ESTIMATES ON THE SUPPORT OF SOLUTIONS OF PARABOLIC VARIATIONAL INEQUALITIES[1]

BY

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1. Introduction

Consider a parabolic Cauchy problem

$$u_t - \Delta u = f \quad (x \in R^n, 0 < t \leq T), \tag{1.1}$$

$$u(x, 0) = u_0(x) \quad (x \in R^n) \tag{1.2}$$

where Δ is the Laplace operator. The solution u does not have compact support in general, even when $f \equiv 0$ and u_0 has compact support. For a parabolic variational inequality consisting of

$$u \geq 0, \qquad (u_t - \Delta u)(v - u) \geq f(v - u) \text{ a.e., for any } v \geq 0, \tag{1.3}$$

and of (1.2), the situation is entirely different: when f is uniformly negative, $u(x, t)$ has compact support whenever $u_0(x)$ has compact support. The object of this paper is to study properties of the support.

In Section 2 we study the variational inequality (1.3), (1.2) when u_0 is any finite measure. Existence and uniqueness are proved.

In Sections 3–6 it is assumed that f is bounded and is uniformly negative.

In Section 3 we show that if $u_0(x)$ has compact support then $u(x, t)$ has compact support. An analogous result for elliptic variational inequalities was proved earlier by Brézis [2] (and then generalized by Redheffer [6]).

In Sections 4 and 5 we study the behavior of the support $S(t)$ of the function $x \to u(x, t)$. In Section 4 we consider the case where u_0 is any function in $L^\infty(R^n)$ with compact support $S = S(0)$; thus u_0 is not required to vanish on ∂S. It is proved that, for all small times t,

$$S(t) \subset S + B(c[t|\log t|]^{1/2})$$

where $+$ denotes the vector sum, $B(\rho) = \{x: |x| \leq \rho\}$, and c is a positive constant. This result is shown to be sharp.

In Section 5 we assume that $u_0(x)$ vanishes together with its first derivatives on ∂S. We then prove that

$$S(t) \subset S + B(C\sqrt{t})$$

for some positive constant C.

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In Section 6 we consider the case where $u_0(x)$ does not have compact support, but $u_0(x) \to 0$ as $|x| \to \infty$. We prove that $S(t)$ is a compact set for any $t > 0$. Thus in sharp contrast with the case of (1.1), the support "shrinks" instantaneously.

2. Existence and uniqueness

Consider the parabolic variational inequality

$$(u_t - \Delta u)(v - u) \geq f(v - u) \text{ a.e.} \quad (x \in R^n, 0 < t < T) \tag{2.1}$$

for any measurable function v, $v \geq 0$,

$$u \geq 0 \quad (x \in R^n, 0 \leq t \leq T), \tag{2.2}$$

$$u(x, 0) = u_0(x) \quad (x \in R^n). \tag{2.3}$$

Let μ be any positive number and introduce the norm

$$|g|_{L^{p,\mu}(R^n)} = \left\{\int_{R^n} e^{-p\mu|x|}|g(x)|^p\, dx\right\}^{1/p}$$

for any $p > 1$. If $|g|_{L^{p,\mu}(R^n)} < \infty$ then we say that $g \in L^{p,\mu}(R^n)$. We let

$$W^{k,p,\mu}(R^n) = \{u \in L^{p,\mu}(R^n);\ D^\alpha u \in L^{p,\mu}(R^n) \quad \text{for } |\alpha| \leq k\}.$$

If u, u_t, u_x, u_{xx} belong to $L^{2,\mu}(R^n)$ for any $t \in (0, T]$, then we can rewrite (2.1) in the form

$$\begin{aligned} &\int_{R^n} e^{-2\mu|x|}u_t(v - u)\, dx + \int_{R^n} e^{-2\mu|x|}\, D_x u \cdot D_x(v - u)\, dx \\ &\quad + \int_{R^n} D_x u \cdot (D_x e^{-2\mu|x|})(v - u)\, dx \geq \int_{R^n} e^{-2\mu|x|} f(v - u)\, dx \end{aligned} \tag{2.4}$$

for $0 < t \leq T$, and for any v such that v, v_x belong to $L^{2,\mu}(R^n)$, $v \geq 0$ a.e.

We shall assume:

$$u_0 \text{ is a measure, } u_0 \geq 0, \int_{R^n} u_0 < \infty, \tag{2.5}$$

$$f \in L^\infty(R^n x(0, T)), f_t \in L^\infty(R^n x(0, T)). \tag{2.6}$$

Denote by $K(x, t, y)$ the fundamental solution of the heat equation. For any function $f(y)$, the integral of f with respect to the measure u_0 is denoted by $\int_{R^n} f(y)u_0(y)\, dy$. The condition (2.3) will be taken, later on, in the sense that

$$\left|u(x, t) - \int_{R^n} K(x, t, y)u_0(y)\, dy\right| \leq Ct \tag{2.7}$$

where C is a constant independent of x. (2.7) implies in particular that $u(x, t) \to u_0(x)$ as $t \downarrow 0$ for the weak*-topology on the space of measures.

THEOREM 2.1. *Let* (2.5), (2.6) *hold. Then there exists a unique solution of* (2.1)–(2.3) *such that, for any* $\delta > 0$,

$$(2.8)\qquad \begin{aligned} &u \in L^\infty[(\delta, T);\ W^{2,p,\mu}(R^n)] \\ &u_t \in L^\infty[(\delta, T);\ L^{p,\mu}(R^n)] \quad \text{for any } 2 \le p < \infty, \mu > 0; \end{aligned}$$

the condition (2.3) *is satisfied in the sense of* (2.7).

Notice that, by the Sobolev inequalities, u is a continuous function for $0 < t \le T$.

