CELL COVERINGS AND RESIDUAL SETS OF CLOSED MANIFOLDS

BY

DANIEL A. MORAN

1. Introduction and preliminaries

Very simple arguments can be combined with recent results of Summerhill and Pedersen to improve the known bounds for the minimal number of open n-cells required to cover an n-dimensional topological manifold. There are two basic methods of proof, both of which depend on the notion of *residual set*, a concept invented by Doyle and Hocking [1] to mean the complement of a dense open n-cell in a closed topological manifold.

The terminology used in this report follows the usage in [6], except where an explicit reference is cited. In what follows, M denotes a closed topological manifold of dimension n. Using the language of residual sets, the recent results which were referred to above are the following:

SUMMERHILL'S THEOREM. *Let $0 \leq k \leq n - 3$. If M is k-connected then there is a strong Z_{k-1}-set which is residual in M.*

Actually, Summerhill proves the converse as well. For the proof, see [8]; for facts about Z_m-sets, also see [7].

PEDERSEN'S THEOREM. *Let $3 \leq k$ and $6 \leq n$. If M is k-connected then there is a polyhedron of dimension $n - k$ locally tamely embedded in M which is residual in M.*

The proof is immediate from corollary 3 of [5] and the topological version of the Tubular Neighborhood Theorem [2].

2. The de Morgan approach

The salient point of this method is the observation that M can be covered by r *dense* open n-cells if and only if M has r residual sets whose common intersection is empty. (This is just de Morgan's law. The reader might enjoy the easy exercise of proving that the product of two spheres can always be covered by three open cells, using this idea.) In order to apply this idea to the problem at hand, we need another result of Summerhill [7], a less technical rendition of which is

GENERAL POSITION THEOREM. *Let X and Y be "tame" closed subsets of E^n having dimensions p and q, respectively. Then arbitrarily close to the identity there exists a self-homeomorphism h of E^n such that $X \cap h(Y)$ has dimension at most $p + q - n$.*

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The "tameness" referred to in the theorem is satisfied by a Z_{k-1}-set in M or by a locally tamely embedded polyhedron, so that this theorem applies to the situations considered in the theorems of Summerhill and of Pedersen.

Restricting our attention to the former case, we note [7] that the dimension of a Z_{k-1}-set in M does not exceed $n - k - 1$; therefore such a set X can be pushed off itself *via* a small homeomorphism h so that $X \cap h(X)$ is a "tame" set of dimension at most $n - 2k - 2$. An iteration of this procedure yields the fact that r such residual sets in general position will have a common intersection of dimension at most $n - rk - r$. Therefore, M can be covered by r dense open n-cells if $n - rk - r \leq -1$, that is, if $(n + 1)/(k + 1) \leq r$. (This is exactly the result proved by Luft [3].) A similar line of reasoning can be applied to the situation of Pedersen's theorem; the end product of both arguments combined is:

THEOREM 1. *Let M be k-connected. If $0 \leq k \leq n - 3$, then M can be covered by $\langle (n + 1)/(k + 1) \rangle$ open n-cells. If $6 \leq n$ and $3 \leq k$, then M can be covered by $\langle (n + 1)/k \rangle$ open n-cells.*

Here $\langle x \rangle$ denotes the least integer not smaller than x.

3. Engulfing the residual set

Perhaps the most naive way to use the idea of residual set to approach the problem of cell coverings is to try to determine the number of cells necessary to cover the residual set; the manifold itself can then certainly be covered with one additional cell (the complement of the residual set). This is essentially the technique used in [4], where it was shown that a sphere bundle over a sphere can be covered by three cells whenever it admits a cross-section. Pedersen's theorem permits the use of an antique engulfing trick in making the initial determination for this method:

THEOREM 2. *Suppose M has as residual set a polyhedron K of dimension $n - k$, locally tamely embedded in M. Then M can be covered by $n - k + 2$ open n-cells.*

Proof. Subdivide the finite simplicial complex which comprises K so finely that each simplex is tamely embedded in M. It is an easy matter to enclose the 0-simplexes of K in a cell C_0. Inductively, the points in the p-simplexes of K which are in the complement of $C_0 \cup \cdots \cup C_{p-1}$ lie in a finite union of mutually disjoint cellular subsets of M, so they can all be enclosed in another cell C_p. Thus K is seen to be covered by the $n - k + 1$ cells C_p.

As mentioned previously, situations which admit the application of this theorem are provided by Pedersen's theorem. However, a minute or two spent with the inequalities involved will convince the reader that the estimates given by Theorem 1 are always better than those given by Theorem 2. Nevertheless, the situation of the latter theorem does provide a setting for the following refinement:

THEOREM 3. *Suppose M has as residual set a polyhedron $|K|$, the underlying point set of a cell complex K with the interior of each cell locally flatly embedded in M. Then M can be covered by $d + 1$ open n-cells, where d is the number of distinct dimensions of the cells which comprise K.*

Proof. Because the interior of each p-cell D of K is locally flatly embedded in M, every closed p-cell contained in the interior of D is flatly embedded in M (see [6, p. 105]) and therefore is cellular in M. The proof now proceeds exactly as does that of Theorem 2, enclosing the sets of cellular cells one dimension at a time instead of doing the same for the sets of subsets of simplexes.

The improvement of the results given by Theorem 3 over those given by Theorem 2 is rather dramatically illustrated by the example of quaternionic projective n-space QP^n of (real) dimension $4n$. If it is recalled that QP^{n-1} is a spine of QP^n, and hence is residual therein, it will be seen that QP^n can be covered by $n + 1$ open $4n$-cells according to Theorem 3; Theorem 2 merely yields the possibility of covering this space with $4n - 3$ such cells.

The local tameness of the polyhedron given in Pedersen's Theorem insures that the structure of any cell complex given to it will satisfy the hypothesis of Theorem 3, so that the latter theorem is always applicable when such a cell structure is known.

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MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN