# COMPLEMENTARY CONES IN DUAL BANACH SPACES

BY

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If $X$ is a compact convex set (in some locally convex space) and $K$ is a closed complemented face (also referred to as a split face) then various order properties are preserved in extending continuous affine functions from $K$ to $X$. For example (Alfsen-Andersen [1, Thm. 3.3]),

(1) if $b_1, b_2 \in A(X)$ and $0 < b_1(x), b_2(x)$ for all $x \in X$ and

$$0 \leq a(x) < b_1(x), b_2(x)$$

for all $x \in K$ then there is an extension $c$ of $a$ such that $0 \leq c < b_1, b_2$ on $X$.

Also [6, Thm. 3.2],

(2) if $a_i \leq b < b_j$ on $X$ and $a_i \leq a < b_j$ on $K$ then $a$ extends to $c$ such that $a_i \leq c < b_j$ on $X$ $(i = 1, \cdots, m; j = 1, \cdots, n)$.

If $A(X)$ (continuous affine functions on $X$) is considered as an ordered Banach space with positive cone $P$ then $X$ is a base for the dual cone $P^*$ and $K$ is a base for a weak$^*$ closed complemented sub-cone $F$. Furthermore if if $Q = \{a \in E : a(x) \geq 0$ for all $x \in F\}$ then $Q$ is a closed cone in $A(X)$ containing $P$ and whose dual cone is $F$. We will refer to $(E, P, Q)$ in this set-up as a *bi-ordered* Banach space. Let $M = Q \cap -Q$. Then

$$M = \{a \in E : a(x) = 0 \quad \text{for all} \quad x \in F\}.$$

We say $a \leq b(P)$ (resp. $(Q)$) if $b - a \in P$ (resp. $Q$). Then (1) can be reformulated as

(3) $0 \leq b_1, b_2(P)$ and $0 \leq a \leq b_1, b_2(Q)$ implies there is $m \in M$ such that $0 \leq a + m \leq b_1, b_2(P)$.

In the following we show that an order condition such as (3) (with a technical modification) provides a necessary and sufficient condition on a bi-ordered Banach space for $Q^*$ to be complemented in $P^*$. Thus in the process we obtain generalizations of the order properties (1) and (2) to cases where the dual cones do not have compact bases. We also apply the results to give somewhat strengthened versions of the order properties for dual cones *with* compact bases (Theorems 2.5 and 2.6). These are analogous to results of Andersen [3] and Alfsen-Hirsberg [2, Thm. 4.5]. Our techniques are based on methods discussed in [4] and [7].

## 1. Preliminaries

Our convention is that an ordered Banach space $E$ is one whose positive cone $P$ is closed and convex and for which $E$ is (i) normal and (ii) positively

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generated. This means that

(i) there is an $M_1 > 0$ such that $y \leq x \leq z$ implies

$$\| x \| \leq M_1(\| y \| \vee \| z \|),$$

and

(ii) there is an $M_2 > 0$ such that each $x = y - z$ with $y, z \in P$ and

$$\| y \| + \| z \| \leq M_2 \| x \|.$$

A bi-ordered Banach space $(E, P, Q)$ is an ordered Banach space $(E, P)$ where $Q$ is a closed convex cone containing $P$. Generally $Q$ will not be pointed and we shall always refer to the subspace $Q \cap -Q$ as $M$. We shall say $a \leq b(P)$ or $a \leq b(Q)$ to distinguish between the two orderings on $E$.

A closed subcone $F$ of $P$ is *complemented* if there is a map $p$ of $P$ onto $F$ such that

(1) $p(x + y) = p(x) + p(y)$
(2) $p(rx) = rp(x)$ $(r \geq 0)$
(3) $p^2x = px \leq x$.

If $F$ is complemented in $P$ then $F$ is extremal as is its complementary subcone $G = \{x \in P : px = 0\}\}$. Also each $x \in P$ has a unique representation $x = y + z$ with $y \in F$ and $z \in G$. Conversely if each $x \in P$ has a unique representation $x = y + z$ with $y, z$ contained in the subcones $F$, $G$ respectively then $F$ and $G$ are complementary with map $px = y$. We will write $P = F \oplus G$ in this case.

If $F$ is complemented in $P$ then $p$ extends to a projection of $E$ onto $N = F - F$ with null space $M = G - G$. Furthermore $p$ is bounded since $x \in P$ implies $0 \leq px \leq x$. By normality $\| px \| \leq M_1 \| x \|$. If $x \in E$ then $x = y - z$ with $y, z \in P$ and $\| y \| + \| z \| \leq M_2 \| x \|$. Thus

$$\| px \| \leq \| py \| + \| px \| \leq M_1(\| y \| + \| z \|) \leq M_1 M_2 \| x \|.$$

In the following proposition we list without proof same facts used later concerning polars of sets. All closures are in the weak, weak* topology on $E$, $E^*$ respectively.

