THE PRODUCT OF CONSECUTIVE INTEGERS IS NEVER A POWER

BY

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We dedicate this paper to the memory of our friends H. Davenport, Ju.V. Linnik, L. J. Mordell, L. Moser, A. Rényi and W. Sierpiński, all of whom were alive when we started our work in 1966 at the University of Illinois at Urbana.

0. Introduction

It was conjectured about 150 years ago that the product of consecutive integers is never a power. That is, the equation

$$(n + 1)\cdots(n + k) = x^l \tag{1}$$

has no solution in integers with $k \geq 2$, $l \geq 2$ and $n \geq 0$. (These restrictions on k, l and n will be implicit throughout this paper.) The early literature on this subject can be found in Dickson's history and the somewhat later literature in the paper of Obláth [5].

Rigge [6], and a few months later Erdös [1], proved the conjecture for $l = 2$. Later these two authors [1] proved that for fixed l there are at most finitely many solutions to (1). In 1940, Erdös and Siegel jointly proved that there is an absolute constant c such that (1) has no solutions with $k > c$, but this proof was never published. Later Erdös [2] found a different proof; by improving the method used, we can now completely establish the old conjecture. Thus we shall prove:

THEOREM 1. *The product of two or more consecutive positive integers is never a power.*

In fact we shall prove a stronger result:

THEOREM 2. *Let k, l, n be integers such that $k \geq 3$, $l \geq 2$ and $n + k \geq p^{(k)}$, where $p^{(k)}$ is the least prime satisfying $p^{(k)} \geq k$. Then there is a prime $p \geq k$ for which $\alpha_p \not\equiv 0 \pmod{l}$, where α_p is the power of p dividing $(n + 1) \cdots (n + k)$.*

Theorem 2 implies Theorem 1, since it is easy to see that $(n + 1)(n + 2)$ is never an lth power and if $n \leq k$ then by Bertrand's postulate the largest prime factor of $(n + 1)\cdots(n + k)$ divides this product to exactly the first power. Moreover, this shows that in proving Theorem 2 it will suffice to assume $n > k$.

One could conjecture the following strengthening of Theorem 2: if $k \geq 4$ and $n + k \geq p^{(k)}$, then there is at least one prime greater than k which divides

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$(n + 1)\cdots(n + k)$ to the first power. This conjecture, if true, seems very deep. The requirement of $k \geq 4$ is motivated by

$$\binom{50}{3} = 140^2.$$

Now we start the proof of Theorem 2. We suppose that Theorem 2 is false for some particular k, l and n, and show that in every case this leads to a contradiction. As noted above, we assume $n > k$.

1. Basic lemmas

First observe that by the well-known theorem of Sylvester and Schur [3] there is always a prime greater than k which divides $(n + 1)\cdots(n + k)$, since $n > k$. Such a prime divides only one of the k factors, so $n + k \geq (k + 1)^l$, whence

$$n > k^l. \tag{2}$$

Furthermore since we suppose Theorem 2 is false, for $1 \leq i \leq k$ we have

$$n + i = a_i x_i^l, \tag{3}$$

where a_i is lth-power free and all its prime factors are less than k.

In the proof [1] for the case $l = 2$, it was shown that $a_i \neq a_j$ if $i \neq j$. In fact for $l > 2$ it is also known that the products $a_i a_j$ are all distinct. In this paper we need the stronger result:

LEMMA 1. *For any $l' < l$, the products $a_{i_1}\cdots a_{i_{l'}}$ $(i_1 \leq \cdots \leq i_{l'})$ are all distinct.*

In fact we prove that the ratio of two such products cannot be an lth power. First we show that (2) ensures

$$(n + i_1)\cdots(n + i_{l-1}) \neq (n + j_1)\cdots(n + j_{l-1}), \tag{4}$$

provided the two products are not identical.

Cancel any equal factors. Since $(n + i, n + j) < k$ and $n > k^l$, it follows that no factor of one member of (4) divides the product of the factors remaining in the other member, so the nonequality in (4) is proved.

