A NOTE ON NORMAL INDEX AND MAXIMAL SUBGROUPS IN FINITE GROUPS

BY

N. P. MUKHERJEE

The concept of normal index was introduced by Deskins [3]. For a maximal subgroup M of a group G, the order of a chief factor H/K of G—where H is minimal in the set of normal supplements of M in G—is known as the normal index of M in G. Both Nyhoff [5] and Beidleman and Spencer [2] have shown that a group G is solvable (p-solvable) iff the normal indices of maximal subgroups satisfy certain conditions. In this note we use the notion of normal index to obtain some more conditions which are both necessary and sufficient for the group theoretic properties solvability, π-solvability, π-supersolvability, and π-nilpotence.

All groups are assumed finite. For the sake of completeness we mention the following results which were established by Beidleman and Spencer [2].

LEMMA 1. *The normal index $\eta(G: M)$ of a maximal subgroup M of a group G is uniquely determined by M.*

LEMMA 2. *If M is a maximal subgroup of a group, G, $N \trianglelefteq G$ and $N \subseteq M$ then $\eta(G/N: M/N) = \eta(G: M)$.*

It follows from the definition of normal index that $[G: M]$ divides $\eta(G: M)$. If H is a minimal normal supplement to M in G and H/K is a chief factor of G then $K \subseteq M$ and $G = MH$. Consequently, $[G: M] = |H|/|M \cap H|$ divides $|H/K| = \eta(G: M)$. For a maximal subgroup which is normal we obtain the following theorem.

THEOREM 1. *If M is a maximal subgroup of a group G and $M \trianglelefteq G$ then $\eta(G: M) = [G: M] =$ a prime.*

Proof. Let N be a minimal normal subgroup of G and distinguish two cases.

Case 1. $N \nsubseteq M$. This implies $G = MN$ and $M \cap N = 1$. Therefore, $\eta(G: M) = |N| = [G: M] =$ a prime.

Case 2. $N \subseteq M$. If $N \subset M$, $N \neq M$, then induction shows that

$$\eta(G/N: M/N) = [G/N: M/N] = \text{a prime}.$$

This implies $\eta(G: M) = [G: M] =$ a prime. Now suppose $N = M$. If L is another minimal normal subgroup of G then $L \cap N = L \cap M = 1$ and $G = ML$. Hence $\eta(G: M) = |L| = [G: M] =$ a prime. Finally, suppose

Received October 2, 1973.

$N = M$ is the unique minimal normal subgroup of G. If R is any other normal subgroup of G then $R \supset N = M$ and the maximality of M implies $R = G$. Therefore, the only normal subgroup of G that supplements M in G is G itself. Since M is maximal and normal in G, clearly $\eta(G: M) = |G|/|M| = [G: M] =$ a prime and the theorem is proved.

The proofs of the remaining results depend on a lemma of Baer [1, Lemma 3, pp. 121] and we state it below for the sake of completeness.

LEMMA 3. *If the group G possesses a maximal subgroup with core* 1 *then the following properties of G are equivalent.*

(1) *The indices in G of all the maximal subgroups with core* 1 *are powers of one and the same prime p.*

(2) *There exists one and only one minimal normal subgroup of G, and there exists a common prime divisor of all the indices in G of all the maximal subgroups with core* 1.

(3) *There exists a soluble normal subgroup, not* 1, *in G.*

For notational purposes, let n_π denote the π-part of n. More precisely, if π is a given set of primes and $n = n_1 n_2$, where $(n_2, q) = 1$ for all primes q in π and n_1 is divisible only by primes in π then $n_\pi = n_1$.