PROPOSITION 1.1. *For $A \subset E$ let $A^0 = \{x \in E^* : x(a) \leq 1, \forall a \in A\}$. If $A \subset E^*$ let $A^0$ be the corresponding set in $E$.*

(1) $A^{00} = \text{cl-conv}(A \cup \{0\})$.

(2) $(A \cup B)^0 = A^0 \cap B^0$. *Thus if $A, B$ are closed convex sets containing* 0 *then*

$$(A \cap B)^0 = (A^{00} \cap B^{00})^0 = (A^0 \cup B^0)^{00} = \text{cl-conv}(A^0 \cup B^0)$$

(3) *If $B$ is a closed subspace than $B^0 = B^\perp$ and*

$$(A \cap B)^0 = (A^0 + B^0)^-.$$

If $B$ is a weak* closed convex set in $E^*$ containing 0 let $B_c$, the *asymptotic* *cone* of $B$, be defined by $B_c = \bigcap_{0<r\leq 1} rB$. Then $B_c$ is the union of rays in $B$ emanating from 0 and is a closed convex cone. Moreover $B = B + B_c$.

PROPOSITION 1.2. *Let $A$, $B$ be weak* closed convex sets in $E^*$ containing* 0 *such that $A$ is strongly bounded. Then*

$$w^*\text{-cl-conv}(A \cup B) = \text{conv}(A + B_c) \cup B.$$

*In particular* $w^*\text{-cl-conv}(A \cup B) = \|\cdot\|\text{-cl-conv}(A \cup B)$.

*Proof.* Let $(x_\alpha)$ be a net in $\text{conv}(A \cup B)$ and $x_\alpha \to x$ (weak*). Then $x_\alpha = \lambda_\alpha y_\alpha + (1 - \lambda_\alpha)z_\alpha$, $0 \leq \lambda_\alpha \leq 1$, $y_\alpha \in A$, $z_\alpha \in B$. Since $A$ is weak* compact we can assume by passing to a sub-net that $y_\alpha \to y \in A$ and $\lambda_\alpha \to \lambda$. If $\lambda < 1$ then eventually $\lambda_\alpha < 1$ so that

$$z_\alpha = (x_\alpha - \lambda_\alpha y_\alpha)/(1 - \lambda_\alpha) \to (x - \lambda y)/(1 - \lambda) \in B.$$

Thus $x = \lambda y + (1 - \lambda)z \in \text{conv}(A \cup B)$. If $\lambda = 1$ let $0 < r \leq 1$. Eventually $1 - \lambda_\alpha < r$. Then

$$(1 - \lambda_\alpha)z_\alpha = x_\alpha - \lambda_\alpha y_\alpha \in B \cap (1 - \lambda_\alpha)B \subset B \cap rB.$$

Thus $x - y \in B \cap rB$ and therefore $x - y \in B_c$. Then

$$x = y + (x - y) \in A + B_c.$$

Since $\text{conv}(A + B_c) \cup B$ is the linear closure of $\text{conv}(A \cup B)$ it is contained in and hence equal to the norm closure of $\text{conv}(A \cup B)$.

We also make use of the following facts on compact convex sets and their affine function spaces. Proposition 1.3 is essentially Lemma 9.6 of [8]. Proposition 1.4 is proved by a standard compactness argument on the graphs of the given functions (see for example [4, Cor. 2]).

PROPOSITION 1.3. *Let $K$ be a compact convex subset of a locally convex space and let $A(K)$ denote the space of continuous affine functions on $K$. Let $p$ be a continuous function on $K$ and let the lower envelope $\hat{p}$ be defined by*

$$\hat{p}(x) = \sup\{a(x) : a \in A(K) \quad \text{and} \quad a \leq p\}.$$

*Then*

$$\hat{p}(x) = \inf\{\textstyle\sum_{i=1}^n \lambda_i p(x_i) : x = \sum_{i=1}^n \lambda_i x_i; x_i \in K, 0 \leq \lambda_i \leq 1, \sum_{i=1}^n \lambda_i = 1\}.$$

PROPOSITION 1.4. *Let $K$ be compact convex and $F$ a closed face of $K$. If $b \in A(K)$, $a \in A(F)$ and $-u$, $v$ are concave lsc on $K$ such that*

$$u < b < v \text{ on } K, \qquad u|_F < a < v|_F \text{ on } F,$$

*then there are continuous concave functions $-u'$, $v'$ such that*

$$u < u' < b < v' < v \text{ on } K, \qquad u|_F < u'|_F < a < v'|_F < v|_F \text{ on } F.$$

PROPOSITION 1.5. *Let $P$ be a closed generating cone in the Banach space $E$ and let $p$ be a weak* continuous non-negative homogeneous function on $P^*$.*

*Define*

$$\bar{p}(x) = \inf\{\textstyle\sum_{i=1}^{n} p(x_i) : x = \sum_{i=1}^{n} x_i;\ x_i \in P^*\}$$

