THE RATIONAL EQUIVALENCE RING OF SYMMETRIC PRODUCTS OF CURVES

BY

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1. Introduction

Let X be a smooth complete irreducible curve of genus g defined over an algebraically closed field, and let J be the Jacobian variety of X. The n-fold symmetric product $X(n)$ is a smooth variety which represents the effective divisors of degree n on the curve. Once a reference point $p \in X$ has been fixed there is a map $q_n: X(n) \to J$ such that $q_n(np)$ is the identity on J, and q_n is uniquely determined up to automorphism of J. If $n \geq 2g - 1$, $X(n)$ becomes in this way a locally trivial projective fibred bundle over J. Both the variety $X(n)$ and the morphism q_n are classical objects of study. There is also recent work about them, for example by Mattuck [8], [9], [10], Schwarzenberger [12], Kempf [5], Kleiman and Laksov [6], and a summary of many results in the book of Gunning [4].

Over the complex field MacDonald [7] determined the structure of the cohomology ring $H^*(X(n), \mathbf{Z})$. His formulae, suitably interpreted, say that for every n, $H^*(X(n), \mathbf{Z})$ is generated via q_n^* as an algebra over $H^*(J, \mathbf{Z})$ by a single element $z \in H^2(X(n), \mathbf{Z})$, and the relations that z satisfies are explicitly given. We will prove in every characteristic a similar result for the Chow ring $A(X(n))$ of cycles with integer coefficients modulo rational equivalence; namely, we show it is generated via q_n^* as an algebra over $A(J)$ by a single element z which represents a cycle of codimension 1. We will also prove that the relations in $q_n^*(A(J))[z]$ are analogous to the relations in $q_n^*(H^*(J, \mathbf{Z}))[z]$.

MacDonald could compute $H^*(X(n), \mathbf{Z})$ directly, using Kunneth formulae and a theorem of Grothendieck which relates the rational cohomology ring of a space to the ring of the quotient space under a finite group of homeomorphisms. He used then the known structure of $H^*(J, \mathbf{Z})$ to establish the fact given above. Now Kunneth formulae do not hold for the Chow ring and not much is known about $A(J)$. Our method is therefore different; it consists in studying the geometry of the natural inclusion $i: X(n-1) \to X(n)$ and in using the Chern relations for the projective bundle $X(2g-1)$ given in [8]. The main point is to show that the morphisms

$$i_*: A(X(n-1)) \to A(X(n)) \quad \text{and} \quad i^*: A(X(n) \to A(X(n-1))$$

are respectively injective and surjective.

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We close this introduction proposing an open question. It is known [12] that $X(n)$ is the projective fibred variety $P(F_n)$ associated to a suitable coherent sheaf of modules F_n over J. Because of our results about $A(X(n))$ it is natural to ask, under what conditions on a base space S and a coherent sheaf of modules F on S is the extension $A(P(F))$ generated over $A(S)$ by a single element? We have some results in this direction for very special situations. Since they are quite incomplete we will not present them here.

2. The injectivity theorem

Let X be a smooth projective curve of genus g defined over an algebraically closed field. The symmetric group on n letters $S(n)$ acts on $X[n]$, the n-fold product of X, by permuting the factors. The symmetric product $X(n)$ is the quotient variety $X[n]/S(n)$. It is smooth and projective, the quotient map

$$\pi_n \colon X[n] \to X(n)$$

is a finite separable morphism of degree $n!$ [8]. We denote by $[x_1, \ldots, x_n]$ a point in $X[n]$ and by $(x_1, \ldots, x_n)$ its image in $X(n)$, where $x_i \in X$. The point $(x_1, \ldots, x_n)$ may be thought of as representing the divisor $\sum_1^n x_i$ on X.

Fix once for all a point $p \in X$ and let $m \leq n$. We have a diagram

$$(1) \qquad \begin{tikzcd} X[m] \arrow[r, shift left, "j_{m,n}"] \arrow[d, "\pi_m"] & X[n] \arrow[l, shift left, "pr_{n,m}"] \arrow[d, "\pi_n"] \\ X(m) \arrow[r, shift left, "i_{m,n}"] & X(n) \arrow[l, shift left, dashed, "g"] \end{tikzcd}$$

Here $pr_{n,m}$ is the projection onto the first m factors, $j_{m,n}$ is the map which sends a point

$$[x_1, \ldots, x_m] \quad \text{to} \quad [x_1, \ldots, x_m, p, \ldots, p],$$

and $i_{m,n}$ is the map which sends

$$(x_1, \ldots, x_m) \quad \text{to} \quad (x_1, \ldots, x_m, p, \ldots, p).$$

We will explain g in a moment. Induced by (1) there is the diagram (2) between the Chow groups

$$(2) \qquad \begin{tikzcd} A(X[m]) \arrow[r, shift left, "(j_{m,n})_*"] \arrow[d, shift left] & A(X[n]) \arrow[l, shift left, "j_{m,n}^*"] \arrow[d, shift left] \\ A(X(m)) \arrow[r, shift left, "(i_{m,n})_*"] \arrow[u, shift left] & A(X(n)) \arrow[l, shift left, "i_{m,n}^*"] \arrow[u, shift left] \end{tikzcd}$$

We drop the subscripts on maps when no confusion exists and will often simply write $X(m)$ for the subvariety $i_{m,n}(X(m))$ of $X(n)$. The composite morphism $pr \circ j$ is the identity, hence

$$(pr \circ j)_* = pr_* j_* \quad \text{and} \quad (pr \circ j)^* = j^* pr^*$$

are both the identity; it follows from this that j_* is injective and j^* surjective.

