TOPOLOGICAL DESCRIPTION OF THE SPACE OF HOMEOMORPHISMS ON CLOSED 2-MANIFOLDS

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In this note we combine many of the facts known about spaces of homeomorphisms on two manifolds with some results from the theory of infinite dimensional manifolds to obtain a topological description of the pair $(H(M), PLH(M))$, for every closed piecewise linear 2-manifold M. In the above $H(M)$ denotes the space of all homeomorphisms of M onto itself under the supremum topology and $PLH(M)$ the subspace of piecewise linear homeomorphisms. The key ingredients are the work of Hamstrom and others [7], [8], [9], [10], [11] concerning the homotopy groups of $H(M)$ (see the proof of Lemma 3), some information obtained by Lickorish and others [1], [2], [15], [16] in their study of the homeotopy groups of 2-manifolds (see the proof of Lemma 2), some infinite dimensional topology theorems obtained by Henderson, West and others [12], [20] (see the proof of the theorem) and the fact that $(H(M), PLH(M))$ is an (l_2, l_2^f)-manifold pair (see Lemma 1 and the comments that follow it).

Notation. The pair (X_1, Y_1) is said to be homeomorphic to the pair (X_2, Y_2) if there exists a homeomorphism f of X_1 onto X_2 such that f restricted to Y_1 is a homeomorphism of Y_1 onto Y_2. Throughout this paper, S^n will denote the n-sphere, T the 2-torus, P^n real projective n-space, K the Klein bottle, Z the countable discrete space, and Z_n the discrete space with n elements.

Let l_2 denote the hilbert space of square-summable sequences and l_2^f the subspace consisting of those sequences having only finitely many nonzero entries. A pair (X, X') is an (l_2, l_2^f)-manifold pair if X is an l_2-manifold (i.e., a separable metric space which is locally homeomorphic to l_2) for which there is an open cover $\mathcal{U}$ and open embeddings $\{f_U\colon U \to l_2 \mid U \in \mathcal{U}\}$ such that for each $U \in \mathcal{U}, f_U(U \cap X') = f_U(U) \cap l_2^f$.

In [5] Geoghegan and Haver proved the following:

LEMMA 1. *Let M be a compact piecewise linear 2-manifold; then $(H(M), PLH(M))$ is an (l_2, l_2^f)-manifold pair.*

The proof makes use of previous work of Luke and Mason [17], Torúnczyk [19], Geoghegan [4], Edwards-Kirby [3], Keesling-Wilson [13], and Haver [6].

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LEMMA 2. *$H(P^2)$ has one component, $H(S^2)$ two components, $H(K)$ four components. For any other closed two manifold, M, $H(M)$ has a countable number of components.*

Proof. Let $H_0(M)$ denote the identity component of $H(M)$. The group $H(M)/H_0(M)$ is known as the homeotopy group of M and has been studied extensively by Lickorish, Chillingworth, Birman, and others [1], [2], [15], [16]. For our purposes we are only interested in the cardinality of the homeotopy group. It is known that $H(S^2)/H_0(S^2)$ has two elements, that $H(P^2)/H_0(P^2)$ is trivial (c.f. [10]) and that $H(K)/H_0(K)$ has four elements [16]. The homeotopy group of any other closed 2-manifold has cardinality $\aleph_0$ as the following argument indicates: Since the homeotopy groups are finitely generated (c.f. [2], [15]), it suffices to show that $H(M)/H_0(M)$ has cardinality at least $\aleph_0$. Since M is not S^2, P^2, or K, there is a 2-manifold N such that M is the connected sum of N and a torus T with the connected sum formed at the disk D. Let a and b be simple closed curves in $T - D$ such that $[a]$ and $[b]$ generate $H_1(T)$. Let ϕ be a map of $T \# N$ onto T which is the identity on $T - D$. Let h be a homeomorphism of M onto itself which is a Lickorish twist (c.f. [2], [15]) about a that is the identity off a neighborhood of a which is contained in $T - D$. We need only show that for $n \neq m$, h^n (the n-fold composition of h) is not isotopic to h^m. If h^n is isotopic to h^m, then $\phi_* h^n_* = \phi_* h^m_*$. But

$$\phi_* h^n_*([\beta]) = (n[\alpha], [\beta]) \quad \text{and} \quad \phi_* h^m_*([\beta]) = (m[\alpha], [\beta])$$

which are different elements of $H_1(T)$.