THEOREM 2. *A group G is π-solvable if and only if $\eta(G: M)_\pi = [G: M]_\pi$ for every maximal subgroup M of G.*

Proof. Suppose G is π-solvable. If G is a π'-group then $\eta(G: M)_\pi = [G: M]_\pi$ holds trivially for each maximal subgroup M. Assume now that $|G|$ is divisible by at least one prime in π and let N be a minimal normal subgroup of G. If a maximal subgroup M of G contains N then, by induction, it follows that $\eta(G/N: M/N)_\pi = [G/N: M/N]_\pi$ and therefore, $\eta(G: M)_\pi = [G: M]_\pi$. If $N \nsubseteq M$ then $G = MN$ and for $|N|_\pi = 1$, clearly $[G: M]_\pi = \eta(G: M)_\pi = 1$. If $|N|_\pi \neq 1$ then π-solvability of G shows that N is a π-group and since N is solvable and minimal normal it is clear that N is elementary abelian. Hence $M \cap N \trianglelefteq G$ and consequently, $M \cap N = 1$. It now follows that $\eta(G: M)_\pi = |N|_\pi = [G: M]_\pi$.

Conversely, let $[G: M]_\pi = \eta(G: M)_\pi$ hold for each maximal subgroup M of G. Observe that G is not simple. For, otherwise, $|G|_\pi = [G: M]_\pi$ holds for every maximal subgroup M of G. If $\pi = \{p_1, p_2, \ldots, p_t\}$ then the indices of the maximal subgroups $M_1, M_2, \ldots, M_t$ (where M_i contains a Sylow p_i-subgroup of G) will be prime to $p_1, p_2, \ldots, p_t$, respectively. Consequently, $|G|_\pi = 1$ and trivially G is π-solvable.

Let N be a minimal normal subgroup of G. By induction, G/N is π-solvable. If L is another minimal normal subgroup of G then G/L is π-solvable and since G is isomorphic to $G/L \cap N$ it follows that G is π-solvable. We may therefore suppose that N is the unique minimal normal subgroup of G. If $|N|_\pi = 1$ then N is a π'-group and since G/N is π-solvable it follows that G is π-solvable. Now suppose $|N|_\pi \neq 1$. If N is contained in each maximal subgroup M of G then $N \subseteq \phi(G)$ and consequently, $G/\phi(G)$ is π-solvable by induction. Hence G is π-solvable. If, however, $N \nsubseteq M$ for some maximal subgroup M of G then $G = MN$, M is core free and $[G: M]_\pi = |N|_\pi$. For any other maximal subgroup M_0 with core 1, $N \nsubseteq M_0$ and $G = M_0N$. By hypothesis, $|N|_\pi = \eta(G: M_0)_\pi = [G: M_0]_\pi$. Hence Lemma 3 shows that N is solvable and this implies G is π-solvable.

With the help of Theorem 2 we can easily establish the following.

THEOREM 3. *In any group G the following are equivalent:*

(1) $\eta(G: M)_2 = [G: M]_2$ *for all maximal subgroups M of G.*
(2) *G is solvable.*
(3) $\eta(G: M)$ *is a power of a prime for all maximal subgroups M of G.*
(4) $\eta(G: M) = [G: M]$ *for all maximal subgroups M of G.*

COROLLARY 1. *If M is a maximal subgroup of a group G and if $\eta(G: M)$ is square free then $\eta(G: M) = [G: M]$.*

COROLLARY 2. *A group G is supersolvable if and only if $\eta(G: M)$ is square free for each maximal subgroup M of G.*

Beidleman and Spencer [2] showed that a group G is solvable whenever all nonnormal maximal subgroups having equal normal index are conjugate. The next theorem improves upon this result. While conjugacy implies equality of the indices, the converse is not necessarily true.

THEOREM 4. *A group G is solvable if and only if any two maximal subgroups M_1 and M_2 of G with $\eta(G: M_1) = \eta(G: M_2)$ satisfy $[G: M_1] = [G: M_2]$.*

Proof. Let G be a solvable group and let M_1 and M_2 be maximal subgroups with $\eta(G: M_1) = \eta(G: M_2)$. By Theorem 3, $\eta(G: M_i) = [G: M_i]$ and therefore $[G: M_1] = [G: M_2]$.

The converse is established in four steps.