We would like analogous results connecting $X(m)$ and $X(n)$. From $X(n)$ to $X(m)$ there is no morphism like $pr_{n,m}$ but there is a correspondence $g = \pi_m pr_{n,m} \pi_n^{-1}$, i.e.,

$$g(x_1, \ldots, x_n) = \sum_{(i)} (x_{i_1}, \ldots, x_{i_m}), \tag{3}$$

the sum being taken over all m-subsets of $1, \ldots, n$. More formally, the correspondence g is given by its graph $\Gamma \subset X(n) \times X(m)$, where

$$\Gamma = [\pi_n \times \pi_m](\Gamma'),$$

Γ' being the graph of the projection map $pr_{n,m}$. The correspondence g gives a morphism $g_*\colon A(X(n)) \to A(X(m))$ defined on representative cycles by

$$g_*(Z) = pr_{X(m)}((Z \times X(m)) \cdot \Gamma).$$

The key Corollary 1 below shows that the composite morphism

$$g_*(i_{m,n})_*\colon A(X(m)) \to A(X(m))$$

is "close" to being the identity map. We study it now.

LEMMA 1. *The morphism $g_*(i_{m,n})_*$ is induced by the cycle $(i_{m,n} \times \mathrm{id})^*\Gamma$ on $X(m) \times X(m)$.*

Proof. We have in fact

$$\begin{aligned} g_* i_* Z &= pr_{X(m)}(((i_* Z) \times X(m)) \cdot \Gamma) \\ &= pr_{X(m)}((i \times \mathrm{id})_*(Z \times X(m)) \cdot \Gamma) \\ &= pr_{X(m)}((Z \times X(m)) \cdot (i \times \mathrm{id})^*\Gamma) \end{aligned}$$

since

$$pr_{X(m)} \circ (i \times \mathrm{id}) = pr_{X(m)}$$

where the two projections are taken respectively on $X(n) \times X(m)$ and $X(m) \times X(m)$.

Let $X_0(m) = X(m) - X(m-1)$ be the complement of $X(m-1)$ in $X(m)$. Let Δ be the diagonal subvariety of $X(m) \times X(m)$. Let Δ_0 be the diagonal subvariety of $X_0(m) \times X_0(m)$ and

$$\alpha\colon X_0(m) \times X_0(m) \to X(n) \times X(m)$$

be the embedding induced by $i_{m,n} \times \mathrm{id}$.

Everything follows from:

PROPOSITION 1. $\alpha^*\Gamma = \Delta_0$.

The proof depends on algebraic-geometric calculations which are deferred to Section 3.

COROLLARY 1. *Let $\rho^*\colon A(X(m)) \to A(X_0(m))$ be the morphism induced by restriction, then $\rho^* g_* i_* = \rho^*$.*

Proof. We prove that the equality holds already in the group of cycles. To begin with, note that by (3),

$$\begin{aligned} gi(x_1, \ldots, x_m) &= g(x_1, \ldots, x_m, p, \ldots, p) \\ &= (x_1, \ldots, x_m) + (p, x_2, \ldots, x_m) + \cdots \end{aligned}$$

where all the points on the right except the first have some of the x_i's replaced by p. Therefore set theoretically, by Lemma 1,

$$(i_{m,n} \times \mathrm{id})^{-1}\Gamma = \Delta + D$$

where $D \subset X(m) \times X(m-1)$. By Proposition 1 it follows that as cycles

$$(i_{m,n} \times \mathrm{id})^*\Gamma = \Delta + Y$$

where $\operatorname{supp} Y \subset X(m) \times X(m-1)$.

Now let Z be an arbitrary cycle in $X(m)$. We have by Lemma 1,

$$\begin{aligned} g_* i_* Z &= pr_{X(m)}[(i_{m,n} \times \mathrm{id})^*\Gamma \cdot (Z \times X(m))] \\ &= pr_{X(m)}[\Delta \cdot Z \times X(m)] + pr_{X(m)}[Y \cdot Z \times X(m)] \\ &= Z + Z_1 \end{aligned}$$

where $\operatorname{supp} Z_1 \subset X(m-1)$. Therefore

$$\rho^* g_* i_* Z = \rho^* Z + \rho^* Z_1 = \rho^* Z$$

which proves the corollary.

THEOREM 1. *For every $m \leq n$, the morphism*

$$(i_{m,n})_*\colon A(X(m)) \to A(X(n))$$

is injective.

Proof. By induction over m. When $m = 0$ we interpret $X(0)$ as a single reduced point and have $i_{0,n}X(0) = (p, \ldots, p)$. Thus the theorem is true for $m = 0$ since $X(n)$ is complete. Assume now the statement to be true for $(m-1)$ and for every $n \geq (m-1)$. There is a diagram, commutative except for g_*, whose rows are right-exact by [1] and exact on the left by the induction hypothesis.

$$\begin{tikzcd}
0 \arrow[r] & A(X(m-1)) \arrow[r, "(i_{m-1,m})_*"] \arrow[d, equal] & A(X(m)) \arrow[r, "\rho^*"] \arrow[d, shift left, "(i_{m,n})_*"] & A(X_0(m)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A(X(m-1)) \arrow[r, "(i_{m-1,n})_*"] & A(X(n)) \arrow[r] \arrow[u, shift left, "g_*"] & A(X(n) - X(m-1)) \arrow[r] & 0
\end{tikzcd}$$

so if $z \in A(X(m))$, then

$$\begin{aligned}(i_{m,n})_*(z) = 0 &\Rightarrow \rho^* g_*(i_{m,n})_*(z) = 0\\ &\Rightarrow \rho^*(z) = 0 \quad \text{by Corollary 1}\\ &\Rightarrow z = (i_{m-1,m})_*(z') \quad \text{for some } z' \in A(X(m-1)) \text{ by exactness}\\ &\Rightarrow (i_{m-1,n})_*(z') = 0\\ &\Rightarrow z' = 0 \quad \text{by induction}\\ &\Rightarrow z = 0.\end{aligned}$$

3. Proof of Proposition 1

The set-theoretical part is given by:

LEMMA 2. $\alpha^{-1}(\Gamma) = \Delta_0$.