Proof. Let $Q_R = \{x; |x| < R\}$, $\varepsilon > 0$, and consider the "truncated problem"

$$(2.9)\qquad u_t - \Delta u + \beta_\varepsilon(u) = f \quad \text{if } x \in Q_R,\ 0 < t < T,$$

$$(2.10)\qquad u(x, 0) = u_0(x) \quad \text{if } x \in Q_R,$$

$$(2.11)\qquad u(x, t) = 0 \quad \text{if } x \in \partial Q_R,\ t > 0.$$

Here the $\beta_\varepsilon(u)$ are C^∞ functions of u, defined for $\varepsilon > 0$, $u \in R^1$, and satisfying:

$$\beta_\varepsilon(u) = 0 \quad \text{if } u > 0,$$

$$\beta_\varepsilon(u) \to -\infty \quad \text{if } u < 0,\ \varepsilon \downarrow 0,$$

$$\beta'_\varepsilon(u) > 0 \quad \text{if } u < 0.$$

Denote the solution of (2.9)–(2.11) by $u_{R,\varepsilon}$. We claim that

$$(2.12)\qquad \min\{\inf f, 0\} \le \beta_\varepsilon(u_{R,\varepsilon}) \le 0.$$

To prove this as well as the existence of $u_{R,\varepsilon}$ it suffices to consider the case where $u_0(x)$ is a (nonnegative) continuous function; for then we can use approximation to handle the general case where u_0 is a measure.

The function $\beta_\varepsilon(u_{R,\varepsilon})$ takes its minimum in $\bar{Q}_R x[0, T]$ at some point $(\bar{x}, \bar{t})$. If $u_{R,\varepsilon}(\bar{x}, \bar{t}) < 0$ then $u_{R,\varepsilon}$ also takes its minimum at $(\bar{x}, \bar{t})$, since $\beta'_\varepsilon(u) > 0$ if $u < 0$. Hence, if $(\bar{x}, \bar{t})$ does not lie on the parabolic boundary, then (2.9) yields

$$\beta_\varepsilon(u_{R,\varepsilon}) \ge f \text{ at } (\bar{x}, \bar{t}), \text{ provided } u_{R,\varepsilon}(\bar{x}, \bar{t}) < 0.$$

If $(\bar{x}, \bar{t})$ lies on the parabolic boundary, then

$$\beta_\varepsilon(u_{R,\varepsilon}) = 0 \text{ at } (\bar{x}, \bar{t}).$$

We have thus proved that if $u_{R,\varepsilon}(\bar{x}, \bar{t}) < 0$ then

$$\beta_\varepsilon(u_{R,\varepsilon}(\bar{x}, \bar{t})) \ge \min(0, \inf f).$$

If $u_{R,\varepsilon}(\bar{x}, \bar{t}) \ge 0$ then this inequality is also (trivially) true. This completes the proof of (2.12).

From (2.9), (2.12) we see that $u = u_{R,\varepsilon}$ satisfies

$$u_t - \Delta u = f - \beta_\varepsilon(u) \in L^\infty(Q_R).$$

Denote by $K_R(x, t, y)$ the Green function of the heat operator in the cylinder $Q_R x(0, T)$. By the maximum principle,

$$0 \le K_R(x, t, y) \le K(x, t, y). \tag{2.13}$$

Using the construction of K_R as $K + h_R$ with a suitable h_R (see [4]), recalling the standard estimates on $D_x K$, and estimating $D_x h_R$ by the interior Schauder estimates (for instance), we conclude that

$$|D_x K_R(x, t, y)| \le \frac{C}{t^{(n+1)/2}} \exp\left[-\frac{|x-y|^2}{2t}\right] \quad \text{if } |x| < R - 1, \tag{2.14}$$

where C is a constant independent of R.

We can represent $u = u_{R,\varepsilon}$ as follows:

$$\begin{aligned} u(x, t) &= \int_{Q_R} K_R(x, t, y) u_0(y)\, dy \\ &\quad + \int_0^t \int_{Q_R} K_R(x, s, y)(f - \beta_\varepsilon(u))(y, s)\, dy\, ds \\ &\equiv u_1 + u_2. \end{aligned} \tag{2.15}$$

Using (2.14) one can show that, for each fixed t,

$$|u_1(x, t)|_{W^{1,\infty}(Q_{R-1})} \le \frac{C}{t^{(n+1)/2}}, \quad |u_2(x, t)|_{W^{1,\infty}(Q_{R-1})} \le C t^{1/2}$$

where C is a constant independent of R and t. Hence

$$|e^{-\mu|x|} u_1(x, t)|_{W^{1,p}(Q_{R-1})} \le \frac{C}{t^{(n+1)/2}}, \quad |e^{-\mu|x|} u_2(x, t)|_{W^{1,p}(Q_{R-1})} \le C t^{1/2} \tag{2.16}$$

for any $\mu > 0$, where C is a constant independent of R, t.

Next, from the L^p estimates of [3], [7], for any $\delta > 0$,

$$\int_\delta^T \int_{Q_R} e^{-p\mu|x|} \left(\left| \frac{\partial}{\partial t} u_2 \right|^p + |D_x u_2|^p + |D_x^2 u_2|^p \right) dx\, dt \le C(\delta) \tag{2.17}$$

where $C(\delta)$ a constant independent of R. Indeed, we write down (2.17) for $u_2 \xi_i$, where $\{\xi_i\}$ is a suitable partition of unity for $\bar{Q}_R$, and sum over i; then, using (2.16), we obtain (2.17) with a constant independent of R (cf. [1], [5]).