Now we prove the lemma. For some rational t, suppose that

$$a_{i_1}\cdots a_{i_{l-1}} = a_{j_1}\cdots a_{j_{l-1}} t^l. \tag{5}$$

We shall show that (5) implies the subscripts must all match. Assume without loss of generality that $(n + i_1)\cdots(n + i_{l-1}) > (n + j_1)\cdots(n + j_{l-1})$, and put $t = u/v$, with $(u, v) = 1$. Then

$$(n + i_1)\cdots(n + i_{l-1}) = a_{i_1}\cdots a_{i_{l-1}} \frac{x^l}{u^l}$$

and

$$(n + j_1)\cdots(n + j_{l-1}) = a_{j_1}\cdots a_{j_{l-1}} \frac{y^l}{v^l},$$

where $x = ux_{i_1} \cdots x_{i_{l-1}}$ and $y = vx_{j_1} \cdots x_{j_{l-1}}$ in the notation of (3). By (5), we may put

$$A = \frac{a_{i_1} \cdots a_{i_{l-1}}}{u^l} = \frac{a_{j_1} \cdots a_{j_{l-1}}}{v^l},$$

so $Ax^l > Ay^l$ and therefore $x \geq y + 1$. Thus

$$(n + i_1) \cdots (n + i_{l-1}) - (n + j_1) \cdots (n + j_{l-1}) \geq A\{(y + 1)^l - y^l\} > Aly^{l-1}. \tag{6}$$

Note that (5) implies A is a positive integer. Also $Ay^l = (n + j_1) \cdots (n + j_{l-1}) > n^{l-1}$, so with (6) we have

$$\begin{aligned}(n + i_1) \cdots (n + i_{l-1}) - (n + j_1) \cdots (n + j_{l-1}) &> Al\left(\frac{n^{l-1}}{A}\right)^{(l-1)/l} \\ &\geq ln^{(l-1)^2/l}.\end{aligned} \tag{7}$$

On the other hand,

$$(n + i_1) \cdots (n + i_{l-1}) - (n + j_1) \cdots (n + j_{l-1}) < (n + k)^{l-1} - n^{l-1} < kln^{l-2}, \tag{8}$$

where the last inequality is obvious if $l = 2$ and for $l \geq 3$ it may be seen as follows. Clearly it suffices to show that

$$kn^{l-2} > \sum_{i=2}^{l-1} \binom{l-1}{i} n^{l-1-i}k^i,$$

that is

$$1 > \sum_{i=2}^{l-1} \binom{l-1}{i} \left(\frac{k}{n}\right)^{i-1}.$$

Now

$$\binom{l-1}{i} < \frac{l^i}{2^{i-1}};$$

also $n > k^l$, $k \geq 3$ and $l \geq 3$, so $n > kl$ and moreover $n > kl^2$. Therefore

$$\sum_{i=2}^{l-1} \binom{l-1}{i} \left(\frac{k}{n}\right)^{i-1} < l \sum_{i=2}^{\infty} \left(\frac{kl}{2n}\right)^{i-1} = \frac{kl^2}{2n - kl} < \frac{kl^2}{n} < 1.$$

The lemma now follows, since (7) and (8) require $k > n^{1/l}$, contrary to (2). Now we prove:

LEMMA 2. *By deleting a suitably chosen subset of* $\pi(k - 1)$ *of the numbers* $a_i (1 \leq i \leq k)$, *we have*

$$a_{i_1} \cdots a_{i_{k'}} \mid (k - 1)! \tag{9}$$

where $k' = k - \pi(k - 1)$.

For each prime $p \leq k - 1$ we omit an a_m for which $n + m$ is divisible by p to the highest power. If $1 \leq i \leq k$ and $i \neq m$, the power of p dividing $n + i$

is the same as the power of p dividing $i - m$. Thus $p^\alpha \| a_{i_1} \cdots a_{i_{k'}}$ implies $p^\alpha \mid (k-m)!(m-1)!$, so $p^\alpha \mid (k-1)!$ and our lemma is proved.