*and*

$$\hat{p}(x) = \sup\{a(x) : a \in E \quad \textit{and} \quad a \leq p \textit{ on } P^*\}.$$

*Then:*

(1) *$\bar{p}$ is sub-additive homogeneous on $P^*$.*

(2) *$\hat{p}$ is sub-additive homogeneous and $\hat{p} \leq \bar{p}$ on $P^*$.*

(3) *If $A = \{x \in P^* : \bar{p}(x) \leq 1\}$ then $A^0 = \{a \in E : a \leq p$ on $P^*\}$.*

(4) *$\{x \in P^* : \hat{p}(x) \leq r\} = w^*$-cl-conv $\{x \in P^* : p(x) \leq r\}$. In particular $\hat{p}$ is weak$^*$ lsc.*

(5) *If $a_1, \cdots, a_m \in P$ and $p = a_1 \wedge \cdots \wedge a_m$ on $P^*$ then $\bar{p} = \hat{p}$.*

(6) *If $P^{**}$ has non-empty interior then $\bar{p} = \hat{p}$.*

(7) *Let $(E, P, Q)$ be bi-ordered and let $p$ be super-additive*

$$(p(x + y) \geq p(x) + p(y)).$$

*Let $\bar{p}_P$, $\bar{p}_Q$ be defined on $P^*$, $Q^*$ resp. If $Q^*$ is complemented in $P^*$ then*

$$\bar{p}_Q = \bar{p}_{P|Q^*}.$$

*If $j$ is the projection of $P^*$ onto $Q^*$ then*

$$\bar{p}_Q \circ j \leq \bar{p}_P.$$

*Proof.* Properties (1), (2) and (3) are straightforward. From the definition of $\hat{p}$, $\{x : \hat{p}(x) \leq r\}$ is weak$^*$ closed and contains $\{x : p(x) \leq r\}$. If $a \in E$ and $a(x) \leq r$ whenever $p(x) \leq r$ then it follows from the homogeneity of $p$ that $a \leq p$ on $P^*$. Thus $a \leq \hat{p}$ and the equality in (4) follows from the separation theorem. For (5) we note that in this case

$$\bar{p}(x) = \inf\{\textstyle\sum_{i=1}^{m} a_i(x_i) : x = \sum_{i=1}^{m} x_i\}.$$

Since $E^*$ is normal and $0 \leq x_i \leq x$ there is a number $\alpha/m$ such that

$$\| x_i \| \leq (\alpha/m) \| x \|.$$

Hence

$$\textstyle\sum_{i=1}^{m} \| x_i \| \leq \alpha \| x \|.$$

Let $K = \{x \in P^* : \| x \| \leq 1\}$. If $x \in K$ and $x \doteq \sum_{i=1}^{m} x_i$ then $x = \sum_{i=1}^{m} \lambda_i y_i$ where

$$y_i = [(\textstyle\sum_{i=1}^{m} \| x_i \|)/\| x_i \|]x_i \quad \text{and} \quad \lambda_i = \| x_i \|/\sum_{i=1}^{m} \| x_i \|.$$

Thus $\| y_i \| = \sum_{i=1}^{m} \| x_i \| \leq \alpha \| x \| \leq \alpha$. Therefore

$$\begin{aligned}
\bar{p}(x) &= \inf\{\textstyle\sum_{i=1}^{m} a_i(x_i) : x = \sum_{i=1}^{m} x_i\} \\
&= \inf\{\textstyle\sum_{i=1}^{m} \lambda_i a_i(y_i) : x = \sum_{i=1}^{m} \lambda_i y_i, \quad y_i \in \alpha K, \quad \sum_{i=1}^{m} \lambda_i = 1\} \\
&= \inf\{\textstyle\sum_{i=1}^{n} \lambda_i p(y_i) : x = \sum_{i=1}^{n} \lambda_i y_i, \quad y_i \in \alpha K, \quad \sum_{i=1}^{n} \lambda_i = 1\} \\
&= \sup\{a(x) : a \in A(\alpha K) \quad \text{and} \quad a \leq p\,|_{\alpha K}\},
\end{aligned}$$

where the last equality is a consequence of Proposition 1.3. Thus $\bar{p}\,|_K$ is weak* lsc and since $\bar{p}$ is homogeneous the Krein-Smulyan Theorem yields $\bar{p}$ weak* lsc on P*. If $x \in \operatorname{conv}\{y : p(y) \leq 1\}$ then $\bar{p}(x) \leq 1$. Hence

$$w^*\text{-cl-conv}\{y : p(y) \leq 1\} = \{y : \hat{p}(y) \leq 1\} \subset \{y : \bar{p}(y) \leq 1\}.$$

Thus $\bar{p} \leq \hat{p}$ and equality follows.

For (6), if $P^{**}$ has non-empty interior then there is an $\alpha > 0$ such that $x = \sum_{i=1}^n x_i$ $(x_i \in P^*)$ then $\sum_{i=1}^n \| x_i \| \leq \alpha \| x \|$.