The inequality (2.17) can be verified directly for u_1. Since $u = u_1 + u_2$, we deduce that

$$\int_\delta^T \int_{Q_R} e^{-p\mu|x|} |u_t|^p\, dx\, dt \le C. \tag{2.18}$$

Let $\xi(t)$ be a C^∞ nonnegative function, $\xi(t) = 0$ if $t < \delta/2$, $\xi(t) = 1$ if $t > \delta$. Differentiating (2.9) with respect to t, we get

$$u_{tt} - \Delta u_t + \beta_\varepsilon'(u) u_t = f_t.$$

Multiplying both sides by $\exp(-p\mu|x|)\xi|u_t|^{p-2}u_t$ and integrating over $Q_R x(0, T)$, we find that we have

$$\frac{1}{p}\int_{Q_R} |e^{-\mu|x|}u_t(x, T)|^p\,dx + \int_0^T\int_{Q_R} \sum_i u_{tx_i}\frac{\partial}{\partial x_i}(\exp(-p\mu|x|))\cdot\xi|u_t|^{p-2}u_t\,dx\,dt$$
$$\leq \int_0^T\int_{Q_R} |f_t|e^{-\mu p|x|}\xi|u_t|^{p-1}\,dx\,dt + \int_0^T\int_{Q_R}\frac{1}{p}|u_t|^p|\xi'|e^{-\mu p|x|}\,dx\,dt$$

But

$$u_{tx_i}|u_t|^{p-2}u_t = \frac{1}{p}\frac{\partial}{\partial x_i}|u_t|^p,$$

so that

$$\int_0^T\int_{Q_R}\sum_i u_{tx_i}\frac{\partial}{\partial x_i}(e^{-p\mu|x|})\cdot\xi|u_t|^{p-2}u_t\,dx\,dt$$
$$= \int_0^T\int_{Q_R}\left(\frac{1}{p}\right)\xi|u_t|^p - \Delta(e^{-p\mu|x|})\,dx\,dt.$$

However

$$\Delta e^{-p\mu|x|} = \left(p^2\mu^2 - \frac{(n-1)p\mu}{|x|}\right)e^{-p\mu|x|}$$
$$\leq p^2\mu^2 e^{-p\mu|x|}.$$

Hence we conclude that

$$\frac{1}{p}\int_{Q_R}|e^{-\mu|x|}u_t(x, T)|^p\,dx \leq \int_0^T\int_{Q_R}|f_t|e^{-\mu p|x|}\xi|u_t|^{p-1}\,dx\,dt$$
$$+ \int_0^T\int_{Q_R}\frac{1}{p}(|u_t|^p)|\xi'|e^{-\mu p|x|}\,dx\,dt$$
$$+ \int_0^T\int_{Q_R} p\mu^2\xi|u_t|^p e^{-\mu p|x|}\,dx\,dt.$$

Recalling (2.18), we conclude that, for any $\delta > 0$,

$$\int_{Q_R}|e^{-\mu|x|}u_t(x, t)|^p\,dx \leq C \quad \text{if } \delta \leq t \leq T \tag{2.19}$$

where C is a constant independent of R. From (2.9), (2.12) we then also have

$$\int_{Q_R}|e^{-\mu|x|}\,\Delta u(x, t)|^p\,dx \leq C \quad \text{if } \delta \leq t \leq T, \tag{2.20}$$

with another constant C, independent of R.

We extend the definition of $u = u_{R,\varepsilon}$ into $R^n x[0, T]$ in such a way that (2.19), (2.20) remain valid with Q_R replaced by R^n, and the $u_{R,\varepsilon}$ remain uniformly bounded.

Using the standard L^p estimates for Δ, we can then choose a sequence $u = u_{R,\varepsilon}$ ($R \to \infty$, $\varepsilon \to 0$) which is convergent uniformly in compact subsets to a function u, such that

$$\frac{\partial}{\partial t} u_{R,\varepsilon} \to \frac{\partial u}{\partial t}, \qquad D_x^\alpha u_{R,\varepsilon} \to D_x^\alpha u \ (1 \le |\alpha| \le 2)$$

weakly in the weak star topology of $L^\infty((\delta_0, T); L^{p,\mu}(R^n))$ for any $\delta_0 > 0$, $2 \le p < \infty$. Thus, u satisfies (2.8).

The fact that u is a solution of the variational inequality (2.1), (2.2) follows by a standard argument. Next, from (2.15) we obtain

$$\left| u_{R,\varepsilon}(x, t) - \int_{Q_R} K_R(x, t, y) u_0(y)\, dy \right| \le |u_2(x, t)| \tag{2.21}$$

where, by (2.13) and the boundedness of $f - \beta_\varepsilon$, $|u_2(x, t)| \le Ct$, C a constant independent of R, ε. Going to the limit in (2.21), we obtain the inequality (2.7). This completes the proof of existence. The proof of uniqueness follows by a standard argument: One writes (2.4) for u and $v = \hat{u}$ and then for $\hat{u}$ and $v = u$, where u, $\hat{u}$ are two solutions. Then, by adding the inequalities, one gets, after some simple manipulations,

$$\frac{d}{dt} \int_{R^n} |e^{-\mu|x|}(\hat{u} - u)(x, t)|^2\, dx \le C \int_{R^n} |e^{-\mu|x|}(\hat{u} - u)|^2\, dx;$$

hence $\hat{u} - u \equiv 0$ by (2.7).

3. Compact support for the solution

We shall now assume that

$$f \in L^\infty(R^n x(0, T)), \quad f_t \in L^\infty(R^n x(0, T)) \text{ for any } T > 0. \tag{3.1}$$

By Theorem 2.1, the variational inequality (3.1)–(3.3) has a unique solution $u(x, t)$ in $R^n x(0, \infty)$ (satisfying (2.8) for any $0 < \delta < T < \infty$). The object of the remaining part of this paper is to study the support of u. We shall henceforth need the condition:

$$f \le -\nu \text{ in } R^n x(0, \infty) \quad (\nu \text{ positive constant}). \tag{3.2}$$

THEOREM 3.1. *Let* (2.5), (3.1), (3.2) *hold. Then there is a positive number* T_0 *such that* $u(x, t) \equiv 0$ *if* $t \ge T_0$.