Proof. Consider a typical point of $X_0(m) \times X_0(m)$:

$$(x_1, \ldots, x_m; y_1, \ldots, y_m), \quad x_i \neq p, y_i \neq p;$$

then

$$\begin{aligned}\alpha(x; y) = (x_1, \ldots, x_m, p, \ldots, p; y_1, \ldots, y_m) &\text{ (which is a point of } \Gamma)\\ &\Leftrightarrow \{y_1, \ldots, y_m\} \subset \{x_1, \ldots, x_m, p, \ldots, p\}\\ &\Leftrightarrow (y_1, y_2, \ldots, y_m) = (x_1, x_2, \ldots, x_m)\\ &\Leftrightarrow (x_1, \ldots, x_m; y_1, \ldots, y_m) \in \Delta_0.\end{aligned}$$

It follows from the lemma that

$$\alpha^*\Gamma = d\Delta_0 \quad \text{for some integer } d > 0. \tag{4}$$

To show $d = 1$ we use the diagram

$$\begin{tikzcd} X_0[m] \times X_0[m] \arrow[r, "\beta"] \arrow[d, "\pi_m \times \pi_m"] & X[n] \times X[m] \arrow[d, "\pi"] \\ X_0(m) \times X_0(m) \arrow[r, "\alpha"] & X(n) \times X(m) \end{tikzcd}$$

in which $X_0[m] = \pi_m^{-1}(X_0(m))$ and $\beta = j_{m,n} \times \text{id}$. We will pull both sides of (4) up via $\pi_m \times \pi_m$ and for this purpose must study this map.

The group $S(n) \times S(m)$ acts on $X[n] \times X[m]$ permuting the coordinates in each factor. The graph Γ' of $pr_{n,m}$ consists of all the points of the form

$$\Gamma' = \{[x_1, \ldots, x_n; x_1, \ldots, x_m]\}.$$

The isotropy groups of Γ', i.e., the subgroup G of $S(n) \times S(m)$ which sends Γ' into itself, is evidently given by

$$G = \{(s, t) \mid s(i) = t(i), i = 1, \ldots, m\}.$$

We choose left coset representatives mod G, $\tau_1, \ldots, \tau_N$ where $N = n!/(n-m)!$, taking as the first $m!$ coset representatives the elements of the subgroup H of $S(n) \times S(m)$ which operates only on the first m coordinates of $X[n]$. (This is possible since $H \cap G = \{1\}$.)

LEMMA 3. $\pi^*(\Gamma) = \sum_1^N \tau_j(\Gamma')$.

Proof. Since G is the isotropy group of Γ', $\tau_i(\Gamma') \neq \tau_j(\Gamma')$ if $i \neq j$ and $\pi^{-1}(\pi(\Gamma')) = \pi^{-1}(\Gamma) = \bigcup_1^N \tau_j(\Gamma')$. Because τ_j is an automorphism of $X[n] \times X[m]$ which commutes with π, the coefficient with which each $\tau_j(\Gamma')$ appears in $\pi^*(\Gamma)$ is equal to the coefficient $h \geq 1$ of Γ'. By the projection formula,

$$n!\, m!\, \Gamma = \pi_*\pi^*(\Gamma) = \pi_*\left(h \cdot \sum_1^N \tau_j(\Gamma')\right).$$

Since each $\tau_j(\Gamma')$ is, as a set, a covering of Γ of degree $m!\,(n-m)!$, it follows that $n!\, m! \geq hNm!\,(n-m)!$. Hence $h = 1$. This completes the proof.

Let Δ' (resp. Δ'_0) denote the diagonal subvariety in $X[m] \times X[m]$ (resp. $X_0[m] \times X_0[m]$). The group $S(m) \times S(m)$ acts on $X[m] \times X[m]$ and $L = \{(s, t) \mid s(i) = t(i)\}$ is the isotropy group of Δ'. Choose coset representatives $\sigma_1, \sigma_2, \ldots, \sigma_{m!}$ which map to $\tau_1, \tau_2, \ldots, \tau_{m!}$ under the natural embedding of $S(m) \times S(m)$ in $S(n) \times S(n)$ as the subgroup which leaves fixed the last $(n-m)$ coordinates of $X[n]$ in $X[n] \times X[m]$.

COROLLARY 2. $(\pi_m \times \pi_m)^*(\Delta_0) = \sum_1^{m!} \sigma_i(\Delta'_0)$.

Proof. It is enough to show

$$(\pi_m \times \pi_m)^*(\Delta) = \sum_1^{m!} \sigma_i(\Delta')$$

and this follows from Lemma 3 since we have $\Gamma = \Delta$ when $m = n$.