Change of notation. In the remainder of this paper it will be convenient to have the a's renumbered so that $a_1 < a_2 < \cdots < a_k$. We shall employ this new notation in Sections 2 and 3.

Note also that to prove Theorem 2 for any particular l it is enough to prove it for some divisor of l, so it suffices to consider only prime l.

2. The case $l > 2$

2.1. *The case $k \geq 30000$.* Now we show that (9) leads to a contradiction for $k \geq 30000$, using only the distinctness of the products $a_i a_j$. It is known [4] that the number of positive integers $b_1 < \cdots < b_r \leq x$ for which the products $b_i b_j$ are all distinct satisfies

$$r < \pi(x) + \frac{c_1 x^{3/4}}{(\log x)^{3/2}}, \tag{10}$$

and this is best possible apart from the value of c_1. However, when r is small this result is not adequate for our needs, so we shall now establish a bound which is sharper for small r.

First we need a graph theoretic lemma. A subgraph of a graph is called a *rectangle* if it comprises two pairs of vertices, with each member of one pair joined to each member of the other. We prove:

LEMMA 3. *Let G be a bipartite graph of s white and t black vertices which contains no rectangles. Then the number of edges of G is at most*

$$s + \binom{t}{2}.$$

Call a subgraph of G comprising one vertex joined to each of two others a *V-subgraph*. Since G contains no rectangle, there can be at most one V-subgraph with any given pair of black vertices as its endpoints. Let s_i be the number of white vertices of valence i, so $\sum_{i\geq 1} s_i = s$. Counting the number of V-subgraphs with black endpoints gives

$$\sum_{i\geq 2} s_i \binom{i}{2} \leq \binom{t}{2}. \tag{11}$$

If E is the number of edges of G, then by (11)

$$E = \sum_{i\geq 1} i s_i = s + \sum_{i\geq 2} (i-1)s_i \leq s + \sum_{i\geq 2} s_i \binom{i}{2} \leq s + \binom{t}{2}$$

which proves Lemma 3.

Now let $u_1 < \cdots < u_s \leq x$ and $v_1 < \cdots < v_t \leq x$ be two sequences of positive integers such that every positive integer up to x can be written in the form $u_i v_j$. If $b_1 < \cdots < b_r \le x$ are positive integers such that all the products $b_i b_j$ are distinct, form the bipartite graph G with s white vertices labelled $u_1, \ldots, u_s$ and t black vertices labelled $v_1, \ldots, v_t$ and an edge between u_i and v_j if $u_i v_j = b_m$ for some m. Distinctness of the products $b_i b_j$ ensures that G has no rectangles so by Lemma 3,

$$r \le s + \binom{t}{2}. \tag{12}$$

Lemma 1 shows in particular that the bound (12) applies to the sequence $a_1 < \cdots < a_k$. Using (12) we next prove that the product of any $k - \pi(k)$ of the a's exceeds $k!$ provided $k \ge 30000$. Because of Lemma 2 this implies Theorem 2 for $k \ge 30000$ and $l > 2$. Evidently it suffices to prove

$$\prod_{i=1}^{k-\pi(k)} a_i > k! \quad \text{if} \quad k \ge 30000. \tag{13}$$

We shall now obtain lower bounds on a_i $(1 \le i \le k)$. We clearly have

$$a_i \ge i, \tag{14}$$

and using (12) we shall show two further inequalities:

$$a_i \ge 3.5694(i - 304), \tag{15}$$

$$a_i \ge 4.3402(i - 1492). \tag{16}$$

Of these, (14) is sharpest for $i \le 422$, (15) is sharpest for $422 < i \le 6993$, and (16) is sharpest for $i > 6993$. With these inequalities, a routine calculation using Stirling's formula suffices to verify (13) when $k = 30000$, and (16) ensures that (13) holds when $k > 30000$.