Proof. From the proof of Theorem 2.1 we infer that $u_{R,\varepsilon}(x, 1) \le M$ where M is a positive constant independent of R, ε. Set $T_0 = 1 + M/\nu$ and consider the function

$$w(x, t) = M - \nu(t - 1) \quad (x \in R^n, 1 \le t \le T_0).$$

Observe that $w > 0$ if $1 \le t \le T_0$, $w(x, T_0) = 0$, and

$$w_t - \Delta w + \beta_\varepsilon(w) = -\nu \quad \text{if } x \in R^n, 1 \le t \le T_0.$$

We can apply the maximum principle to $w - u_{R,\varepsilon}$ in the strip $1 \leq t \leq T_0$, and thus conclude that $w - u_{R,\varepsilon} \geq 0$ in this strip. In particular,

$$u_{R,\varepsilon}(x, T_0) \leq 0.$$

Taking $R \to \infty$, $\varepsilon \to 0$, we conclude that $u(x, T_0) \equiv 0$. By uniqueness, $u(x, t) \equiv 0$ if $t \geq T_0$.

THEOREM 3.2. *Let the conditions* (2.5), (3.1), (3.2) *hold and suppose that* u_0 *has compact support. Then there is a positive constant* R_0 *such that* $u(x, t) = 0$ *if* $|x| > R_0$.

Proof. Let ρ be a positive number such that supp $u_0 \subset \{x; |x| < \rho\}$. From the proof of Theorem 2.1 we infer that

$$|u_{R,\varepsilon}(x, T)| \leq N \quad \text{if } x \in R^n,\ \rho \leq |x| \leq R,\ 0 < t < T_0.$$

Consider the function

$$w(x) = \begin{cases} \mu(R_0 - r)^2 & \text{if } 0 < r < R_0, \\ 0 & \text{if } r > R_0 \end{cases}$$

where $r = |x|$ and μ, R_0 are positive constants. Choosing μ, R_0 such that $2\mu \leq v$, $\mu(R_0 - \rho)^2 \geq N$, we find that

$$\begin{aligned} &w_t - \Delta w + \beta_\varepsilon(w) = -\Delta w \geq -v && \text{if } |x| > \rho, \\ &w \geq N && \text{if } |x| = \rho. \end{aligned}$$

We can now apply the maximum principle to $w - u_{R,\varepsilon}$ and conclude that $w - u_{R,\varepsilon} \geq 0$ if $\rho < |x| < R$, $0 < t < T_0$. In particular,

$$u_{R,\varepsilon}(x, t) = 0 \quad \text{if } R_0 \leq |x| \leq R,\ 0 \leq t \leq T_0.$$

Taking $R \to \infty$, $\varepsilon \to 0$ the assertion of the theorem follows.

We conclude this section with a standard comparison lemma that will be needed in the following sections.

LEMMA 3.3. *Denote by* u *and* $\hat{u}$ *two functions satisfying* (2.1) *and* (2.2) *with*

$$u, \hat{u} \in L^\infty(\delta, T; W^{2,2,\mu}(R^n)), \qquad u_t, \hat{u}_t \in L^\infty(\delta, T; L^{2,\mu}(R^n))$$

for some μ *and any* $\delta > 0$. *Assume* $u(\cdot, t) \to u_0(\cdot)$ *and* $\hat{u}(\cdot, t) \to \hat{u}_0(\cdot)$ *in* $L^{2,\mu}(R^n)$ *as* $t \to 0$. *If* $u_0 \leq \hat{u}_0$ *a.e. on* R^n *and* $f \leq \hat{f}$ *a.e. on* $R^n x(0, T)$, *then* $u \leq \hat{u}$ *a.e. in* $R^n x(0, T)$.

Proof. Let $w = (u - \hat{u})$; substituting $v = \operatorname{Min}\{u, \hat{u}\}$ and then $\hat{v} = \operatorname{Max}\{u, \hat{u}\}$ in (2.1) we obtain after addition:

$$w_t w^+ - \Delta w \cdot w^+ \leq (f - \hat{f})w^+ \leq 0 \text{ a.e.}$$

Multiplying through by $e^{-2\mu|x|}$ and integrating by parts, we get, after some simple calculations,

$$\frac{1}{2}\frac{d}{dt}\int_{R_n} e^{-2\mu|x|}|w^+(x, t)|^2\, dx \le \int_{R^n} \tfrac{1}{2}|w^+(x, t)|^2\, \Delta(e^{-2\mu|x|})\, dx$$
$$\le 2\mu^2 \int_{R^n} |w^+(x, t)|^2 e^{-2\mu|x|}\, dx.$$

On the other hand $w^+(\cdot, t) \to 0$ in $L^{2,\mu}(R^n)$ as $t \to 0$. We conclude that $w^+ = 0$ and thus $u \le \hat{u}$ a.e. as $R^n x(0, T)$.

4. Estimates on the support

In what follows we use the notation $B(\rho) = \{x;\, |x| < \rho\}$. If A and B are sets in R^n, we denote by $A + B$ their vector sum.

We shall denote by $S(t)$ the support of the function $x \to u(x, t)$, and write $S = S(0)$, i.e., S is the support of the measure u_0 (S is a closed set).

THEOREM 4.1. *Let f satisfy* (3.1), (3.2) *and let $u_0(x) \ge 0$ be a function in $L^\infty(R^n)$. Assume that the support S of u_0 consists of a finite union of disjoint bounded closed domains, with C^1 boundary. Then, there is a positive constant c such that*

$$S(t) \subset S + B(c\sqrt{t}|\log t|) \tag{4.1}$$

if t is sufficiently small.