LEMMA 4. *With the above notations*

$$\beta^{-1}(\tau_j(\Gamma')) = \emptyset \qquad \text{if } j > m!,$$
$$\beta^{-1}(\tau_j(\Gamma')) = \sigma_j(\Delta'_0) \quad \text{if } j \leq m!.$$

Proof. Assume first $j > m!$. Consider a point of $X_0[m] \times X_0[m]$:

$$z = [x_1, \ldots, x_m; y_1, \ldots, y_m], \quad x_i \neq p,\ y_i \neq p.$$

Then

$$\beta(z) = [x_1, \ldots, x_m, p, \ldots, p; y_1, \ldots, y_m].$$

Since $j > m!$, it follows that τ_j does not belong to the subgroup of $S(n) \times S(m)$ which permutes the last $(n-m)$ coordinates of $X[n]$; hence some p appears among the first m coordinates of $\tau_j^{-1}(\beta(z))$. Now

$$\beta(z) \in \tau_j(\Gamma') \Leftrightarrow (\tau_j)^{-1}(\beta(z)) \in \Gamma',$$

so the first m coordinates of $(\tau_j)^{-1}(\beta(z))$ are equal to the last m if $\beta(z) \in \tau_j(\Gamma')$; but the last m coordinates of $(\tau_j)^{-1}(\beta(z))$ are just the y's and therefore contain no p. We have then a contradiction with the above statement, hence

$$\beta^{-1}(\tau_j(\Gamma')) = \emptyset.$$

On the other hand if $j \leq m!$, then

$$\begin{aligned}\tau_j([x_1, \ldots, x_n; x_1, \ldots, x_m]) &\\ &= [x_{\tau_j(1)}, \ldots, x_{\tau_j(n)}; x_1, \ldots, x_m] \\ &= x_{\sigma_j(1)}, \ldots, x_{\sigma_j(m)}, \ldots, x_{m+1}, \ldots, x_n; x_1, \ldots, x_m]\end{aligned}$$

from which it is clear $\beta^{-1}(\tau_j(\Gamma')) = \sigma_j(\Delta_0')$.

LEMMA 5.
$$\beta^*(\tau_j(\Gamma')) = 0 \qquad \textit{if } j > m!,$$
$$\beta^*(\tau_j(\Gamma')) = \sigma_j(\Delta_0') \quad \textit{if } j \leq m!.$$

Proof. From Lemma 4 all we need to prove is that $\tau_j(\Gamma')$ intersects transversally $\beta(X_0[m] \times X_0[m])$ for $j \leq m!$, which is obvious since $\tau_j(\Gamma')$ is the graph of a projection from $X[n]$ to $X[m]$.

We now complete the proof of Proposition 1. Applying $(\pi_m \times \pi_m)^*$ to the left side of (4) we get

$$\begin{aligned}(\pi_m \times \pi_m)^*\alpha^*\Gamma &= \beta^*\pi^*\Gamma \\ &= \beta^*\left(\sum_1^N \tau_j(\Gamma')\right) \quad \text{by Lemma 3} \\ &= \sum_1^{m!} \sigma_j(\Delta_0') \qquad \text{by Lemma 5.}\end{aligned}$$

Applying $(\pi_m \times \pi_m)^*$ to the right side of (4) we get, by Corollary 2,

$$(\pi_m \times \pi_m)^*(d\Delta_0) = d \cdot \sum_1^{m!} \sigma_j(\Delta_0').$$

Comparing with the above we see that $d = 1$, hence Proposition 1 is proved.

4. The surjectivity theorem

To prove the surjectivity of $i_{m,n}^*: A(X(n)) \to A(X(m))$ a natural approach would be to consider the morphism

$$g^*: A(X(m)) \to A(X(n))$$

induced by the transpose of Γ in $X(m) \times X(n)$. However this is not convenient to work with since there are multiplicities involved which are awkward to compute. We therefore use a somewhat different method from the one used for the injectivity theorem.

We continue to let $X(j)$ denote not only the jth symmetric product, but also its image in $X(m)$ via the immersion $i_{j,m}$. As before $X(0)$ represents a point, so that in an expression like $X(k) \times X(0)$ it can be omitted; if $n < 0$ we take $X(n) = \emptyset$. We define

$$D(m) = \{(x_1, \dots, x_m) \in X(m) \mid x_i = x_j \text{ for some } i \neq j\}.$$

$D(m)$ is a closed subvariety of $X(m)$; it is empty if $m \leq 1$, otherwise $\dim D(m) = m - 1$. By abuse of notation we often write $D(s)$ instead of $i_{s,m}(D(s))$.

DEFINITION 1. We say a subvariety Y of $X(m)$ is in regular position if every component intersects all the subvarieties $X(i)$ and $D(i)$ properly, $i = 0, \dots, m$.

It follows that if Y is of codimension t and in regular position, then for $i = 0, \dots, m$,

$$\begin{aligned} &\dim Y \cap X(m-i) = m - t - i &&\text{or}\quad Y \cap X(m-i) = \emptyset \\ &\dim Y \cap D(m-i) = m - t - i - 1 &&\text{or}\quad Y \cap D(m-i) = \emptyset. \end{aligned} \tag{5}$$

Remark 1. If Y is in regular position in $X(m)$ then $Y \cap X(m-i)$ is in regular position in $X(m-i)$. This follows easily from (5) and the isomorphisms

$$\begin{aligned} &i_{m-i,m}\colon [Y \cap X(m-i)] \cap X(m-i-j) \simeq Y \cap X(m-i-j), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq j \leq m-i \\ &i_{m-i,m}\colon [Y \cap X(m-i)] \cap D(m-i-j) \simeq Y \cap D(m-i-j), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq j \leq m-i \end{aligned} \tag{6}$$

where the varieties on the left are to be interpreted as lying in $X(m-i)$ and those on the right in $X(m)$.

Let $\pi_m^r\colon X(m-r) \times X(r) \to X(m)$ be the finite morphism of degree $C(m, r)$ defined by

$$(x_1, \dots, x_{m-r}) \times (y_1, \dots, y_r) \to (x_1, \dots, x_{m-r}, y_1, \dots, y_r).$$

By our conventions, if $r = 0$ then π_m^0 is the identity map.

DEFINITION 2. We say a subvariety V of $X(m)$ is of *type* r, if for some subvariety $A \subset X(m-r)$,

$$V = \pi_m^r(A \times X(r)). \tag{7}$$

Remark 2. If V is irreducible and of type r, the variety A in (7) can be taken to be irreducible. Every subvariety V is of type 0, according to our conventions with $A = V$. If V has type m, then $V = X(m)$ and $A = X(0)$.

