FOURIER-STIELTJES TRANSFORMS WITH SMALL SUPPORTS

BY
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Abstract

Let G be a LCA group with dual Γ. Suppose $S \subseteq \Gamma$ is a Borel set such that $S \cap (\gamma - S)$ has finite Haar measure for a dense set of $\gamma \in \Gamma$ (or $S \cap (S - \gamma)$ does). If μ and ν are regular Borel measures whose Fourier-Stieltjes transforms vanish off S, then $|\mu| * |\nu| \in L^1(G)$ ($|\mu|$ denotes the total variation measure). This generalizes to non-metrizable groups a result of Glicksberg. Related results are given; the proofs are elementary.

1. Introduction

Glicksberg's result appears in [G], where he shows that if μ, ν and S obey the hypothesis then $\mu * \nu \in L^1(G)$, and, if G is metrizable, $|\mu| * |\nu| \in L^1(G)$ also. See [M], [PS], [W] for similar and/or related results. I use the notation of [R] and prove:

Theorem 1. *Let G be a LCA group with dual Γ and $S \subseteq \Gamma$ a Borel set such that either*

(a) $\{\gamma : S \cap (S - \gamma)$ *has finite Haar measure*$\}$ *is dense in* Γ, *or*
(b) $\{\gamma : S \cap (\gamma - S)$ *has finite Haar measure*$\}$ *is dense in* Γ.

Let μ, ν be regular Borel measures on G whose Fourier-Stieltjes transforms $\hat{\mu}$, $\hat{\nu}$ vanish off S. Then $|\mu| * |\nu| \in L^1(G)$.

Glicksberg's proof uses disintegration of measures. (See remarks at the end of this paper.) An iteration of the method here yields many varients. Theorem 2 is a sample. Its proof is left to the reader.

Theorem 2. *Let G be a LCA group with dual Γ, and $S \subseteq \Gamma$ a Borel set such that for some integers, $m, n \geq 1$,*

$$\{(\gamma_1, \cdots, \gamma_{n+m}) : S \cap (S - \gamma_1) \cap \cdots \cap (S - \gamma_n) \cap (\gamma_{n+1} - S) \cap \cdots \cap (\gamma_{n+m} - S) \text{ has finite measure}\}$$

is dense in Γ^{n+m}.

Let $\mu_0, \mu_1, \cdots, \mu_{n+m}$ be regular Borel measures whose Fourier-Stieltjes transforms vanish off S. Then $|\mu_0| * \cdots * |\mu_{n+m}| \in L^1(G)$.

There do not seem to be any known examples of sets S for which $\hat{\mu} = 0$ off S implies $\mu^2 \in L^1(G)$, while for *some* $\mu \notin L^1(G)$, $\hat{\mu} = 0$ off S.

I make some general remarks before proceeding to the proof of Theorem 1.

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For a measure σ on G and a bounded continuous function f on G let $f\sigma$ denote the measure whose value at $g \in C_0(G)$ is $\int g\,d(f\sigma) = \int gf\,d\sigma$. Note that if $\gamma_\alpha \in \Gamma$ are characters which converge to $\gamma \in \Gamma$, then $\gamma_\alpha\sigma$ converges to $\gamma\sigma$ in measure norm. (This is because $\gamma_\alpha \to \gamma$ uniformly on each compact set, and boundedly everywhere.) Finally, if $\{\gamma : \hat{\sigma}(\gamma) \neq 0\}$ has finite Haar measure, then the inversion formula implies $\sigma \in L^1(G)$.