To prove (15), we take $v_1 < \cdots < v_t$ to be the $t = 25$ positive integers up to 36 which have no prime factor greater than 7 (so $v_1 = 1$ and $v_{25} = 36$). Next we obtain a suitable set of positive integers $u_1 < \cdots < u_s \le x$ so that every positive integer $m \le x$ is expressible in the form $u_i v_j$. For convenience, let V denote the set of v's. Clearly any positive integer $m \le x$ with all prime factors greater than 7 must be included in the u's: let U_1 denote the set of such numbers. Next, suppose $m \le x$ is a positive multiple of 7 and $m = dd'$, where d is the largest divisor of m with no prime factor greater than 7. If $d \notin V$ then $d \ge 42$, since $7 \mid d$. Thus $x \ge m = dd' > 42d'$, so $7d' \le x/6$. Hence we include in the u's all positive integers of the form $7d' \le x/6$ with least prime factor 7: let U_2 denote this set of numbers. Similarly, if $m \le x$ is a positive multiple of 5 and $m = dd'$, where d is the largest divisor of m with no prime factor greater than 5, then $d \notin V$ requires $d \ge 40$ and $5d' \le x/8$. Hence we include in the u's all positive integers of the form $5d' \le x/8$ with least prime factor 5, and let U_3 denote this set. Likewise we include in the u's all positive integers of the form $3d' \le x/14$ with least prime factor 3, and all positive integers of the form $2d' \le x/20$, denoting these sets by U_4 and U_5 respectively.

Now every positive integer $m \leq x$ is expressible in the form $m = u_i v_j$ for some $u_i \in U$ and $v_j \in V$, where U denotes the union of $U_1, \ldots, U_5$.

The number of u's in each U_i can readily be calculated. For example

$$|U_2| = \frac{\phi(30)}{30} \cdot \frac{x}{42} + \varepsilon_2(x) = \frac{2x}{315} + \varepsilon_2(x),$$

where the error term has the bound $\varepsilon_2(x) \leq 14/15$. Likewise $\varepsilon_1(x) \leq 53/35$, $\varepsilon_3(x) \leq 2/3$, $\varepsilon_4(x) \leq 1/2$ and $\varepsilon_5(x) \leq 0$. Thus the total number of u's is

$$s = |U| = \sum_{i=1}^{5} |U_i| = \left(\frac{8}{35} + \frac{2}{315} + \frac{1}{120} + \frac{1}{84} + \frac{1}{40}\right) x$$
$$+ \sum_{i=1}^{5} \varepsilon_i(x) < \frac{353}{1260} x + 4.$$

Now (12) implies that the number of a's up to x is less than $353x/1260 + 304$, whence (15).

To prove (16), we take the v's to be the $t = 55$ positive integers up to 100 with no prime factor greater than 11, and the u's to be all positive integers up to x with all prime factors greater than 11, together with all those up to $x/10$ with least prime factor 11, all those up to $x/15$ with least prime factor 7, all those up to $x/21$ with least prime factor 5, all those up to $x/35$ with least prime factor 3, and finally all even integers up to $x/54$. The first of these subsets of u's contains $16x/77 + \varepsilon_0(x)$ numbers, where $\varepsilon_0(x) \leq 194/77$. The error terms in counting the other subsets of u's are the same as before, so the total error is less than 7. With (12), this leads to (16). Now we shall work upwards from small k to resolve the cases with $k < 30000$.