The proof of (6) is now identical to (5).

For (7) note that if $Q^*$ is complemented in $P^*$ then it is extremal. Thus $x \in Q^*$ and $x = \sum_{i=1}^m x_i$ implies $x_i \in Q^*$. Hence $\bar{p}_Q = \bar{p}_{P|Q^*}$. If $x \in P^*$ then $0 \leq jx \leq x$. Since $p$ is super-additive it is monotonic, that is, $p(jx) \leq p(x)$. Thus if $x = \sum_{i=1}^n x_i$ then

$$\bar{p}_Q \circ j(x) \leq \textstyle\sum_{i=1}^n p(jx_i) \leq \sum_{i=1}^n p(x_i).$$

Therefore

$$\bar{p}_Q \circ j(x) \leq \bar{p}_P(x).$$

Proposition 1.6. *Let $(E, P, Q)$ be bi-ordered such that $Q^*$ is complemented in $P^*$. Then $N = Q^* - Q^*$ is weak* closed and $N^0 = M = Q \cap -Q$. Every weak* continuous homogeneous additive function $a$ on $Q^*$ extends to an element $c \in E$..*

*Proof.* Let $K = \{x \in Q^* : \| x \| \leq 1\}$ and $X = \{x \in P^* : \| x \| \leq 1\}$. Since $Q^*$ is complemented $N$ is the range of a continuous projection and hence norm closed. Since $N \cap X = K$ the conclusions now follow from [5, Thm. 3.1].

## 2. Duality results

We give first an order property for $(E, P, Q)$ analogous to (3) in the introduction that is necessary and sufficient for $Q^*$ to be complemented in $P^*$. We prove sufficiency first.

Theorem 2.1. *Let $(E, P, Q)$ be a bi-ordered Banach space such that if $0 \leq b_1, b_2(P)$ and $0 \leq a \leq b_1, b_2(Q)$ then for any $\varepsilon > 0$ there is an*

$$m \in M = Q \cap -Q$$

*and $c$ with $\| c \| < \varepsilon$ for which $0 \leq a + m \leq b_1 + c, b_2 + c(P)$. Then $Q^*$ is complemented in $P^*$.*

*Proof.* If $a, b \in P$ let us say $a \approx b$ if and only if $a - b \in M$. Given $x \in P^*$ and $a \in P$ define

$$(px)(a) = \inf\{x(b) : b \approx a\}.$$

(i) $(px)(ra) = r(px)a$ $(r \geq 0, a \in P, x \in P^*)$.

(ii) $(px)(a_1 + a_2) = px(a_1) + px(a_2)$.

If $b_i \approx a_i$ then $b_1 + b_2 \approx a_1 + b_2$ and hence

$$px(a_1 + a_2) \leq x(b_1 + b_2) = x(b_1) + x(b_2).$$

Thus $px(a_1 + a_2) \leq px(a_1) + px(a_2)$. If $b \approx a_1 + a_2$ then $0 \leq b(P)$, $0 \leq a_1 \leq b(Q)$ and so there is a $b_1 \approx a_1$ with $0 \leq b_1 \leq b + c(P)$ and $\| c \|$ arbitrarily small. Since $(E, P)$ is positively generated we can assume $c \in P$. Now

$$0 \leq b + c - b_1(P) \quad \text{and} \quad 0 \leq a_2 \approx b - b_1 \leq b + c - b_1(Q).$$

Thus $a_2 \approx b_2$ with $0 \leq b_2 \leq b + c - b_1 + c'$, $\| c' \|$ arbitrarily small. Therefore $0 \leq b_1 + b_2 \leq b + c + c'$ and

$$x(b) \geq x(b_1) + x(b_2) - x(c) - x(c') \geq px(a_1) + px(a_2) - x(c + c').$$

Thus (ii) follows.

(iii) $p(rx)(a) = r(px)(a)$ $(r \geq 0, x \in P^*, a \in P)$.

(iv) If $x_1, x_2 \in P^*$ than $p(x_1 + x_2)(a) = px_1(a) + px_2(a)$ for all $a \in P$.

Given $a \in P$ and $b \approx a$,

$$(x_1 + x_2)(b) = x_1(b) + x_2(b) \geq px_1(a) + px_2(a).$$

Thus $p(x_1 + x_2)(a) \geq px_1(a) + px_2(a)$. Choose $b_1, b_2 \approx a$ such that $x_i(b_i) < px_i(a) + \varepsilon/2$. Now $0 \leq b_1, b_2(P)$ and $0 \leq a \leq b_1, b_2(Q)$ so that by (1) there is $b \approx a$ with $0 \leq b \leq b_1 + c, b_2 + c(P)$. Then

$$\begin{aligned} p(x_1 + x_2)(a) &\leq (x_1 + x_2)(b) \\ &= x_1(b) + x_2(b) \\ &\leq x_1(b_1) + x_2(b_2) + \| c \| (\| x_1 \| + \| x_2 \|) \\ &< px_1(a) + px_2(a) + \varepsilon + \| c \| (\| x_1 \| + \| x_2 \|). \end{aligned}$$

(v) $0 \leq (px)(a) \leq x(a)$.

