STACKING TRANSFORMATIONS AND DIOPHANTINE APPROXIMATION

BY

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Introduction

The stacking method (see [1] and [6, Section 6]) has been used with great success in ergodic theory to construct a wide variety of examples of ergodic transformations (see, for example, [1], [4], [5], [6], [10]). However, very little is known in general about the class $\mathcal{S}$ of transformations obtained by the stacking method using single stacks. In particular there is no simple characterization of the class $\mathcal{S}$. In [1], the following question is raised: is every transformation with simple spectrum an $\mathcal{S}$-transformation? (The converse is true by [2, Theorem 1].) As a particular case the following question is also raised: is the translation by an irrational number α in $[0, 1)$ an $\mathcal{S}$-transformation? In Section 1 of this paper we answer this question affirmatively for α in a set E of Lebesgue measure 1, as well as giving a partial negative result for α in E^c. We also consider certain products of translations. Section 2 is concerned with giving an explicit stacking construction having $e^{2\pi i\alpha}$ as an eigenvalue. We show this is possible for almost all α, and for all α in E^c. All these results depend on various conditions connected with the goodness of approximation by rationals of the irrationals involved and we prove several results asserting the existence of irrationals satisfying these conditions.

The methods of this paper can also be used to show that the examples considered in [8], Sections 8 and 9, belong to $\mathcal{S}$ thereby also furnishing examples of transformations with continuous spectrum and mixed continuous and discrete spectrum respectively (other than examples actually constructed by the stacking method). We shall not give the proofs here.

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Section 0

All measure spaces $(X, \mathcal{F}, \mu)$ will be isomorphic to $[0, 1]$ with Borel sets and Lebesgue measure. A transformation (automorphism) of $(X, \mathcal{F}, \mu)$ is an invertible, bimeasurable, measure preserving mapping of X onto X. A partition of X is a finite collection of mutually disjoint elements of $\mathcal{F}$. If $\{P_n\}$ is a sequence of partitions, $P_n \to \varepsilon$ means $\mu(A \Delta P_n(A)) \to 0$ for all $A \in \mathcal{F}$, where $P_n(A)$ denotes any union of atoms of P_n such that $\mu(P_n(A)\Delta A)$ is minimal. If T is a trans-

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formation of X a stack for T (or T-stack) is an ordered partition $S = \{S_0, \ldots, S_{n-1}\}$ of X such that $T(S_j) = S_{j+1}$ for $0 \leq j \leq n - 2$. S_0 is called the base of S and n is its height. S_i is called the ith level of S (so the base is the 0*th* level).

We shall need the stacking method for constructing transformations for which the reader is referred to [6]. The class $\mathscr{S}$ is the class of transformations isomorphic to one constructed by the stacking method (using single stacks). This is just the class of transformations for which there exists a sequence of stacks $\{S_n\}$ such that $S_n \to \varepsilon$ and the base of S_n is a union of levels of S_{n+1}. The following characterization, due to Baxter [2], shows that the last requirement is unnecessary.

THEOREM 0.1 (Baxter). *A transformation T belongs to $\mathscr{S}$ iff there is a sequence $\{S_n\}$ of T-stacks such that $S_n \to \varepsilon$.*

We shall also need some elementary facts about continued fractions for which the reader is referred to [9] or [3, Section 4]. We will use the notation of [3]. We shall also use the notation $f(x) = o(g(x))$ ("little oh" notation) in the usual way and we shall write $f(x) \asymp g(x)$ to mean that there exist constants c and C, both greater than 0, such that $cf(x) < g(x) < Cf(x)$ for all values of x.

Section 1

We begin with a simple result which gives a sufficient condition for a transformation to belong to $\mathscr{S}$.

LEMMA 1.1. *Let T be a transformation of $(X, \mathscr{F}, \mu)$. Suppose that there exists a sequence $\{P_i\}$ of partitions of $(X, \mathscr{F}, \mu)$ and that for each i there is a permutation σ_i of P_i and an atom P_{i0} of P_i such that:*

(1) $P_i \to \varepsilon$;
(2) $\mu(T^{q_i-1}P_{i0}\Delta\sigma_i^{q_i-1}P_{i0}) = O(1/q_i)$ *where q_i is the number of atoms of P_i;*
(3) $T^j(P_{i0} \cap T^{-(q_i-1)}\sigma^{q_i-1}P_{i0}) \subset \sigma^j P_{i0}$ *for $0 \leq j \leq q_i - 1$.*

(Of course P_{i0} and σ_i serve for nothing more than to order P_i, but this statement of Lemma 1.1 is the most convenient for our applications.)

