_2(w) \\ &\leq \varphi_1(\log c) + \sum_{k=k_0}^{p} \int_{A_k} \varphi_1(\log|f_p(w)|)\, dm_2(w) \\ &\leq \varphi_1(\log c) + (4/\pi)\textstyle\sum_{k=k_0}^{p} \varphi_1(1 + \log R_k)\sqrt{(1-r_k)} \\ &\leq c_1 \end{aligned} \tag{3.34}$$

for some constant c_1 which is independent of p by (3.6). Since $\varphi_1(\log|f_p(z)|)$ is 2-subharmonic, we have for all $r < 1$ and all $p \geq k_0$,

$$\int_{T^2} \varphi_1(\log|f_p(rw)|)\, dm_2(w) \leq c_1. \tag{3.35}$$

For each $r < 1$, $f_p(rw)$ converges uniformly to $f(rw)$ on T^2 as $p \to \infty$. Thus

$$\int_{T^2} \varphi_1(\log|f(rw)|)\, dm_2(w) \leq c_1, \quad 0 \leq r < 1 \tag{3.36}$$

and $f \in H_{\varphi_1}(U^2)$.

We now consider a holomorphic function g on U^2, $g \not\equiv 0$, such that $Z(g) \supset Z(f)$. Suppose g has a zero of multiplicity m at the origin where if

$g(0) \neq 0$ we of course take $m = 0$. Let the homogeneous polynomial of degree m in the Taylor series for g be

$$a_0 z_1^m + a_1 z_1^{m-1} z_2 + \cdots + a_m z_2^m. \tag{3.37}$$

For $k \geq k_0$ let $C_k \subset T^2$ be

$$C_k = \{(e^{i\alpha}, e^{i\beta}) : |\beta_k + \beta - \alpha| \leq 1/\sqrt{n_k}\}. \tag{3.38}$$

Clearly $C_k \subset B_k$ and thus the C_k are disjoint. By an argument identical to that used in establishing (3.24) we see that

$$m_2(C_k) = 1/\pi\sqrt{n_k}\,. \tag{3.39}$$

Suppose $w = (w_1, w_2) \in C_k$. Let α and β be real numbers such that $w_1 = e^{i\alpha}$, $w_2 = e^{i\beta}$, and

$$\beta_k + \beta - \alpha = \varepsilon_k(w) = \varepsilon_k \quad \text{where} \quad |\varepsilon_k| \leq 1/\sqrt{n_k}\,. \tag{3.40}$$

The coefficient of λ^m in the Taylor series for $g_w(\lambda)$ is thus

$$e^{im\alpha}(a_0 + a_1 e^{i(\varepsilon_k - \beta_k)} + \cdots + a_m e^{im(\varepsilon_k - \beta_k)}). \tag{3.41}$$

Because the polynomial $a_0 + a_1 z + \cdots + a_m z^m$ has only finitely many zeros and $\{\beta_k\}$ has infinitely many limit points in $(0, 2\pi)$ by (3.16), we see there exist an integer j, a positive number δ, and an integer k_1 such that if $k \in I_j$, $k \geq k_1$, and $w \in C_k$ then $g_w(\lambda)/\lambda^m$ has absolute value at $\lambda = 0$ exceeding δ. For the rest of the discussion we let j have this fixed value.

If $w = (w_1, w_2) \in C_k$ and if α and β are chosen as in (3.40) then

$$\begin{aligned} |(w_1 + e^{i\beta_k} w_2)/2| &= \tfrac{1}{2}\,|1 + e^{i(\beta_k + \beta - \alpha)}| \\ &= \cos \tfrac{1}{2}(\beta_k + \beta - \alpha) \\ &\geq 1 - (\beta_k + \beta - \alpha)^2/8 \\ &\geq 1 - 1/8n_k \\ &> e^{-1/n_k}, \end{aligned} \tag{3.42}$$

where we have used (3.10). For such a value of w, it follows from the definition of f and the fact that $Z(g) \supset Z(f)$ that $g_w(\lambda)/\lambda^m$ has at least n_k zeros at values of λ of modulus

$$(1/R_k)^{1/n_k}\,|2/(w_1 + e^{i\beta_k} w_2)| \leq (e/R_k)^{1/n_k}. \tag{3.43}$$

If we let $n_w(t)$ be the number of zeros of $g_w(\lambda)/\lambda^m$ in $|\lambda| \leq t$ and $x_k = (e/R_k)^{1/n_k}$, then for $x_k < r < 1$,

$$\int_{x_k}^{r} (n_w(t)/t)\,dt \geq n_k(\log r - 1/n_k + (\log R_k)/n_k). \tag{3.44}$$

If in addition $w \in C_k$ for some $k \in I_j$, $k \geq k_1$, then because $g_w(\lambda)/\lambda^m$ has absolute value at $\lambda = 0$ exceeding δ, Jensen's theorem implies

$$\begin{aligned} &(1/2\pi) \int_0^{2\pi} \log |g_w(re^{i\theta})|\,d\theta \\ &\qquad \geq \int_0^r (n_w(t)/t)\,dt + \log \delta + m \log r. \end{aligned} \tag{3.45}$$

Thus for each $k \in I_j$, $k \geq k_1$, there exists $\rho(k)$ such that $\rho(k) < r < 1$ implies for all $w \in C_k$ that

$$(3.46) \qquad (1/2\pi) \int_0^{2\pi} \log | g_w(re^{i\theta}) | \, d\theta > \log R_k + \log \delta - 2.$$

From Jensen's inequality we conclude for $k \in I_j$, $k \geq k_1$, $w \in C_k$, and $\rho(k) < r < 1$ that

$$(3.47) \qquad \begin{aligned} \varphi_2 (\log R_k + \log \delta - 2) &\leq \varphi_2 \left(\frac{1}{2\pi} \int_0^{2\pi} \log | g_w(re^{i\theta}) | \, d\theta \right) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \varphi_2(\log | g_w(re^{i\theta}) |) \, d\theta . \end{aligned}$$

Suppose $M > 0$. By (3.18) there exists an integer N such that

$$(3.48) \qquad \sum\nolimits_{k \in I_j, k_1 \leqq k \leqq N} \varphi_2 (\log R_k + \log \delta - 2)/\sqrt{n_k} > \pi M.$$

Suppose $r > \max \{\rho(k) : k \in I_j \text{ and } k_1 \leq k \leq N\}$. Then by Lemma 3.3.2 of [2], the disjointness of the C_k, and (3.39),

$$(3.49) \qquad \begin{aligned} &\int_{T_2} \varphi_2 (\log | g(rw) |) \, dm_2 (w) \\ &= \int_{T_2} dm_2 (w) \left\{ \frac{1}{2\pi} \int_0^{2\pi} \varphi_2 (\log | g_w (re^{i\theta}) |) \, d\theta \right\} \\ &\geq \sum_{k \in I_j, k_1 \leqq k \leqq N} \int_{C_k} dm_2 (w) \left\{ \frac{1}{2\pi} \int_0^{2\pi} \varphi_2 (\log | g_w (re^{i\theta}) |) \, d\theta \right\} \\ &\geq \sum_{k \in I_j, k_1 \leqq k \leqq N} \varphi_2 (\log R_k + \log \delta - 2) m_2 (C_k) \\ &= \sum_{k \in I_j, k_1 \leqq k \leqq N} \frac{\varphi_2 (\log R_k + \log \delta - 2)}{\pi \sqrt{n_k}} \\ &> M. \end{aligned}$$

Thus $g \notin H_{\varphi_2}(U^2)$ and the proof is complete.

References

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