LEMMA 6. *If V is of type r every component is of type r.*

Proof. Let $A = \bigcup A_i$, $V = \bigcup V_j$ be the decompositions in irreducible components. Then

$$\pi_m^r(A \times X(r)) = \bigcup_i \pi_m^r(A_i \times X(r)) = \bigcup_j V_j.$$

Since the $\pi_m^r(A_i \times X(r))$ are irreducible, for each j there exists an $i(j)$ such that $V_j = \pi_m^r(A_{i(j)} \times X(r))$. Therefore V_j is of type r.

From now on V denotes an equidimensional subvariety of type r and cod t in $X(m)$. We may take A equidimensional in (7), according to Remark 2. With these assumptions we have by (7), dim $V = m - t$ and dim $A = m - t - r$.

PROPOSITION 2. *V and A being as described above, V is in regular position in $X(m)$ if and only if A is in regular position in $X(m - r)$.*

Proof. If either $r = m$ or $r = 0$ the assertion is trivial by Remark 2. Thus we assume $0 < r < m$ and proceed in several steps.

V intersects properly all the $X(m - i)$, $0 \le i \le m$

$\Leftrightarrow A$ intersects properly all the $X(m - r - j)$, $0 \le j \le m - r$.

This is by definition equivalent to

$$\text{(8)} \qquad \dim V \cap X(m - i) \le m - t - i$$
$$\Leftrightarrow \dim A \cap X(m - r - j) \le m - r - t - j$$

The proof is as follows. A point y in V has the form

$$y = (a_1, \ldots, a_{m-r}, x_1, \ldots, x_r) \quad \text{where } a = (a_1, \ldots, a_{m-r}) \in A.$$

If $y \in V \cap X(m - i)$, a subset of i p's appears in

$$(a_1, \ldots, a_{m-r}, x_1, \ldots, x_r).$$

Say $j \ge 0$ of these p's occur in a. Then

$$a \in A \cap X(m - r - j) = \text{(by definition) } A_j.$$

The other $i - j$ p's appear in $(x_1, \ldots, x_r)$, which belongs therefore to $X(r - i + j)$, hence

$$y \in \pi_{m-i}^{r-i+j}(A_j \times X(r - i + j)).$$

Conversely every point in $\pi_{m-i}^{r-i+j}(A_j \times X(r - i + j))$ is a point of $V \cap X(m - i)$ so that

$$\text{(9)} \qquad V \cap X(m - i) = \bigcup_j \pi_{m-i}^{r-i+j}(A_j \times X(r - i + j)).$$

Now the π's are finite, hence

$$\dim (V \cap X(m - i)) = \max_{j \le i,\, j \le m-r} \dim (A_j \times X(r - i + j)),$$

that is,

$$\dim (V \cap X(m - i)) \leq m - t - i$$
$$\Leftrightarrow \dim A_j \leq (m - t - i) - (r - i + j) = m - r - t - j,$$
$$\text{for all } j \leq \min (i, m - r).$$

This proves (8).

(10) If A intersects properly $D(m - r)$, then V intersects properly $D(m)$.

To see this it is enough to prove $V \not\subset D(m)$ because our assumptions on r imply that $D(m)$ has codimension 1. Since $A \not\subset D(m - r)$ there is a point $a = (a_1, \ldots, a_{m-r}) \in A$ such that $a_i \neq a_j$ for $i \neq j$. Fix a point

$$(x_1, \ldots, x_r) \in X(r)$$

so that $x_i \neq x_j$, $x_i \neq a_k$ for all i, j, k. Then $(a, x) \notin D(m)$, hence $V \not\subset D(m)$.

We may now complete the proof of the proposition. We have, according to the hypotheses, still to show that

$$\dim (V \cap D(m - i)) \leq m - t - i - 1, 0 \leq i \leq m$$
$$\Leftrightarrow \dim (A \cap D(j)) \leq j - t - 1, 0 \leq j \leq m - r.$$

We begin with the reverse implication. First of all, we have

$$V \cap D(m - i) = (V \cap X(m - i)) \cap D(m - i), \quad \text{by (6)},$$
$$\dim (V \cap X(m - i)) \leq m - t - i, \quad \text{by (8)},$$

hence it is enough to prove that no component, say T, of $V \cap X(m - i)$ is contained in $D(m - i)$. Now we have from (9).

$$V \cap X(m - i) = \bigcup_j \pi_{m-i}^{r-i+j}(A_j \times X(r - i + j)),$$

therefore using the argument in Lemma 6 there is a component, S say, of some A_j such that $T = \pi_{m-i}^{r-i+j}(S \times X(r - i + j))$. Since by hypothesis, A is in regular position in $X(m - r)$, then by Remark 1, S is in regular position in $A(m - r - j)$; hence by (10) taking V and A to be T and S respectively, T intersects properly $D(m - i) \subset X(m - i)$, that is $T \not\subset D(m - i)$.

To prove the other implication, notice that $i_{m-r,m}(A \cap D(j)) \subset V \cap D(j)$ hence

$$\dim (A \cap D(j)) \leq \dim (V \cap D(j)) \leq j - t - 1.$$

This completes the proof.

The hypotheses about V and A continue as they were for Proposition 2. We define $W \subset X(m + 1)$ and $V^* \subset X(m)$ by

$$\begin{aligned} W &= \pi_{m+1}^{1}(V \times X) = \pi_{m+1}^{r+1}(A \times X(r + 1)), \\ V^* &= \pi_m^{r+1}((A \cap X(m - r - 1)) \times X(r + 1)) \end{aligned} \tag{11}$$

where the first factor is being viewed as a subvariety of $X(m - r - 1)$.