2. Proof of Theorem 1

(A) Suppose (a) holds and consider the measures ω_γ whose Fourier-Stieltjes transforms are

$$\hat{\omega}_\gamma(\rho) = \hat{\mu}(\rho)\hat{\nu}(\rho - \gamma) \quad (\gamma \in \Gamma). \tag{2.1}$$

The hypotheses on S and μ, ν imply that $\omega_\gamma \in L^1(G)$ for a dense set of γ. If γ is any element of Γ, there exist $\gamma_\alpha \to \gamma$ with $\omega_{\gamma_\alpha} \in L^1(G)$. Of course $\hat{\nu}(\rho - \gamma_\alpha) = (\gamma_\alpha\nu)^\wedge(\rho)$ so $\gamma_\alpha\nu \to \gamma\nu$ in norm, and therefore $\omega_\gamma = \mu * (\gamma\nu)$ belongs to $L^1(G)$. Hence $\omega_\gamma \in L^1(G)$ for all $\gamma \in \Gamma$.

An easy computation shows

$$\gamma_1\,\omega_{-\gamma_1+\gamma_2} = (\gamma_1\,\mu) * (\gamma_2\,\nu)$$

so $(\gamma_1\,\mu)*(\gamma_2\,\nu) \in L^1(G)$ for all $\gamma_1, \gamma_2 \in \Gamma$. By replacing each of the γ's with finite linear combinations of characters, we easily obtain $|\mu|*|\nu| \in L^1(G)$.

(B) Suppose (b) holds. For a measure σ define $\sigma^\#$ and $\tilde{\sigma}$ by

$$\int f(x)\,d\sigma^\#(x) = \int f(-x)\,d\sigma, \quad \text{and} \quad \int f(x)\,d\tilde{\sigma}(x) = \int f(-x)\,d\bar{\sigma} \tag{2.2}$$

where an overbar denotes complex conjugation. It is then easy to see that the Fourier-Stieltjes transforms obey

$$(\sigma^\#)^\wedge(\rho) = \hat{\sigma}(-\rho), \qquad \tilde{\sigma}^\wedge(\rho) = \hat{\sigma}(\rho)^-. \tag{2.3}$$

Also note that

$$|\sigma|^\sim = |\tilde{\sigma}| = |\sigma^\#| = |\sigma|^\# \quad \text{and} \quad |\sigma|^{\sim\#} = |\sigma|. \tag{2.4}$$

We define ω_γ by

$$\omega_\gamma = \mu * ((\gamma\nu)^{\sim\#}) = \mu * (\gamma(\tilde{\nu})^\#).$$

A straightforward calculation using (2.3) shows

$$\hat{\omega}_\gamma(\rho) = \hat{\mu}(\rho)(\hat{\nu}(\gamma - \rho))^-.$$

The argument of (A) using the hypothesis (b) now shows $|\mu| * |(\tilde{\nu})^\#| \in L^1(G)$. By (2.4), $|\mu| * |(\tilde{\nu})^\#| = |\mu| * |\nu|$, so Theorem 1 is proved.

Remarks. The use of the limit argument to show $\omega_\gamma \in L^1(G)$ for all γ appears implicitly in [G; p. 421]. The passing to $|\mu| * |\nu|$ was suggested to me by [W].

By using the method of [P] one may weaken the hypotheses of Theorem 1:

instead of requiring that $\hat{\mu}$, $\hat{\nu}$ be zero off S, we may suppose there exist f, $g \in \bigcup_{1\leq p\leq 2} L^p(G)$ with $\hat{f} = \hat{\mu}$ a.e. on $\Gamma\backslash S$ and $\hat{g} = \hat{\nu}$ a.e. on $\Gamma\backslash S$. (In the proof, one finds, as in [P], f_1, $g_1 \in L^1(G)$ with $\hat{f}_1 = \hat{\mu}$, $\hat{g}_1 = \hat{\nu}$ a.e. on $\Gamma\backslash S$. Let $\mu_1 = \mu - f_1\, dx$, $\nu_1 = \nu - g_1\, dx$, so μ_1, $\nu_1 = 0$ a.e. off S. The proof of Theorem 1 shows $|\mu_1| * |\nu_1| \in L^1(G)$, and, of course, $|\mu| * |\nu|$ is absolutely continuous with respect to $(|\mu_1| + |f_1|\, dx)*(|\nu_1| + |g_1|\, dx \in L^1(G).)$

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