(i) G is not simple. Otherwise, the normal index of every maximal subgroup is $|G|$. By hypothesis the indices of all maximal subgroups are then equal. Suppose $\{p_1, p_2, \ldots, p_n\}$ is the set of prime divisors of $|G|$ and assume $n > 1$. Every maximal subgroup containing a Sylow p_i-subgroup of G must have its index in G prime to p_i. This implies that the index of every maximal subgroup must be prime to p_i for all i. Therefore $n \leq 1$ and G is a p-group for some prime p. Hence G is not simple unless it is of order p in which case G is solvable and we are done.

(ii) G contains a unique minimal normal subgroup. Let N be a minimal normal subgroup of G. By induction, G/N is solvable and if $L \neq N$ is another minimal normal subgroup of G then G/L is solvable. Therefore $G \cong G/N \cap L$ is solvable and hence we may suppose that N is the unique minimal normal subgroup of G.

(iii) $\phi(G) = 1$. If $\phi(G) \neq 1$ then it follows by induction that $G/\phi(G)$ is solvable and therefore G is solvable.

(iv) G is solvable. It follows from (iii) that for some maximal subgroup M of G, $N \nsubseteq M$ and $G = MN$. Evidently, M is core free. If M_0 is any other core free maximal subgroup of G then $G = M_0N$ and $\eta(G: M_0) = \eta(G: M) = |N|$. By hypothesis, $[G: M] = [G: M_0] = |N|/|M \cap N|$. Therefore there exists a common prime divisor of the indices of all the maximal subgroups with core 1. By Lemma 3 N is solvable and consequently, G is solvable since G/N is solvable.

In view of Theorem 4 it seems natural to ask if a group G is π-solvable when maximal subgroups whose π-parts of the normal indices coincide have indices with equal π-parts. For $\pi = \{3\}$, each maximal subgroup M in $G = \mathrm{PSL}(2, 7)$ satisfies $\eta(G: M)_3 = 3$ and $[G: M]_3 = 1$ but G is not 3-solvable. For π-solvability, the equality of the common values of $[G: M]_\pi$ and $\eta(G: M)_\pi$ is crucial.

In the next two theorems we obtain conditions which are both necessary and sufficient for a group to be π-solvable. The necessity in both cases follow from Theorem 2. The sufficiency is established by applying the methods used in Theorem 4 and Theorem 2.

THEOREM 5. *A group G is π-solvable if and only if for maximal subgroups M_1 and M_2 having $\eta(G: M_1)_\pi = \eta(G: M_2)_\pi$, $[G: M_1]_\pi = [G: M_2]_\pi = \eta(G: M_1)_\pi$ holds.*

THEOREM 6. *A group G is π-solvable if and only if the following hold.*

(1) *G has a π-solvable maximal subgroup M with $\eta(G: M)_\pi = [G: M]_\pi$.*

(2) *If M_1 and M_2 are maximal subgroups with $\eta(G: M_1)_\pi = \eta(G: M_2)_\pi$ then $[G: M_1]_\pi = [G: M_2]_\pi$.*

In Theorems 7 and 8 below we characterize respectively, the π-supersolvable and π-nilpotent groups. Recall that a group G is called π-supersolvable if every chief factor of G is of prime order for some prime in π or is a π'-group and G is π-nilpotent if and only if it is the product of a nilpotent Hall π-subgroup and a normal Hall π'-subgroup.

We omit the proof of Theorem 7 since it is quite similar to the proof of Theorem 2.

THEOREM 7. *If π is a given set of primes then G is π-supersolvable if and only if for each maximal subgroup M of G, $\eta(G: M)_\pi = [G: M]_\pi = 1$ or $p \in \pi$.*

COROLLARY 1. *G is a π'-group if and only if $\eta(G: M)_\pi = 1$ for all maximal subgroups M of G.*

COROLLARY 2. *A group G is π-supersolvable if and only if for maximal sub-*

groups M_1 and M_2 with $\eta(G\colon M_1)_\pi = \eta(G\colon M_2)_\pi$, both $[G\colon M_1]_\pi = [G\colon M_2]_\pi$ and $\eta(G\colon M_i)_\pi = [G\colon M_i]_\pi = 1$ or $p \in \pi$ hold.