Thus for each $x \in P^*$, $px$ is a linear form on $P$ and hence extends to a linear functional on $E$. Moreover by (v) $px$ is bounded and contained in $P^*$. Also by (iii) and (iv), $p : P^* \to P^*$ is positive homogeneous and additive. Again, by (v) the canonical extension of $p$ to an operator on $E^*$ is bounded.

(vi) $p^2 = p$ on $P^*$.

For $x \in P^*$, $a \in P$,

$$\begin{aligned} (p^2x)(a) &= \inf \{px(b) : b \approx a\} \\ &= \inf \{\inf \{x(c) : c \approx b\} : b \approx a\} \\ &= px(a). \end{aligned}$$

Let $F = \{px : x \in P^*\}$ and $G = \{x \in P^* : px = 0\}$. Then (v) and (vi) show that $P^* = F \oplus G$. It remains to show that $F = Q^*$. We show first that $M = M \cap P - M \cap P$. Let $a \in M$. Choose $b \in P$ such that $b \geq a(P)$. Then

$$0, a \leq b(P) \quad \text{and} \quad 0, a \leq 0 \leq b(Q).$$

Thus an application of the hypothesis yields $m_1 \in M$ and $c_1 \in P$ with $\| c_1 \| < \frac{1}{2}$ and $-c_1, a - c_1 \leq m_1 \leq b(P)$. Thus

$$0, a, m_1 \leq m_1 + c_1(P) \quad \text{and} \quad 0, a, m_1 \leq 0 \leq m_1 + c_1(Q).$$

Another application yields $m_2 \in M$, $c_2 \in P$ with $\| c_2 \| < \frac{1}{4}$ and

$$0, a, m_1 \leq m_2 + c_2 \leq m_1 + c_1 + c_2(P).$$

Continuing by induction we obtain sequences $(m_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ with $m_n \in M$, $\| c_n \| < \frac{1}{2}^n$, and

$$0, a, m_n \leq m_{n+1} + c_{n+1} \leq m_n + c_n + c_{n+1}(P).$$

It follows that

$$-c_{n+1} \leq m_{n+1} - m_n \leq c_n$$

and hence by normality $\| m_{n+1} - m_n \|$ is bounded by a constant times $2^{-n}$. Therefore $m_n \to m \in M$ and $c_n \to 0$ so that $0, a \leq m(P)$ and hence

$$a = m - (m - a) \in M \cap P - M \cap P.$$

If $y \in F$ and $a \in M \cap P$ then $a \approx 0$ and hence $y(a) = py(a) = 0$. Thus $y \in F$ and $a \in M$ implies $y(a) = 0$. If $0 \leq a(Q)$ then the hypothesis implies there is $m \in M$ such that $0 \leq a + m(P)$. Then

$$y(a) = y(a) + y(m) = y(a + m) \geq 0.$$

Thus $F \subset Q^*$. On the other hand $y \in Q^*$ and $a \in M$ implies $y(a) = 0$. Thus for $a \in P$ $(py)(a) = \inf\{y(b) : b \approx a\} = y(a)$. Hence $py = y \in F$ so $F = Q^*$.

COROLLARY. *Let $(E, P)$ be an ordered Banach space with $P = F \oplus G$ and let $Q = P + (F - F)$. Then $(E, P, Q)$ is bi-ordered with $Q^*$ complemented in $P^*$.*

*Proof.* If $p : P \to F$ and $q : P \to G$ are extended to complementary projections on $E$ with $M = F - F = q^{-1}(0)$ then $Q = q^{-1}(0)$ and hence is closed. Moreover $Q \cap -Q = M$. If $a \in Q$ then $qa \in P$. Thus $a \leq b(Q)$ implies $qa \leq qb(P)$. Thus if $0 \leq b_1, b_2(P)$ and $0 \leq a \leq b_1, b_2(Q)$ then

$$0 \leq qa = a + (qa - a) \leq qb_1, qb_2(P).$$

But $0 \leq b_i(P)$ implies $qb_i \leq b_i(P)$ and hence $0 \leq a + m \leq b_1, b_2(P)$ where $m = qa - a \in M$. Therefore by Theorem 2.1, $Q^*$ is complemented in $P^*$.

In the next theorem we give the converse of Theorem 1.1 with a formally stronger conclusion.

THEOREM 2.2. *Let $(E, P, Q)$ be a bi-ordered Banach space for which $Q^*$ is complemented in $P^*$. Let $a_1, \cdots, a_m, b_1, \cdots, b_n, b$ be elements of $E$ and let $a$ be a $w^*$ continuous homogeneous additive function on $Q^*$ such that*

(1) $a_1, \cdots, a_m \leq b \leq b_1, \cdots, b_n(P)$.