Lemma 1.1 deals with a special type of approximation by cyclic transformations which is in the spirit of the kind of approximations introduced by Katok and Stepin in [8]. For interest's sake we also state without proof a sufficient condition for T to belong to $\mathscr{S}$ which is exactly of the type considered in [8].

PROPOSITION 1.2. *If T admits a cyclic approximation with speed $o(1/n^2)$ in the sense of [8] then T belongs to $\mathscr{S}$.*

The proof of Proposition 1.2 is very similar to that of Lemma 1.1.

Proof of Lemma 1.1. Let $\bar{P}_{i0} = P_{i0} \cap T^{-(q_i-1)}\sigma^{q_i-1}P_{i0}$. By (3), $T^j\bar{P}_{i0} \subset \sigma^j P_{i0}$, so the $T^j\bar{P}_{i0}$, $0 \le j \le q_i - 1$, are mutually disjoint. Thus we may define a new partition

$$\bar{P}_i = \{\bar{P}_{i0}, \ldots, T^{q_i-1}\bar{P}_{i0}\}.$$

By (1) and (2) if i is sufficiently large $\mu(\bar{P}_{i0}) > (1 - \varepsilon)\mu(P_{i0})$. It follows from this and (1) that $\bar{P}_i \to \varepsilon$. Thus Theorem 0.1 implies that $T \in \mathscr{S}$.

For an irrational $\alpha \in [0, 1)$ we denote by T_α the transformation T_α: $x \mapsto x + \alpha \pmod 1$ on $[0, 1)$.

THEOREM 1.3. *Suppose α is an irrational number and there exists a sequence P_i/q_i of irreducible fractions such that $\alpha - P_i/q_i = o(1/q_i^2)$. Then $T_\alpha \in p$. Indeed, the levels for the stacks of T may be taken to be intervals.*

Proof. Let P_i be the partition of [0, 1]) into intervals

$$P_{im} = [m/q_i, (m + 1)/q_i), \quad 0 \le m \le q_i - 1.$$

Define σ_i on $\{0, \ldots, q_i - 1\}$ by $\sigma_i(m) = m + p_i \pmod{q_i}$ and denote by the same letter the permutation σ_i of P_i defined by $\sigma_i(P_{im}) = P_{i\sigma_i(m)}$. σ_i is cyclic since p_i and q_i are co-prime. Note that

$$\mu[T^{q_i-1}P_{i0}\Delta\sigma_i^{q_i-1}(P_{i0})] \le 2(q_i - 1)|\alpha - p_i/q_i| = o(1/q_i)$$

by the hypotheses. It is easy to see that condition (3) of Lemma 1.1 is also satisfied so the theorem follows from that lemma. It is clear, moreover that in this case Lemma 1.1 yields stacks whose levels are intervals.

COROLLARY *For almost all $\alpha \in [0, 1)$ (with respect to Lebesgue measure), $T_\alpha \in \mathscr{S}$.*

(This follows immediately from [3, Theorem 4.2], by taking $f(q) = 1/q \log q$.

The question now arises whether or not T_α belongs to $\mathscr{S}$ if α is a number which is not approximable to order $o(1/q^2)$. This seems to be a very difficult question. However Theorem 1.4 at least asserts that such a T_α cannot have stacks whose levels are intervals, in contrast to the assertion of Theorem 1.3.

THEOREM 1.4. *Suppose that α is an irrational number and that for some $c > 0$ the inequality $|\alpha - p/q| < c/q^2$ has no solutions in integers p and q. Then there does not exist a sequence of stacks S_i for T_α such that $S_i \to \varepsilon$ and the levels of S_i are intervals* (mod 1).

Proof. Suppose that I is an interval of length l which is the base of a T_α-stack S of height h which covers more than $1 - \eta$ (in measure) of [0, 1). We shall show that if η is small enough we have a contradiction.

Clearly we may assume $I = [0, l)$. Let p_i/q_i denote the ith convergent in the continued fraction expansion for α. Let i be the unique integer such that

$$1/q_{i+1} < l \le 1/q_i. \tag{1}$$

Set $p_i/q_i - \alpha = \varepsilon_i$ and assume for simplicity that $\varepsilon_i > 0$ or, what is the same thing, that i is odd (see [3, p. 41]). The argument is similar in case i is even. Note that by [3, p. 42] we have

$$\frac{1}{q_i(q_i + q_{i+1})} < |\varepsilon_i| < \frac{1}{q_i q_{i+1}}. \tag{2}$$

(Strict inequality holds in (2) since α is irrational.)