THEOREM 8. *A group G is π-nilpotent if and only if the following conditions are satisfied.*

(1) *$\eta(G\colon M)_\pi = [G\colon M]_\pi = 1$ or $p \in \pi$ for all maximal subgroups M of G.*

(2) *If $\eta(G\colon M)_\pi = [G\colon M]_\pi = p \in \pi$ for some maximal subgroup M then $M \trianglelefteq G$.*

Proof. Let G be a π-nilpotent group. Then G is π-supersolvable and (1) holds by Theorem 7. Now suppose M is a maximal subgroup of G with $\eta(G\colon M)_\pi = [G\colon M]_\pi = p \in \pi$ and $G = ST$, where S is a nilpotent Hall π-subgroup and T is a normal Hall π'-subgroup of G. If $T \nsubseteq M$ then $G = MT$ and $[G\colon M]_\pi = 1$, a contradiction. Hence $T \subseteq M$ and $M = (S \cap M)T$. It therefore follows that $|S|/|S \cap M| = p$ since $[G\colon M]_\pi = p$. Consequently, $S \cap M \trianglelefteq S$ since S is nilpotent and so $M \trianglelefteq G$.

Now suppose (1) and (2) both hold. By Theorem 7, G is π-supersolvable. Let N be a minimal normal subgroup of G. If $L \neq N$ is another minimal normal subgroup of G then, by induction, both G/N and G/L are π-nilpotent groups and therefore $G \cong G/N \cap L$ is π-nilpotent. Hence we may regard N as the unique minimal normal subgroup of G. If $N \subseteq M$ for each maximal subgroup M of G then $N \subseteq \phi(G)$ and, by induction, $G/\phi(G)$ is π-nilpotent. Consequently, G is π-nilpotent. Now suppose $N \nsubseteq M$ for some maximal subgroup M of G. Then $G = MN$ and $\eta(G\colon M) = |N|$. From (1) we have $\eta(G\colon M)_\pi = [G\colon M]_\pi = |N|_\pi = 1$ or p. Distinguish two cases.

Case 1. $|N|_\pi = 1$. Since G/N is π-nilpotent, $G/N = S/N.\ T/N$, where S/N is a nilpotent Hall π-subgroup and T/N is a normal Hall π'-subgroup of G/N. Therefore, $T \trianglelefteq G$ and $S = CN$, where C is a Hall π-subgroup of S. Thus $G = CT$, C is a nilpotent Hall π-subgroup and T is a normal Hall π'-subgroup of G. Hence G is π-nilpotent.

Case 2. $|N|_\pi = p$. This implies $|N| = p$, since G is π-supersolvable, and hence $M \cap N = 1$. Consequently, $G/N = MN/N$ is isomorphic to M. Since G/N is π-nilpotent, we can write $M = RW$ where R is a nilpotent Hall π-subgroup and W is a normal Hall π'-subgroup of M. Hence $G = (RN)W$, where W is a normal Hall π'-subgroup and RN is a nilpotent Hall π-subgroup of G. Therefore G is π-nilpotent.

REFERENCES

1. R. BAER, *Classes of finite groups and their properties*, Illinois J. Math., vol. 1 (1957), pp. 115–187.
2. J. C. BEIDLEMAN AND A. E. SPENCER, *Maximal subgroups in finite groups*, Illinois J. Math., vol. 16 (1972), pp. 95–101.
3. W. E. DESKINS, *On maximal subgroups*, Proceedings of Symposia in Pure Mathematics, American Math. Soc., vol. 1 (1959), pp. 100–104.

4. B. Huppert, *Endlichen Gruppen I*, Springer, N.Y., 1967.
5. L. R. Nyhoff, *The influence on a finite group of the cofactors and subcofactors of its subgroups*, Trans. Amer. Math. Soc., vol. 154 (1971), pp. 459–491.

West Virginia University
Morgantown, West Virginia