2.2. *The case* $k = 3$. It is easy to see that (1) has no solution when $k = 3$, for $(n + 1)(n + 2)(n + 3) = m(m^2 - 1)$, where $m = n + 2$, shows that the product could only be an lth power if m and $m^2 - 1$ are lth powers, but $m^2 - 1$ and m^2 cannot both be lth powers. But for Theorem 2 we need to show $\alpha_p \not\equiv 0 \pmod{l}$ for some prime $p \geq 3$, where α_p is the power of p in $(n + 1)(n + 2)(n + 3)$. Suppose there is no such p. If n is even, $(n + 1, n + 3) = 1$ ensures $a_1 = a_2 = 1$, contradicting Lemma 1. If n is odd, $(n + 1, n + 3) = 2$ ensures $a_1 = 1$, $a_2 = 2$ and $a_3 = 2^{\alpha}$, with $1 < \alpha < l$, and Lemma 1 is contradicted by $a_1^{\alpha-1} a_3 = a_2^{\alpha}$.

2.3. *The case* $4 \leq k \leq 1000$, $l = 3$. Here we restrict attention to those a's with no prime factor greater than the mth prime, say $f(k, m)$ in number. If u and v are positive integers with prime factors similarly restricted, there are 3^m rationals u/v no two of which differ by a factor which is the cube of a rational. The number of formally distinct expressions a_i/a_j is $f(k, m)\{f(k, m) - 1\}$, so there are two whose quotient yields a solution to (5), thus contradicting Lemma 1, if

$$f(k, m)\{f(k, m) - 1\} > 3^m. \tag{17}$$

Since the a's arise as divisors of k consecutive integers, and have all prime factors less than k, it is straightforward to calculate a lower bound for $f(k, m)$. Thus we verify (17) for $4 \leq k \leq 10$ with $m = 2$, for $10 < k \leq 28$ with $m = 3$, for $28 < k \leq 77$ with $m = 4$, for $77 < k \leq 143$ with $m = 5$, for $143 < k \leq 340$ with $m = 6$, for $340 < k \leq 646$ with $m = 7$, and for $646 < k \leq 1000$ with $m = 8$.

This method could be continued beyond $k = 1000$, but certainly fails before reaching $k = 10000$. Fortunately we have an improvement available, and we now proceed with it.

2.4. *The case* $1000 < k < 30000$, $l = 3$. Let $q_1 < \cdots < q_r$ be the r largest primes satisfying $q_i \leq k^{1/2}$, where r is to be suitably chosen. We now restrict attention to those a's, say $F(k, r)$ in number, which have no prime factor greater than $k^{1/2}$, and at most one prime factor (counting multiplicity) among the q's. If u and v are positive integers with prime factors similarly restricted, there are $3^{\pi(q_1)-1} R$ rationals u/v no two of which differ by a factor which is the cube of a rational. In this count the factor $R = r^2 + r + 1$ arises from the fact that u and v each contain at most one of the q's as a divisor. As in (17), the number of formally distinct expressions a_i/a_j is enough to ensure that there are two whose quotient yields a solution to (5), and therefore contradicts Lemma 1, if

$$F(k, r)\{F(k, r) - 1\} > 3^{\pi(q_1)-1}(r^2 + r + 1). \tag{18}$$

To obtain a lower bound for $F(k, r)$, note that for each prime p in $(k^{1/2}, k)$ we omit at most $[k/p] + 1$ of the a's; similarly for the products q_i^2 and q_iq_j, so

$$\begin{aligned} F(k, r) &\geq k - \sum_{k^{1/2}<p<k} \left(\left[\frac{k}{p}\right] + 1\right) - \sum_{1 \leq i \leq j \leq r} \left(\left[\frac{k}{q_iq_j}\right] + 1\right) \\ &\geq k - \sum_{k^{1/2}<p<k} \left(\left[\frac{k}{p}\right] + 1\right) - \left[\frac{k}{2}\left\{\sum_{i=1}^{r} \frac{1}{q_i^2} + \left(\sum_{i=1}^{r} \frac{1}{q_i}\right)^2\right\}\right] - \binom{r+1}{2}. \end{aligned}$$

For example, with $k = 175^2 = 30625$ and $r = 31$ (so $q_1 = 29$) this bound is adequate to verify (18). Indeed, for $1000 < k < 30000$ we can readily verify (18), in each case taking q_1 around $k^{0.3}$.