The proof of Theorem 4.1 relies on the following lemmas.

LEMMA 4.2. *There exists a function $w(x, t)$, $x \in R$, $t \in (0, 1)$ such that*

$$w \in L^\infty(Rx(\delta, 1), \tag{4.2}$$

$$w_t,\, w_x,\, w_{xx} \in L^\infty(Rx(\delta, 1) \quad \text{for each } 0 < \delta < 1, \tag{4.3}$$

$$w \ge 0 \quad \text{as } Rx(0, 1), \tag{4.4}$$

$$\text{as } t \to 0,\ w(x, t) \to 0 \text{ for } x > 0 \text{ and } w(x, t) \to 1 \text{ for } x < 0, \tag{4.5}$$

$$|w_t - w_{xx}| \le kt^{1/2}|\log t|^{3/2} \quad \text{for } x \in R,\ t \in (0, 1) \text{ and } k \text{ some constant}, \tag{4.6}$$

$$w(x, t) = 0 \quad \text{for } x > \sqrt{6t|\log t|} \text{ and } t \in (0, 1) \tag{4.7}$$

Proof of Lemma 4.2. Let $s(t) = \sqrt{6t|\log t|}$ and define for $x \in R$, $t \in (0, 1)$:

$$v(x, t) = \begin{cases} Ax^2 + Bt + Ct\log t + \dfrac{D}{\sqrt{t}}e^{-x^2/4t} & \text{when } |x| < s(t), \\ 0 & \text{when } |x| > s(t). \end{cases} \tag{4.8}$$

We determine the constants A, B, C, and D in such a way that

$$v(s(t), t) = 0,\ v_x(s(t), t) = 0 \quad \text{for } t \in (0, 1). \tag{4.9}$$

Therefore it is required that

$$-6At\log t + Bt + Ct\log t + Dt = 0 \quad \text{and} \quad 2s(t)(A - D/4) = 0,$$

i.e.

$$A = D/4,\ B = -D,\ C = 3D/2. \tag{4.10}$$

It is easy to verify that when $D > 0$, then $v \geq 0$. Define now for $x \in R$ and $t \in (0, 1)$,

$$w(x, t) = \int_x^{s(t)} v(\xi, t)\, d\xi, \tag{4.11}$$

so that $w(x, t) = 0$ when $x > s(t)$ and hence $w(x, 0) = 0$ for $x > 0$. Next let $x < 0$; if t is small enough to insure $s(t) < -x$, then

$$w(x, t) = \int_{-s(t)}^{+s(t)} v(\xi, t)\, d\xi.$$

Therefore

$$\begin{aligned} w(x, t) &= 2\int_0^{s(t)} \left(A\xi^2 + Bt + Dt\log t + \frac{D}{\sqrt{t}}\, e^{-\xi^2/4t}\right) d\xi \\ &= \tfrac{2}{3}As^3(t) + 2(Bt + Dt\log t)s(t) + \frac{2D}{\sqrt{t}}\int_0^{s(t)} e^{-\xi^2/4t}\, d\xi. \end{aligned}$$

The last term equals

$$2D\int_0^{\sqrt{6|\log t|}} e^{-\eta^2/4}\, d\eta,$$

and thus as $t \to 0$ we see that, for $x < 0$,

$$w(x, t) \to 2D\int_0^{+\infty} e^{-\eta^2/4}\, d\eta.$$

We fix now D in such a way that

$$2D\int_0^{+\infty} e^{-\eta^2/4}\, d\eta = 1$$

and next A, B, and C are determined by (4.10).

In order to compute $Lw = w_t - w_{xx}$ we distinguish three regions.

Region I. $x > s(t)$, where $w = 0$ and so $Lw = 0$.

Region II. $x < -s(t)$ where

$$w(x, t) = \int_{-s(t)}^{+s(t)} v(\xi, t)\, d\xi,$$

$$w_t = v(s(t), t)s'(t) + v(-s(t), t)s'(t) + \int_{-s(t)}^{+s(t)} v_t(\xi, t)\, d\xi,$$

$$w_{xx} = 0.$$

By (4.9) we get

$$w_t(x, t) = 2\int_0^{s(t)} (B + C\log t + C)\, d\xi + 2D\int_0^{s(t)} \zeta_t(\xi, t)\, d\xi$$

where

$$\zeta(x, t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}.$$

Since $\zeta_t = \zeta_{xx}$ we have

$$\int_0^{s(t)} \zeta_t(\xi, t)\, d\xi = \int_0^{s(t)} \zeta_{xx}(\xi, t)\, d\xi = \zeta_x(s(t), t) - \zeta_x(0, t) = -\frac{s(t)}{2}.$$

Finally

$$Lw = w_t(x, t) = 2s(t)(B + C\log t + C) - Ds(t) = 3Ds(t)\log t.$$

Region III. $-s(t) < x < +s(t)$ where $w(x, t) = \int_x^{s(t)} v(\xi, t)\, d\xi$. Thus

$$w_t(x, t) = \int_x^{s(t)} v_t(\xi, t)\, d\xi,$$

$$w_x(x, t) = -v(x, t), \qquad w_{xx}(x, t) = -v_x(x, t) = \int_x^{s(t)} v_{xx}(\xi, t)\, d\xi.$$

Consequently

$$\begin{aligned} Lw &= \int_x^t (v_t - v_{xx})(\xi, t)\, d\xi \\ &= \int_x^{s(t)} (B + C + C\log t + -2A)\, d\xi \\ &= (s(t) - x)C\log t. \end{aligned}$$

In the three regions we conclude that $|Lw| \leq 3Ds(t)|\log t|$.