(2) $a_1, \cdots, a_m \leq a \leq b_1, \cdots, b_n(Q)$.

*Then given* $\varepsilon > 0$ *there is an extension* $c \in E$ *of* $a$ *such that*

(3) $a_1, \cdots, a_m \leq c \leq b_1 + z, \cdots, b_n + z(P); \| z \| < \varepsilon.$

*Proof.* We can assume by Proposition 1.6 that $a \in E$ and satisfies (2). Thus we must find $c$ satisfying (3) where $c = a + m$, $m \in M = Q \cap -Q$. We show first that given $\varepsilon > 0$, $\delta > 0$ that

(4) $a = c + m + w; m \in M, \| w \| < \delta$ and

$$a_1 \vee \cdots \vee a_m(x) \leq c(x) \leq b_1 \wedge \cdots \wedge b_n(x) + z(x) \quad (x \in P^*)$$

where $\| z \| < \varepsilon$.

We assume without loss of generality that $b = 0$. Then let

$$v = b_1 \wedge \cdots \wedge b_n, \quad u = -(a_1 \vee \cdots \vee a_m)$$

on $P^*$ with $\bar{v}_P, \bar{v}_Q, \bar{u}_P, \bar{u}_Q$ defined as in Proposition 1.5. By 1.5 (5) these functions are lsc on $P^*$, $Q^*$. Let

$$V_P = \{x \in P^* : \| x \| \leq r, \bar{v}_P(x) \leq 1\}, \quad V_Q = \{x \in Q^* : \bar{v}_Q(x) \leq 1\},$$

$$U_P = \{x \in -P^* : \bar{u}_P(-x) \leq 1\}, \quad U_Q = \{x \in -Q^* : \bar{u}_Q(-x) \leq 1\}.$$

Let

$$A_P = \{c \in E : c \geq a_1, \cdots, a_m(P)\}, \quad A_Q = \{c \in E : c \geq a_1, \cdots, a_m(Q)\},$$

$$B_P = \{c \in E : c \leq b_1, \cdots, b_n(P)\}, \quad B_Q = \{c \in E : c \leq b_1, \cdots, b_n(Q)\}.$$

Each is a weakly closed convex set containing the origin and using the notation $E_s = \{x \in E : \| x \| \leq s\}$ we have

$$(V_P)^0 = \text{cl-conv}(B_P \cup E_{1/r}) \subset B_P + E_{2/r}, \quad (U_P)^0 = A_P,$$

$$(V_Q)^0 = B_Q, \qquad (U_Q)^0 = A_Q.$$

We have for $N = Q^* - Q^*$,

(5) $[w^*\text{-cl-conv}(V_P \cup U_P)] \cap N \subset w^*\text{-cl-conv}(V_Q \cup U_Q).$

For, if $z \in \text{conv}(V_P \cup U_P)$ and $p$ is the projection of $P^*$ onto $Q^*$ then by 1.5 (7) $pz \in \text{conv}(V_Q \cup U_Q)$. Since by Proposition 1.2, $w^*$ closure is norm closure on the left we consider $z \in N$ with $z = \lim z_n$, $(z_n)_{n=1}^{\infty} \subset \text{conv}(V_P \cup U_P)$. Then $pz_n \to pz = z$ and so $z \in w^*\text{-cl-conv}(V_Q \cup U_Q)$. This proves (5).

By taking the polar of both sides in (5) we obtain

$$\begin{aligned} B_Q \cap A_Q &= (V_Q)^0 \cap (U_Q)^0 \\ &= [w^*\text{-cl-conv}(V_Q \cup U_Q)]^0 \\ &\subset \{[w^*\text{-cl-conv}(V_P \cup U_P)]^0 + M\}^- = [V_P^0 \cap U_P^0 + M]^- \\ &\subset [(B_P + E_{2/r}) \cap A_P + M]^-. \end{aligned}$$

Thus, since $a \in B_Q \cap A_Q$, (4) follows if $r$ is chosen greater than $2/\varepsilon$. To establish (3) first use (4) to find

$$a = c_1 + m_1 + w_1, \quad m_1 \in M,$$

where (since $(E, P)$ is positively generated) $w_1 = w_{11} - w_{12}$ with $w_{1i} \in P$, $\| w_{1i} \| < \frac{1}{2}$ and

$$a_1, \cdots, a_m \leq c_1 \leq b_1 + z_1, \cdots, b_n + z_1(P); \quad \| z_1 \| < \varepsilon/2.$$

Then

$$a_1, \cdots, a_m, c_1 - w_{12} \leq c_1 \leq b_1 + z_1, \cdots, b_n + z_1, c_1 + w_{11}(P),$$

$$a_1, \cdots, a_m, c_1 - w_{12} \leq a \leq b_1 + z_1, \cdots, b_n + z_1, c_1 + w_{11}(Q).$$