2.5. *The case* $4 \leq k \leq 30000$, $l > 3$. Here it is inconvenient to work with ratios of products of a's, so we work directly with the products themselves, since we do not need the extra sharpness.

With the a's selected as in Section 2.3, the inequality corresponding to (17) is

$$\binom{f(k, m) + l - 2}{l - 1} > l^m. \tag{19}$$

The left member of (19), derived by counting the number of nondecreasing sequences of $l - 1$ a's, is the number of formally distinct products of $l - 1$ a's, and the right member is the number of lth-power free positive integers with all prime factors among the first m primes. When (19) holds, (5) has a solution, contradicting Lemma 1. It is easy to verify by direct computation that (17) implies (19) for $4 \leq k \leq 1000$ and m chosen as in Section 2.3.

Similarly, with the a's selected as in Section 2.4, the inequality corresponding to (18) is

$$\binom{F(k, r) + l - 2}{l - 1} > l^{\pi(q_1) - 1} \binom{l + r - 1}{l - 1}. \tag{20}$$

The left member of (20) is the number of formally distinct products of $l - 1$ a's, and the right member is the number of lth-power free positive integers with no prime factor greater than $k^{1/2}$ and at most $l - 1$ prime factors among the q's (counted by multiplicity). When (20) holds, (5) has a solution, contradicting Lemma 1. For $1000 < k < 30000$ and the values of r chosen as in Section 2.4, (20) easily holds when (18) holds.

This completes the proof of Theorem 2 for $l > 2$. It seems certain that one could get a more general inequality than (19) and (20), leading to a more elegant method valid for all k.

3. The case $l = 2$

It remains to prove that $(n + 1) \cdots (n + k)$ always contains a prime $p \geq k$ to an odd exponent. (We already know that the product is not a square, by the results of Rigge and Erdös cited earlier.)

The a's are now square-free and, by Lemma 1, all distinct. So, by Lemma 2,

$$\prod_{i=1}^{k} a_i \,\Big|\, (k - 1)! \prod_{p<k} p \tag{21}$$

We shall now show that for $k \geq 71$ this leads to a contradiction.

3.1. *The case $k \geq 71$.* Since 12 of every 36 consecutive integers are divisible by 4 or 9, at most 24 of any 36 consecutive integers are square-free. Thus for $k \geq 64$ we have

$$\prod_{i=1}^{k} a_i > k! \left(\frac{3}{2}\right)^k. \tag{22}$$

For any positive integer m and prime p, the power to which p divides $p^m!$ is $(p^m - 1)/(p - 1)$. From this we can deduce that if the powers to which 2 and 3 divide $(k - 1)!$ are α and β respectively, then

$$\alpha \geq k - 1 - \log_2 k \quad \text{and} \quad \beta \geq \tfrac{1}{2}(k - 1) - \log_3 k.$$

On the other hand, since the a's arise from k consecutive integers and are square-free, we calculate that if the powers to which 2 and 3 divide $a_1 \cdots a_k$ are γ and δ respectively, then

$$\gamma \leq \tfrac{1}{3}\{k + \log_2 (3k + 1)\} \quad \text{and} \quad \delta \leq \tfrac{1}{4}\{k + 1 + 2 \log_3 (2k + 1)\}.$$

Since (21) implies that $a_1 \cdots a_k \leq (k-1)!2^{\gamma-\alpha}3^{\delta-\beta} \prod_{p<k} p$, we now deduce from (22) that for $k \geq 64$,

$$\left(\frac{3}{2}\right)^k 2^{2k/3}3^{k/4} < \frac{14}{3} k^2 \prod_{p<k} p. \tag{23}$$

However $\prod_{p<k} p < 3^k$, so (23) fails for $k \geq 297$. Indeed, $\prod_{p<k} p < e^k$ for $k \leq 10^8$ by Theorem 18 of [7], so (23) fails for $k \geq 71$.