LEMMA 4.3. *Let C be the cube $(-\theta, +\theta)^n$ $(\theta > 0)$. There exists a function $z(x, t)$, $x \in R^n$, $t \in (0, 1)$ such that*

$z \in L^\infty(R^n x(0, 1))$,

$z_t, z_{x_i}, z_{x_i x_j} \in L^\infty(R^n x(\delta, 1))$ *for each* $0 < \delta < 1$,

$z \geq 0$ *on* $R^n x(0, 1)$,

as $t \to 0$, $z(x, t) \to 0$ *for* $x \in C$, *and* $z(x, t) \to$ limit ≥ 1 *for* $x \notin \bar{C}$,

$|z_t - \Delta z| \leq k' t^{1/2} |\log t|^{3/2}$ *for* $x \in R^n$, $t \in (0, 1)$ *and k is some constant,*

$z(x, t) = 0$ *for* $\mathrm{Max}_{1 \leq i \leq n} |x_i| < \theta - \sqrt{6t|\log t|}$ *(t small).*

Proof. It is clear from Lemma 4.2 that the function

$$z(x, t) = \sum_{i=1}^n [w(x_i + \theta, t) + w(\theta - x_i, t)]$$

satisfies all the required properties.

Proof of Theorem 4.1. Let $\alpha = \text{ess sup}_S u_0$. We denote by $v(x_0)$ the unit outward normal at every point $x_0 \in \partial S$ and by $C(x_0, 2\theta)$ an open cube centered on $v(x_0)$ whose side has length 2θ and such that x_0 is one of the vertices.

Since ∂S is C^1 there exists a fixed $\theta > 0$, independent of x_0, sufficiently small such that $C(x_0, 2\theta) \cap S = \emptyset$ for every $x_0 \in \partial S$. By shifting the origin we can always assume that $C(x_0, 2\theta)$ is centered at the origin and has the form $(-\theta, +\theta)^n$. It follows from the comparison Lemma 3.3 that $u \leq \alpha z$ on $R^n x(0, t_0)$ where t_0 is small enough to insure that

$$kt^{1/2}|\log t|^{3/2} \leq v \quad \text{for } 0 < t < t_0.$$

Therefore we conclude that $u(x, t) = 0$ for t small enough and for x of the form $x = x_0 + \lambda v(x_0)$, $\sqrt{6nt}|\log t| < \lambda < \theta$.

The conclusion of the theorem follows.

Remark. The proof of Theorem 4.1 applies also in cases where ∂S is not in C^1; for instance in case S is a convex set.

Let S be a closed set in R^n. Suppose for any $x \in \partial S$ there exists a cone V_x with vertex x and with opening σ and height h independent of x such that $V_x \subset S$; then we say that S satisfies the *uniform cone property.*

In the next theorem we derive a lower bound on $S(t)$.

THEOREM 4.2. *Let f satisfy* (3.1), *and let $f \geq -v_0 > 0$, v_0 constant. Let u_0 be a bounded measurable function whose support S satisfies the uniform cone property. If there is a positive constant β such that $u_0(x) \geq \beta$ for $x \in S$, then there is a positive constant c such that*

$$S(t) \supset S + B(c\sqrt{t}|\log t|) \quad \textit{for all } t \textit{ sufficiently small.} \tag{4.12}$$

Proof. Consider the function

$$w(x, t) = \frac{\beta}{(2\pi t)^{n/2}} \int_S \exp\left[-\frac{|x - \xi|^2}{4t}\right] d\xi - v_0 t. \tag{4.13}$$

It satisfies $w_t - \Delta w = -v_0$, $w(x, 0) \leq u_0(x)$. Since $u_t - \Delta u \geq f \geq -v_0$, the maximum principle can be applied to $u - w$. It gives

$$u(x, t) \geq w(x, t). \tag{4.14}$$

Denote by $d(y)$ the distance of a point y to S. If we can prove that

$$w(y, t) > 0 \quad \text{whenever } y \notin S,\ d(y) \leq c\sqrt{t}|\log t|, \tag{4.15}$$

then, by (4.14), also $u(y, t) > 0$ and, consequently, the assertion (4.12) would follow.

In order to prove (4.15), let x_0 be a point on ∂S such that $d(y) = |y - x_0|$. Integrating in (4.13) only over the cone with vertex x_0, opening σ, and height η $(0 < \eta < h)$ which lies in S, we find that

$$w(y, t) > \beta_0 \frac{\eta^n}{t^{n/2}} \exp\left[-\frac{\mu\, d^2(y)}{t} - \frac{\mu\eta^2}{t}\right] - v_0 t$$

for any $0 < \eta < h$, where β_0, μ are positive constants. If t is sufficiently small then we can take $\eta = \sqrt{t}$. Hence, $w(y, t) > 0$ if

$$\beta_1 \exp\left[-\frac{\mu\, d^2(y)}{t}\right] \geq v_0 t$$

where β_1 is a positive constant. Taking the logarithm we see that $w(y, t) > 0$ if

$$\frac{\mu\, d^2(y)}{t} \leq |\log t| + \text{const.}$$

This gives (4.12) with $c < 1/\sqrt{\mu}$.

5. Estimates on the support (continued)

THEOREM 5.1. *Let* (3.1), (3.2) *hold and let* $S = \operatorname{supp} u_0$ *be a finite disjoint union of bounded closed domains with* C^2 *boundary. Assume that*

$$u_0 \in C^2(S), \quad u_0 = 0, \quad D_x u_0 = 0 \text{ on } \partial S. \tag{5.1}$$

Then there exists a positive constant α, *depending only on the data, such that*

$$S(t) \subset S + B(\alpha\sqrt{t}) \quad \text{for all } t \geq 0. \tag{5.2}$$

Proof. Let y be any point outside S. Let $\delta = \text{dist.}\,(y.\ S)$. For simplicity we take $y = 0$.