Hence applying (4) again

$$a = c_2 + m_2 + w_2; \quad m_2 \in M, \quad w_2 = w_{21} - w_{22} \quad (w_{2i} \in P, \| w_{2i} \| < \tfrac{1}{4})$$

and

$$a_1, \cdots, a_m, c_1 - w_{12} \leq c_2 \leq b_1 + z_1 + x_2, \cdots, b_n + z_1 + z_2, c_1 + w_{11}(P).$$

By induction we have sequences $(c_k)$, $(m_k)$, $(z_k)$, $(w_k)$ such that

$$w_k = w_{k1} - w_{k2} \quad (w_{ki} \in P, \| w_{ki} \| < \tfrac{1}{2}^k)$$

$$a = c_k + m_k + w_k, \quad m_k \in M,$$

$$a_1, \cdots, a_m \leq c_k \leq b_1 + z_1 + \cdots + z_k, \cdots, b_n + z_1 + \cdots + z_k;$$

$$\| z_k \| < \varepsilon/2^k,$$

$$c_k - w_{k2} \leq c_{k+1} \leq c_k + w_{k1} + z_{k+1}.$$

Since $P$ is normal the last inequality shows that $\| c_{k+1} - c_k \|$ is on the order of $\frac{1}{2}^k$ and hence $(c_k)_{k=1}^{\infty}$ converges to $c \in E$ such that

$$a_1, \cdots, a_m \leq c \leq b_1 + z, \cdots, b_n + z; \quad \| z \| \leq \sum \| z_k \| < \varepsilon.$$

Since $w_k \to 0$ we have $a = c + m$ with $m = \lim m_k \in M$.

If Theorems 2.1 and 2.2 are combined we have a characterization of complemented dual cones.

THEOREM 2.3. *Let $(E, P, Q)$ be a bi-ordered Banach space. Then $Q^*$ is complemented in $P^*$ if and only if the order condition of Theorem 2.1 holds.*

If $P^{**}$ has non-empty interior then making use of Proposition 1.5(6) a stronger version of Theorem 2.2 is possible. The proof follows the same lines as Theorem 2.2 and is omitted.

THEOREM 2.4. *Let $(E, P, Q)$ be a bi-ordered Banach space for which $P^{**}$ has non-empty interior and $Q^*$ is complemented in $P^*$. Let $-u$, $v$ be weak$^*$ continuous homogeneous super-additive functions on $P^*$ and let $a$ be a weak$^*$ continuous homogeneous additive function on $Q^*$ such that*

(1) *there is $b \in E$ for which $u \leq b \leq v(P)$,*

(2) *$u \leq a \leq v(Q)$.*

*Then given $\varepsilon > 0$ there is an extension $c \in E$ of $a$ such that*

$$u \leq c \leq v + z(P), \quad z \in E \quad and \quad \| z \| < \varepsilon.$$

If $P$ has non-empty interior then $P^*$ is weak* locally compact and hence has a compact base $K$. In this case the space $E$ is isomorphic to $A(K)$ and $Q^* \cap K$ is a complemented or split face of $K$. We will use the convention that $a < b(P)$ mean $b - a$ is in the *interior* of $P$. We can apply the technique of Theorem 2.2 together with a modified iteration (related to Andersen's method [3]) to obtain a stronger version of Theorem 2.2.

THEOREM 2.5. *Let $(E, P, Q)$ be bi-ordered with $Q^*$ complemented in $P^*$ and the interior of $P$ non-empty. If $-u$, $v$ are weak* continuous super-additive forms (homogeneous) on $P^*$ with*
(1) *$u \leq b < v(P)$ for some $b \in E$,*
(2) *$u \leq a \leq v(Q)$ for a weak* continuous additive form on $P^*$,*
*then $a$ extends to $c \in E$ such that $u \leq c \leq v(P)$.*

*Proof.* Assume without loss that $b = 0$ and that $a \in E$. Let

$$V_P = \{x \in P^* : \bar{v}_P(x) \leq 1\}, \quad V_Q = \{x \in Q^* : \bar{v}_Q(x) \leq 1\}$$

$$U_P = \{x \in -P^* : (-u)_P^-(-x) \leq 1\}, \quad U_Q = \{x \in -Q^* : (-u)_Q^-(-x) \leq 1\}.$$