LEMMA 7. *If V is of type r and in regular position, then either V^* is of type $r + 1$, in regular position, and* $\dim V^* = \dim V$, *or else* $V^* = \emptyset$.

Proof. If not empty, then clearly V^* is of type $r + 1$ and

$$\dim V^* = \dim (A \cap X(m - r - 1)) + (r + 1).$$

By Proposition 2, A is in regular position in $X(m - r)$, thus $A \cap X(m - r - 1)$ is in regular position in $X(m - r - 1)$ and of dimension $(m - r - t - 1)$ or empty. This implies by Proposition 2 that either $V^* = \emptyset$ or V^* is in regular position, of dim $(m - t) = \dim V$.

PROPOSITION 3. *If V is in regular position and irreducible, then W is irreducible, and as cycles we have $i^*_{m,m+1} W = V + V_1$ where* supp $V_1 \subseteq V^*$.

Proof. We have the diagram

$$\begin{tikzcd} X(m) \arrow[r, "f"] \arrow[dr, "i = i_{m,m+1}"'] & X(m) \times X \arrow[d, "\pi_{m+1}{}^1 = \pi"] \\ & X(m+1) \end{tikzcd}$$

where $f(x) = (x) \times p$, therefore $i^{-1}(W) = f^{-1}\pi^{-1}(W)$ and since we will show the cycles are all defined,

$$i^*W = f^*\pi^*W. \tag{12}$$

We study first $\pi^* W$. Let

$$\pi^{-1}(W) = \bigcup T_i \tag{13}$$

be the decomposition in irreducible components. Because V is irreducible, W is also; $\pi^* W$ is defined, since π is finite, and $\pi^{-1}(W)$ is equidimensional.

We have then $\pi^* W = \sum d_i T_i$, $d_i \geq 1$. We prove now that

$$\pi^* W = \sum T_i. \tag{14}$$

Because π is a finite surjective morphism between smooth varieties it is flat [3, IV, vol. 2, 6.1.5.]. Denoting for simplicity's sake $X(m) \times X = Y$, $X(m + 1) = Z$, we have then

$$\pi^*W = \sum l(O_{Y,T_i}/m_w O_{Y,T_i})\, T_i \tag{15}$$

where m_w denotes the maximal ideal in the local ring $O_{Z,W}$ and l denotes the length of an artinian module. By Proposition 2, A is in regular position; again by Proposition 2 so is W. Hence W is not contained in $D(m + 1)$, which is easily seen to be the branch locus of π. Because π is unramified at W we have therefore $m_w O_{Y, T_i} = m_{T_i}$, the maximal ideal of O_{Y, T_i}; hence

$$l(O_{Y, T_i}/m_w O_{Y, T_i}) = 1,$$

which proves (14).

The next step is to show that set-theoretically

$$i^{-1}(W) = V \cup V^*. \tag{16}$$

To begin with, it is easy to check that

$$\pi^{-1}(W) = (V \times X) \cup S, \tag{17}$$

where we put

$$S = \{(a_1, \ldots, \hat{a}_i, \ldots, a_{m-r}, x_1, \ldots, x_{r+1}) \times a_i \mid (a_1, \ldots, a_{m-r}) \in A\}. \tag{18}$$

We now apply f^{-1} to both sides of (17). Evidently

$$f^{-1}(V \times X) = V, \tag{19}$$

and also

$$f^{-1}(S) = V^*, \tag{20}$$

since

$$\begin{aligned} & f(y_1, \ldots, y_m) \in S \\ & \quad \Leftrightarrow (y_1, \ldots, y_m) \times p \in S \\ & \quad \Leftrightarrow (y_1, \ldots, y_m) \times p = (a_1, \ldots, \hat{a}_i, \ldots, a_{m-r}, x_1, \ldots, x_{r+1}) \times p, \\ & \qquad\qquad \text{where } (a_1, \ldots, a_{i-1}, p, a_{i+1}, \ldots, a_{m-r}) \in A \quad \text{by (18)}, \\ & \quad \Leftrightarrow (a_1, \ldots, \hat{a}_i, \ldots, a_{m-r}, x_1, \ldots, x_{r+1}) \in V^*. \end{aligned}$$

(17), (19), and (20) prove (16).

Comparing (17) with (13) we may set $T_1 = V \times X$ and thus $T_1 \subseteq S$ for $i \geq 2$. Now by Lemma 7, dim $V^* =$ dim V, by (11), dim $W =$ dim $V + 1$; hence f^* is well defined on the cycle $\pi^* W$ and

$$i^* W = f^* \left(\sum T_i\right), \quad \text{by (12) and (14)},$$

or

$$i^* W = f^*(V \times X) + f^* \left(\sum_{i \geq 2} T_i\right),$$

where supp $f^* (\sum_{i \geq 2} T_i) \subseteq V^*$ by (20). Now $V \times X$ intersects $X(m) \times p$ transversally, hence $f^*(V \times X) = V$, and therefore

$$i^* W = V + f^* \left(\sum_{i \geq 2} T_i\right), \quad \text{where} \quad \operatorname{supp} f^* \left(\sum_{i \geq 2} T_i\right) \subseteq V^*.$$

This completes the proof.

Remark 3. With the above hypotheses and notations, if V is of type r but not $r+1$, then $V \not\subset V^*$, since V^* is of type $r+1$ and therefore so is every component by Lemma 6. It follows that $V \times X \not\subset S$ because $f^{-1}(V \times X) = V$ while $f^{-1}(S) = V^*$. Hence by (17), $S = \bigcup T_i$ for $i \geq 2$, and thus supp $f^*(\sum_{i \geq 2} T_i) = V^*$.