Using (5.1) we find that, for any $x \in S$,

$$u_0(x) = |u_0(x) - u_0(x')| \leq C_0|x - x'|^2 \leq C_0(|x| - \delta)^2 \tag{5.3}$$

where x' is the first point where the ray from x to y intersects ∂S.

Setting $r = |x|$, $\lambda = (r - \delta)/\sqrt{t}$, we shall construct a comparison function

$$w(x, t) = \begin{cases} tF(\lambda) & \text{if } \delta - \alpha\sqrt{t} \leq r < \infty, \\ 0 & \text{if } r < \delta - \alpha\sqrt{t} \end{cases}$$

for $0 < t < \sigma$, σ sufficiently small, where F is a nonnegative function defined on $[-\alpha, +\infty)$. We require that

$$F(-\alpha) = 0, \quad F'(-\alpha) = 0, \tag{5.4}$$

so that w is continuously differentiable across $r = \delta - \alpha\sqrt{t}$. We also require that $w(x, 0) \geq u_0(x)$. In view of (5.3), the last inequality holds if

$$\lim_{\lambda \to +\infty} \frac{F(\lambda)}{\lambda^2} \geq C_0. \tag{5.5}$$

Finally, we require that $w_t - \Delta w \geq -v$; in terms of F this means that

$$F - \tfrac{1}{2}\lambda F' - \frac{n-1}{r}\sqrt{t}\, F' - F'' \geq -v \tag{5.6}$$

where the argument in F, F', F'' is λ.

We seek $F(\lambda)$ of the form

$$F(\lambda) = \begin{cases} \mu(\lambda + \alpha)^2 & \text{if } -\alpha < \lambda < 0, \\ A\lambda^2 + B\lambda + C & \text{if } \lambda > 0. \end{cases}$$

Then w is continuously differentiable across $\lambda = 0$ if

$$\mu\alpha^2 = C,\ 2\mu\alpha = B. \tag{5.7}$$

If we take

$$A \geq C_0 \tag{5.8}$$

then (5.5) holds. The conditions in (5.4) are clearly satisfied.

We now turn to verifying the inequality (5.6). In the region where $-\alpha < \lambda < 0$, (5.6) reduces to

$$\mu(\lambda + \alpha)^2 - \mu\lambda(\lambda + \alpha) - \mu\frac{2(n-1)}{r}\sqrt{t}\,(\lambda + \alpha) - 2\mu \geq -v.$$

If t is sufficiently small then $\delta/2 < r < \delta$; the last inequality is then a consequence of

$$\mu\left[\lambda\alpha + \alpha^2 - \frac{4(n-1)}{\delta}\sqrt{t}\,(\lambda + \alpha) - 2\right] \geq -v,$$

or, a consequence of

$$\mu\left(2 + \frac{4(n-1)}{\delta}\sqrt{\sigma}\,\alpha - \alpha^2\right) \leq v \quad (0 < t \leq \sigma). \tag{5.9}$$

In the region where $\lambda > 0$, (5.6) holds if

$$A\lambda^2 + B\lambda + C - \tfrac{1}{2}\lambda(2A\lambda + B) - \frac{n-1}{r}\sqrt{t}\,(2A\lambda + B) - 2A \geq -v.$$

Since $r \geq \delta$, this inequality holds, for all $0 < t < \sigma$, if

$$\frac{B}{2} - 2\frac{n-1}{\delta}\sqrt{\sigma}\,A \geq 0, \tag{5.10}$$

$$C - \frac{n-1}{\delta}\sqrt{\sigma}\,B - 2A \geq -v. \tag{5.11}$$

From (5.7) we find that

$$\alpha = 2C/B, \quad \mu = B^2/4C. \tag{5.12}$$

Taking $C \geq 2A - v + 1$ we see that (5.11) holds if σ is sufficiently small. If we further choose B, C to be positive, then α and μ are positive. If we also take C/B to be sufficiently large, then α becomes so large that the left-hand side of (5.9) is negative. Thus (5.9) is satisfied. Notice that also (5.10) is satisfied if σ is sufficiently small.

Thus, with the above choice of B, C, and A, and with the definitions of α, μ

by (5.12), we have established that the function w is a comparison function, i.e., it satisfies the conditions of Lemma 3.3. Consequently, $u(x, t) \leq w(x, t)$ in $R^n x(0, \sigma)$. The conclusion of Theorem 5.1 follows.

Remark 1. Theorem 5.1 extends to the case where S consists of a finite disjoint union of closed convex domains with C^1 boundary.

Remark 2. If $f_t \leq 0$ and

$$f + \Delta u_0 < 0 \quad \text{in } S \tag{5.13}$$

then one can show that $u_t \leq 0$. Consequently, $S(t) \subset S(t')$ if $t > t' > 0$.

6. Instantaneous shrinking of the support

In this section we consider the case where u_0 need not have compact support, but $u_0(x) \to 0$ if $|x| \to \infty$. We shall show that the support $S(t)$ of $x \to u(x, t)$ is compact, for any $t > 0$.

THEOREM 6.1. *Let f satisfy* (3.1), (3.2) *and assume that*

$$u_0 \in L^\infty(R^n) \cap L^1(R^n), \quad u_0(x) \to 0 \textit{ if } |x| \to \infty. \tag{6.1}$$

Then $S(t)$ is a compact set, for any $t > 0$.