3.2. *The case $k < 71$.* For $k = 3$, it is impossible for the a's to be distinct. For $k = 4$ the only possibility is $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 6$. Then $a_1a_2a_3a_4 = 6^2$, so

$$(n+1)(n+2)(n+3)(n+4)$$

must be a square; but this product equals $(n^2 + 5n + 5)^2 - 1$ and we have a contradiction.

For $5 \leq k \leq 20$, we will count the number of a's with no prime factor greater than 3; if this is at least 5, it is impossible for the a's to be distinct, and we have a contradiction. This works unless $k = 6$ and $5 \mid n + 1$, or $k = 8$, $7 \mid n + 1$ and $5 \mid n + 2$. But in either of these cases we have four consecutive integers whose product is a square, and this was shown above to be impossible.

Similarly we obtain a contradiction for $20 < k \leq 56$ by noting that there are at least 9 a's with no prime factor greater than 5, and for $56 < k \leq 176$, where there are at least 21 a's with no prime factor greater than 7. (This method could be extended. For example, with $176 < k \leq 416$ there are at least 42 a's with no prime factor greater than 11, and with $416 < k < 823$ there are at least 65 a's with no prime factor greater than 13.)

This completes the proof of Theorem 2.

4. Remarks and further problems

No doubt our method would suffice to show that the product of consecutive odd integers is never a power, in the sense of (1). In fact, the proof would probably be simpler. More generally, for any positive integer d there must be an integer t_d such that $(n + d)(n + 2d) \cdots (n + td)$ is never a perfect power if $t > t_d$. Without t_d this result fails since $x(x + d)(x + 2d) = y^2$ has infinitely many solutions.

By our methods we can prove that for fixed t,

$$(n + d_1) \cdots (n + d_k) = x^l, \quad 1 = d_1 < \cdots < d_k \leq k + t \tag{24}$$

has only a finite number of solutions. Our theorem shows that there is no solution with $t = 0$. With $t = 1$ we have the solutions $4!/3$, $6!/5$ and $10!/7$; perhaps there are no others. Suppose that t is a function of k, or of k and l. How fast must t grow to give an infinite number of solutions to (24)? The Thue-Siegel theorem implies that (24) has only a finite number of solutions when d_k and l are fixed, with $l > 2$. For fixed k it seems probable that $\lim_{l\to\infty} d_k = \infty$.

Another question which arises naturally from our method is the following. Let $a_i^{(l)}$ be the largest divisor of $n + i$ which is lth-power free and has all prime factors less than k. Our proof for $l = 2$ implies that for $1 \leq i \leq k$, the $a_i^{(2)}$ are not all distinct when $k \neq 4, 6, 8$. An easy argument also shows that the $a_i^{(2)}$ cannot all be distinct when $k = 8$. To what extent do these results extend to $l > 2$? For how many consecutive values of i can the $a_i^{(l)}$ be distinct?

We mention one final problem. Let a_i be the largest divisor of $n + i$ which has all prime factors less than k. Our proof of Theorem 2 shows that for any $n \geq 0$ and $k \geq 30000$, the products $a_i a_j$ cannot all be distinct. Very likely this holds for much smaller values of k, perhaps as small as $k \geq 16$. To see that it does not hold for $3 \leq k < 16$, it suffices to check for $k = 3, 5, 7, 11, 13, 15$. We conclude with a table of examples for these cases.

k	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
3	2	1	2^{α}												
5	12	1	2	3^{α_1}	2^{α_2}										
7	60	1	2	3	2^{α_1}	5^{α_2}	2.3^{α_3}								
11	90	7^{α_1}	2^{α_2}	3	2	5^{α_3}	12	1	14	3^{α_4}	40				
13	90	11^{α_1}	2^{α_2}	3	14	5^{α_3}	12	1	2	3^{α_4}	40	7^{α_5}	66		
15	104	11^{α_1}	18	7^{α_2}	20	3	2	1	3.2^{α_3}	5	14	3^{α_4}	44	13^{α_5}	6.5^{α_6}

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