THEOREM 2. *$i^*_{m,n}: A(X(n)) \to A(X(m))$ is surjective.*

Proof. It is enough to show $i^*_{m,m+1}$ surjective for every m. By Chow's moving lemma [1] the irreducible subvarieties V in regular position generate $A(X(m))$. Hence it suffices to prove that for such a V,

$$i^*_{m,m+1} Y = V \quad \text{for some cycle } Y \text{ in } X(m+1). \tag{21}$$

We fix the dimension and prove (21) for all subvarieties of that dimension by descending induction over the type r of the subvariety.

If $r > \dim V$, the result is trivially true, since no such V exists.

If $r \leq \dim V$, then by Proposition 3,

$$i^*_{m,m+1} W = V + V_1 \quad \text{where} \quad \operatorname{supp}(V_1) \subseteq V^*. \tag{22}$$

Now V^* is either empty or else it is in regular position and of type $r+1$ by Lemma 7, hence all of its components V^*_i are such by Lemma 6. If $V^* = \emptyset$ then (21) is proved since $i^* W = V$. Otherwise, by the induction step,

$$V^*_i = i^*_{m,m+1}(Z_i) \quad \text{for some cycle } Z_i \subseteq X(m+1).$$

Since supp $(V_1) \subseteq V^*$ and they have the same dimension, $V_1 = \sum n_i V^*_i$, n_i integer; hence $V_1 = \sum i^*_{m,m+1}(n_i Z_i)$ and therefore $V = i^*_{m,m+1}(W - n_i Z_i)$, by (22), which proves (21).

5. The structure of $A(X(n))$

Let J be the Jacobian variety of X and $p \in X$ be the point fixed in Section 2. Identifying J with the divisor classes of degree 0, there is a morphism

$$q_n: X(n) \to J$$

defined by

$$q_n(x_1, \ldots, x_n) = \operatorname{cl}\left(\sum x_i - np\right)$$

The fiber $q_n^{-1}(y)$, $y \in J$, is therefore a projective space which represents the complete linear system of effective divisors whose class is $(y + n \cdot \operatorname{cl}(p))$. The dimension of such space is determined by Riemann-Roch theorem. For $m \leq n$ we have the diagram

$$\begin{tikzcd} X(m) \arrow[r, "i_{m,n}"] \arrow[d, "q_m"'] & X(n) \arrow[d, "q_n"] \\ J \arrow[r, equal] & J \end{tikzcd} \tag{23}$$

Induced by (23) there is the corresponding diagram of Chow rings

$$\begin{array}{ccc} A(X(m)) & \xrightarrow{i_{m,n}^*} & A(X(n)) \\ {\scriptstyle q_m^*}\uparrow & & \uparrow{\scriptstyle q_n^*} \\ A(J) & = & A(J) \end{array}$$

which shows that $i_{m,n}^*$ is a morphism of $A(J)$ algebras.

We want to determine the structure of $A(X(n))$ as an extension of $A(J)$.

In the case $n \geq 2g - 1$ this was done by Mattuck [8] who showed that $X(n)$ is then a projective bundle over J and computed the Chern relations. In order to describe his results we write $W_i = q_i(X(i))$, $U_i = \Theta(W_i)$ where $\Theta: J \to J$ is the morphism which maps y to $(-y + c)$, c being the canonical point, image of the canonical divisor, and we let

$$\begin{aligned} u_i &= \mathrm{cl}\,(U_{g-i}) \in A(J) \quad \text{if } 0 \leq i \leq g \\ u_i &= 0 \qquad\qquad\qquad\quad \text{if } i > g \text{ or } i < 0 \qquad (24)\\ \zeta_n &= \mathrm{cl}\,(i_{n-1,n}(X(n-1))) \in A(X(n)). \end{aligned}$$

On J there is a vector bundle F_n, $n \geq 2g - 1$, whose associated projective bundle is $X(n)$. The total Chern class is given by

$$c(F_n) = \sum_{0}^{n-g+1} (-1)^i u_i$$

and $i_{n-1,n}(X(n-1))$ is a divisor in the class of the fundamental sheaf $0(1)$. We put

$$\alpha = \sum_{0}^{g} (-1)^i u_i z^{g-i} \qquad (25)$$

PROPOSITION 4 (Mattuck). *The structure of the extension $A(X(n))$ of $A(J)$ is given for $n \geq 2g - 1$ by the exact sequence*

$$0 \longrightarrow (\alpha z^{n-2g+1}) \longrightarrow A(J)[z] \xrightarrow{\phi_n} A(X(n)) \longrightarrow 0,$$

where ϕ_n is the $A(J)$ morphism defined by $\phi_n(z) = \zeta_n$.

We proceed now to deduce the structure of $A(X(n))$ as an extension of $A(J)$ in the general case.

LEMMA 8. *In $A(X(n))$ we have* $\mathrm{cl}\,(i_{n-m,m}(X(n-m)) = \zeta_n^m$.

The proof is given in [8] and consists essentially in lifting $i_{n-1,n}(X(n-1))$ from $X(n)$ to $X[n]$ and in computing the self-intersections there.

LEMMA 9. $i_{m,n}^* \zeta_n = \zeta_m$.

Proof. We have

$$(i_{m,n})_*(i_{m,n}^* \zeta_n) = \zeta_n^{n-m+1}, \qquad (26)$$

by the projection formula and Lemma 8; also

$$(i_{m,n})_*\zeta_m = \mathrm{cl}\,(i_{m-1,n}(X(m-1)) = \zeta_n^{n-m+1} \tag{27}$$

by Lemma 8. Comparing (26) with (27) we deduce the lemma because $(i_{m,n})_*$ is injective by Theorem 1. Note that a different proof of Lemma 9 is given in [12].

We can now prove our main result.