Proof. The assertion of the theorem follows from the assertion that there exists a function $\phi(r)$ and a positive number R, such that

$$\phi(r) > 0, \quad \phi(r) \downarrow 0 \text{ if } r \uparrow \infty, \tag{6.2}$$

$$u(x, t) = 0 \quad \text{if } t > \phi(|x|), |x| > R. \tag{6.3}$$

In view of Theorem 3.2 it suffices to prove (6.3) just for $t < t_0$, where t_0 is a sufficiently small positive number. We first establish that

$$u(x, t) \to 0 \text{ if } |x| \to \infty, \quad \text{uniformly in } t. \tag{6.4}$$

Let z be the bounded solution of

$$z_t - \Delta z = 0 \; (x \in R^n, t > 0), \qquad z(x, 0) = u_0(x) \; (x \in R^n).$$

Representing z in terms of the fundamental solution and using (6.1), we find that for any $T > 0$,

$$z(x, t) \to 0 \text{ if } |x| \to \infty, \quad \text{uniformly in } t, 0 \leq t \leq T. \tag{6.5}$$

Since $z \geq 0$, we can verify that the function $\hat{u} = z$ satisfies (2.1) with $\hat{f} = 0$. Noting that $\hat{f} \geq f$, we can apply Lemma 3.3 to conclude that $z \geq u$. But then (6.4) is a consequence of (6.5).

Let η be any small positive number. By (6.4), there is an $R > 0$ sufficiently large such that

$$u(x, t) < \eta \quad \text{if } |x| > R, 0 \leq t \leq T. \tag{6.6}$$

We shall estimate $u(x, t)$ more precisely in a region

$$|x| > R, \quad 0 < t < t_0 \tag{6.7}$$

where t_0 is a sufficiently small positive number.

Let $r = |x|$ and

$$w(x, t) = \begin{cases} (\phi(r) - t)^2 & \text{if } |x| > R, 0 < t < \phi(r), \\ 0 & \text{if } |x| > R, t > \phi(r). \end{cases}$$

Then w satisfies $w \geq u$ if $|x| = R$, $0 < t < t_0$, or if $|x| > R$, $t = 0$ provided

$$(\phi(R) - t_0)^2 \geq \eta, \tag{6.8}$$

$$\phi^2(r) \geq u_0(x) \quad (r = |x| > R). \tag{6.9}$$

Also w satisfies the variational inequality (2.1) on $|x| > R$, $0 < t < t_0$ with $f \geq -\gamma$ provided

$$\begin{aligned} &-2(\phi(r) - t) - 2(\phi(r) - t)\phi''(r) - 2(\phi'(r))^2 \\ &-2\frac{n-1}{r}(\phi(r) - t)\phi'(r) \geq -\gamma \qquad (6.10) \\ &\qquad\qquad (r > R, 0 < t < \phi(r)). \end{aligned}$$

Since the last inequality is linear in t, it suffices to verify it at $t = \phi(r)$ and $t = 0$, i.e.,

$$(\phi'(r))^2 \leq \tfrac{1}{2}\gamma \quad (r > R), \tag{6.11}$$

$$\phi(r) + \phi(r)\phi''(r) + (\phi'(r))^2 + \frac{n-1}{r}\phi(r)\phi'(r) \leq \tfrac{1}{2}\gamma. \tag{6.12}$$

We shall now construct a function ϕ satisfying (6.8), (6.9), (6.11), and (6.12). Since $u_0(x) \to 0$ as $|x| \to \infty$, we can find an increasing sequence (a_n) such that $a_1 = R$ and

$$\sqrt{u_0(x)} \leq \sqrt{\eta} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \chi_{[a_n, a_{n+1}]}(|x|) \quad \text{for } |x| > R$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$. We can always assume that $a_{n+1} - a_n \geq 1$ for $n \geq 1$.

Let $\zeta(t)$, $t \in R^1$, be a smooth function with compact support such that $\zeta \geq 0$, $\zeta(t) = 1$ for $0 \leq t \leq 1$, $|\zeta(t)| \leq 1$, $|\zeta'(t)| \leq 1$ and $|\zeta''(t)| \leq 1$ for $t \in R^1$.

Define

$$\phi(r) = \rho \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \zeta\left(\frac{r - a_n}{a_{n+1} - a_n}\right), \ \rho > 0.$$

Clearly ϕ is a smooth function and $\phi(r) \to 0$ as $r \to +\infty$; we are going to see that for η small enough, it is possible to choose ρ in such a way that ϕ satisfies (6.8), (6.9), (6.11), and (6.12).

Since $\phi(R) \geq \rho$, the conditions

$$\rho \geq 2\sqrt{\eta} \quad \text{and} \quad t_0 \leq \sqrt{\eta} \tag{6.13}$$

imply (6.8).

We have, for $|x| > R$,

$$\sqrt{u_0(x)} \leq \frac{\sqrt{\eta}}{\rho} \phi(|x|)$$

and therefore (6.13) also implies (6.9).

On the other hand $|\phi'(r)| \leq 2\rho$ and $|\phi''(r)| \leq 2\rho$ for $r \in R$. Thus (6.11) and (6.12) are consequences of the following

$$4\rho^2 \leq \tfrac{1}{2}\gamma \tag{6.14}$$

$$2\rho + 4\rho^2 + 4\rho^2 + 4\rho^2 \leq \tfrac{1}{2}\gamma. \tag{6.15}$$

Conclusion: we first choose a $\rho > 0$ satisfying (6.14), (6.15); next η and t_0 are obtained from (6.13). Finally, we choose R, $\{a_n\}$ and construct ϕ.

From a variant of Lemma 3.3 we deduce that

$$u(x, t) \leq w(x, t) \quad \text{if } |x| > R, 0 < t < t_0$$

and the assertion (6.3) follows.

Remark. If u_0 has compact support, then in the above proof we can take $\phi(r)$ to vanish if r is sufficiently large. Thus $u(x, t)$ will vanish if $|x| \geq R_0$ for some R_0 sufficiently large. This gives another proof of Theorem 3.2.

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