LEMMA 3. *For each closed 2-manifold, M, there is a CW complex K_M and a homotopy equivalence between K_M and $H_0(M)$.*

Proof. Since $H_0(M)$ is an l_2-manifold (and hence has the homotopy type of a CW-complex), it suffices to show that for each M, there is a weak homotopy equivalence between $H_0(M)$ and a CW complex K_M (c.f. [18, p. 405]). Kneser [14] showed that $SO(3)$, the space of rotations of S^2, is a deformation retract of $H_0(S^2)$. Since $SO(3)$ is homeomorphic to P^3, this implies immediately that P^3 and $H_0(S^2)$ are homotopy equivalent. Hamstrom proved directly in [10] that $H_0(P^2)$ is also weakly homotopy equivalent to $SO(3) \approx P^3$.

In [9] Hamstrom showed that $\pi_1(H_0(T)) = \pi_1(T)$ and that $\pi_k(H_0(T)) = 0 = \pi_k(T)$, $k > 1$. Hence T is an Eilenberg-Maclane CW complex and $H_0(T)$ is an Eilenberg-Maclane space, both of type $(\pi_1(T), 1)$. Therefore T and $H_0(T)$ are weakly homotopy equivalent (c.f. Whitehead [22]). A similar argument holds for K, since $\pi_1(H_0(K)) = Z$ and $\pi_k H_0(K) = 0$ for $k > 1$ [10]. Hence S^1 and $H_0(K)$ are both of type $(\pi_1(S^1), 1)$ and hence are weakly homotopy equivalent. Finally, if M is orientable of genus >1 or nonorientable of genus >2, Hamstrom [11] showed that for all n, $\pi_n(H_0(M)) = 0$ and it follows that $H_0(M)$ is homotopy equivalent to a point.

THEOREM.

$$(H(S), PLH(S)) \approx (Z_2 \times P^3 \times l_2, Z_2 \times P^3 \times l_2^f),$$
$$(H(T), PLH(T)) \approx (Z \times T \times l_2, Z \times T \times l_2^f),$$
$$(H(P), PLH(P)) \approx (\{pt\} \times P^3 \times l_2, \{pt\} \times P^3 \times l_2^f),$$
$$(H(K), PLH(K)) \approx (Z_4 \times S^1 \times l_2, Z_4 \times S^1 \times l_2^f),$$
$$(H(M), PLH(M)) \approx (\{pt\} \times Z \times l_2, \{pt\} \times Z \times l_2^f),$$

for any other closed 2-manifold M.

Proof. We first shall prove the absolute version of the theorem. Note that the first factor in each product on the right is a discrete space of the same cardinality as the homeotopy group (Lemma 2) and hence that the product of the first two factors is a finite simplicial complex of the same homotopy type as the space of homeomorphisms (proof of Lemma 3). But the product of any finite simplicial complex with l_2 is an l_2-manifold [20]. The absolute case then follows since in each case the space of homeomorphisms is an l_2-manifold and two l_2-manifolds of the same homotopy type are homeomorphic [12].

By Lemma 1, $(H(M), PLH(M))$ is an (l_2, l_2^f)-manifold pair for any 2-manifold M. Since for any complex K, $(K \times l_2, K \times l_2^f)$ is an (l_2, l_2^f)-pair, in each case the pair on the right is also an (l_2, l_2^f)-pair. The relative version then follows immediately since by [21] if (X, Y) and (X, Z) are (l_2, l_2^f)-pairs then (X, Y) and (X, Z) are pair homeomorphic.

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