Let $A_P = \{c \in E : c \geq u(P)\}$ with $A_Q$, $B_P$, $B_Q$ defined analogously. Then by (1), $B_P$ has non-empty interior and hence $V_P = B_P^0$ is bounded and therefore weak* compact. As in Theorem 1.2 we have

$$w^*\text{-cl-conv}(V_P \cup U_P) \cap N = w^*\text{-cl-conv}(V_Q \cup U_Q).$$

The polar then becomes

$$B_Q \cap A_Q = [B_P \cap A_P + M]^-.$$

Thus given $\varepsilon > 0$, $a = c + m + w$ with $m \in M$, $\| w \| < \varepsilon$ and $u \leq c \leq v(P)$. Choose $e \in \operatorname{int} P$ such that $w \in E$ and $\| w \| \leq 1$ implies $-e \leq w \leq e$. Now assume (without loss) $\| a \| \leq 1$. Then

$$u \leq 0 < v/2(P), \quad u \leq a/2 \leq v/2(Q).$$

Thus using the above we have

$$a/2 = c_1 + m_1 + w_1, \quad \| w_1 \| \leq \tfrac{1}{4}, \quad m_1 \in M \quad \text{and} \quad u \leq c_1 \leq v/2(P).$$

We show by induction there are sequences $(c_n)$, $(m_n)$, $(w_n)$ such that

$$\| w_n \| \leq \tfrac{1}{2}^{n+1}, \quad m_n \in M,$$

and

$$(1 - \tfrac{1}{2}^n)a = c_n + m_n + w_n$$

with

$$u \leq c_n \leq (1 - \tfrac{1}{2}^n)v(P) \quad \text{and} \quad -e/2^n \leq c_{n+1} - c_n \leq e/2^n.$$

Suppose $c_n$ has been chosen as required. Then

$$\begin{aligned} c_n - e/2^n \leq c_n + w_n - e/2^{n+1} &= (1 - \tfrac{1}{2}^n)a - m_n - e/2^{n+1} \\ &\leq (1 - \tfrac{1}{2}^n)a + a/2^{n+1} \\ &= (1 - \tfrac{1}{2}^{n+1})a \\ &\leq c_n + w_n + a/2^{n+1} \\ &\leq c_n + e/2^n (Q). \end{aligned}$$

Since $u \leq c_n < (1 - \frac{1}{2}^{n+1})v(P)$ we have

$$u \vee (c_n - e/2^n) \leq c_n < (1 - \tfrac{1}{2}^{n+1})v \wedge (c_n + e/2^n)(P)$$

and

$$u \vee (c_n - e/2^n) \leq (1 - \tfrac{1}{2}^{n+1})a \leq (1 - \tfrac{1}{2}^{n+1})v \wedge (c_n + e/2^n)(Q).$$

Thus $c_{n+1}$ can be chosen as claimed. Thus $(c_n)$ is Cauchy and in the limit

$$a = c + m; \quad u \leq c \leq v(P).$$

Finally, again in case int $P$ is not empty we give an extension result where the dominating functions are only assumed to be semi-continuous.

THEOREM 2.6. *Let $F$ be a closed split face of the compact convex set $K$ and let $-u$, $v$ be lsc concave functions on $K$, $a \in A(F)$ and $b \in A(K)$ such that*

$$u < b < v \text{ on } K, \qquad u \leq a \leq v \text{ on } F.$$

*Then $a$ extends to $c \in A(K)$ such that $u \leq c \leq v$.*

*Proof.* Extend $a$ to a function in $A(K)$ also referred to as $a$. We assume without loss that $\| b - a \| \leq 1$. By Proposition 1.4 and Theorem 2.5 if

$$u < b < v \text{ on } K \quad \text{and} \quad u < a < v \text{ on } F$$

there is $c \in A(K)$ such that $c|_F = a|_F$ and $u < c < v$ on $K$. We construct a sequence $(c_n)$ in $A(K)$ such that

(1) $u < c_n < v$ on $K$,
(2) $c_n|_F = b + (1 - \frac{1}{2}^n)(a - b)|_F$, $|c_{n+1} - c_n| < \frac{1}{2}^n$ on $K$.

Since $u < b < v$ on $K$ and $u < b + \frac{1}{2}(b - a) < v$ on $F$ there is $c_1 \in A(K)$ satisfying (1) and (2) above. Assume $c_n$ has been chosen satisfying (1) and (2). Then on $F$,

$$\begin{aligned} c_n - \tfrac{1}{2}^n &= b + (1 - \tfrac{1}{2}^n)(a - b) + \tfrac{1}{2}^{n+1}(a - b) \\ &= b + (1 - \tfrac{1}{2}^{n+1})(a - b) \\ &< c_n + \tfrac{1}{2}^n. \end{aligned}$$

Hence

$$(c_n - \tfrac{1}{2}^n) \vee u < c_n < (c_n + \tfrac{1}{2}^n) \wedge v \quad \text{on } K$$

and

$$(c_n - \tfrac{1}{2}^n) \vee u < b + (1 - \tfrac{1}{2}^{n+1})(a - b) < (c_n + \tfrac{1}{2}^n) \wedge v \quad \text{on } F.$$

Thus $c_{n+1}$ can be chosen as required. Let $c = \lim_{n\to\infty} c_n$. Then

$$u \leq c \leq v \quad \text{on } K \quad \text{and} \quad c|_F = b + (a - b)|_F = a|_F.$$

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