THEOREM 3. *Let α be as in* (25), *z a variable, and let I_n denote the ideal in $A(J)[z]$:*

$$I_n = \begin{cases} ((\alpha): z^{2g-1-n}) & \text{if } n < 2g-1 \\ (\alpha \cdot z^{n-2g+1}) & \text{if } n \geq 2g-1 \end{cases}$$

Then for every n the structure of the extension $A(X(n))$ of $A(J)$ is determined by the exact sequence

$$0 \longrightarrow I_n \longrightarrow A(J)[z] \xrightarrow{\phi_n} A(X(n)) \longrightarrow 0$$

where ϕ_n is the morphism of $A(J)$ algebras defined by $\phi_n(z) = \zeta_n$, with ζ_n as in (24).

Proof. If $n \geq 2g-1$ the theorem is given by Proposition 4. If $n \leq 2g-2$ we define

$$\phi_n = i^*_{n,2g-1}\phi_{2g-1}. \tag{28}$$

We have then

$$\begin{aligned} \phi_n(z) &= i^*_{n,2g-1}\phi_{2g-1}(z) \\ &= i^*_{n,2g-1}(\zeta_{2g-1}) && \text{by Proposition 4} \\ &= \zeta_n && \text{by Lemma 9.} \end{aligned}$$

The morphism $i^*_{n,2g-1}$ is surjective by Theorem 2 and ϕ_{2g-1} is surjective by Proposition 4, hence ϕ_n is surjective. To finish the proof we determine the kernel of ϕ_n. Let $y \in A(J)[z]$; we have

$$\begin{aligned} \phi_n(y) = 0 &\Leftrightarrow i^*_{n,2g-1}\phi_{2g-1}(y) = 0 && \text{by (28)} \\ &\Leftrightarrow (i_{n,2g-1})_*(i^*_{n,2g-1}\phi_{2g-1}(y)) = 0 && \text{by Theorem 1} \\ &\Leftrightarrow \phi_{2g-1}(z^{2g-1-n})\phi_{2g-1}(y) = 0 \\ & && \text{by the projection formula and Lemma 9} \\ &\Leftrightarrow z^{2g-1-n} \cdot y \in (\alpha) && \text{by Proposition 4} \\ &\Leftrightarrow y \in I_n. \end{aligned}$$

In [7] MacDonald has given an analogous result for the cohomology ring which we wish to make explicit here. There are natural maps

$$A(J) \to H^*(J, \mathbf{Z}), \quad A(X(n)) \to H^*(X(n)\mathbf{Z}), \quad A(J)[z] \to H^*(J, \mathbf{Z})[\eta] \tag{29}$$

(To conform to MacDonald's notation we use the letter η in place of z.) We continue to denote the images of u_i, ζ_n, α, I_n under (29) by the same letters. Thus

$$u_i \in H^{2i}(J, \mathbf{Z}), \quad \zeta_n \in H^2(X(n), \mathbf{Z}),$$

$$I_n = \begin{cases} ((\alpha) : \eta^{2g-1-n}), & n < 2g - 1 \\ (\alpha\eta^{n-2g+1}), & n \geq 2g - 1 \end{cases}$$

We set formally $\deg \eta = 2$.

Then the structure of $H^*(X(n), \mathbf{Z})$ as an extension of $H^*(J, \mathbf{Z})$ is determined by the exact sequence

$$(30) \quad 0 \longrightarrow I_n \longrightarrow H^*(J, \mathbf{Z})[\eta] \xrightarrow{\psi_n} H^*(X(n), \mathbf{Z}) \longrightarrow 0 \quad \text{where} \quad \psi_n(\eta) = \zeta_n.$$

Actually MacDonald never states his result precisely in this form. He can describe $H^*(X(n), \mathbf{Z})$ directly as follows: let $\xi_1, \dots, \xi_g, \xi'_1, \dots, \xi'_g$ be the elements of degree 1 which generate the exterior algebra $H^*(J, \mathbf{Z})$ and let η be as in (29). Define R_n to be the ideal generated by the set

$$\mathscr{R}_n = \{(\xi_{i_1} \cdots \xi_{i_a} \xi'_{j_1} \cdots \xi'_{j_b} (\xi_{k_1} \xi'_{k_1} - \eta) \cdots (\xi_{k_c} \xi'_{k_c} - \eta)\eta^q)\}$$

where $i_1, \dots, i_a, j_1, \dots, j_b, k_1, \dots, k_c$ are any distinct integers and a, b, c, q are any integers such that $a + b + 2c + q = n + 1$. Then according to MacDonald the structure of $H^*(X(n), \mathbf{Z})$ is given by the exact sequence

$$0 \longrightarrow R_n \longrightarrow H^*(J, \mathbf{Z})[\eta] \xrightarrow{\psi_n} H^*(X(n), \mathbf{Z}) \longrightarrow 0$$

where ψ_n is as in (30).

If $n \geq 2g - 1$, then [7] there is essentially only one relation, i.e.,

$$R_n = (\eta^{n-2g+1} \textstyle\prod_1^g (\xi_i \xi'_i - \eta)), \quad n \geq 2g - 1.$$

To see that R_n coincides with I_n for $n \geq 2g - 1$, it suffices therefore to show that

$$(31) \qquad \alpha = \pm \textstyle\prod_1^g (\xi_i \xi'_i - \eta).$$

This could be done explicitly by expressing the u_i's in terms of the ζ_i's; however it is clear since evidently $\alpha \in R_n$, and both sides of (31) are monic polynomials of degree g in η.

For $n \leq 2g - 2$, one shows easily for the given generators of R_n that

$$x \in \mathscr{R}_n \Leftrightarrow x\eta^{2g-1-n} \in \mathscr{R}_{2g-1};$$

it follows therefore that $R_n = I_n$, $n \leq 2g - 2$; hence the two ideals are equal for all n.

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