Now by equation 4.4 of [3] and since i is odd we have

$$q_{i-1}p_i \equiv 1 \pmod{q_i}. \tag{3}$$

By (2) and the hypotheses of the theorem we have q_{i-1}/q_i bounded away from 0 and this together with the relation $q_i = a_i q_{i-1} + q_{i-2}$ implies that q_{i-1}/q_i is also bounded away from 1. Thus if η is sufficiently small (1) implies that the height h of S must be greater than q_{i-1}. In particular

$$T_\alpha^{q_i - 1} I \cap I = \emptyset. \tag{4}$$

Now

$$\begin{aligned} T_\alpha^{q_i-1}(I) &= [q_{i-1}\alpha, q_{i-1}\alpha + l) \pmod 1 \\ &= [q_{i-1}(p_i/q_i - \varepsilon_i), q_{i-1}(p_i/q_i - \varepsilon_i) + l) \pmod 1 \\ &= [1/q_i - q_{i-1}\varepsilon_i, 1/q_i - q_{i-1}\varepsilon_i + l) \quad \text{by (3)}. \end{aligned}$$

(Notice that $q_{i-1}\varepsilon_i < 1/q_i$.) This together with (4) implies that

$$l \leq (1 - q_{i-1}q_i\varepsilon_i)\frac{1}{q_i}. \tag{5}$$

Now since $q_{i-1} \asymp q_i$ and $\varepsilon_i \asymp 1/q_i^2$, $q_{i-1}q_i\varepsilon_i$ is bounded away from 0, so if η is sufficiently small (5) implies that $h > q_i$. In particular, $T_\alpha^{q_i}(I)$ and I are disjoint. But

$$T_\alpha^{q_i}(I) = [-q_i\varepsilon_i, -q_i\varepsilon_i + l) \pmod 1.$$

It follows that $l \leq q_i\varepsilon_i < 1/q_{i+1}$ by (2). But this contradicts (1), so the theorem is proved.

We next prove two results which together imply that certain products of translations belong to $\mathscr{S}$.

THEOREM 1.5. *Suppose $\alpha_0, \ldots, a_n$ are irrational numbers and that there exist $n + 1$ sequences $p_{i0}/q_{i0}, \ldots, p_{in}/q_{in}$ of irreducible fractions such that:*

$$\alpha_r - \frac{p_{ir}}{q_{ir}} = o\left(\frac{1}{q_{ir}\prod_{j=0}^{n} q_{ij}}\right) \quad \textit{as } i \to \infty \textit{ for each } r; \tag{1}$$

(2) *$q_{i0}, \ldots, q_{in}$ are pairwise co-prime for each i.*
Then $T_{\alpha_0} \times \cdots \times T_{\alpha_n}$ belongs to $\mathscr{S}$.

Proof. For each i let P_{ir} be the partition of $[0, 1)$ into intervals

$$I_{irm} = \left[\frac{m}{q_{ir}}, \frac{m+1}{q_{ir}}\right)$$

and let P_i be the partition $P_{i0} \times \cdots \times P_{in}$ of $[0, 1)^{n+1}$. As in the proof of Theorem 1.3 let σ_{ir} be the permutation of $\{0, \ldots, q_{ir} - 1\}$ defined by $\sigma_{ir}(m) = m + p_{ir} \pmod{q_{ir}}$ and denote by the same letter the permutation induced on P_{ir}. Since $\sigma_{i0}, \ldots, \sigma_{in}$ are cyclic permutations with orders that are pairwise co-prime the permutation $\sigma_i = \sigma_{i0} \times \cdots \times \sigma_{in}$ of P_i is also cyclic. Finally it is easy to see that condition (3) of Lemma 1.1 is satisfied and that condition (2) of Lemma 1.1 is guaranteed by (2) of our hypotheses so the result follows by Lemma 1.1.

It is interesting to note in passing the following purely number theoretic result which can be obtained via Theorem 1.5. If $\alpha_1, \ldots, \alpha_n$ satisfy the conditions of Theorem 1.5 then $T_{\alpha_0} \times \cdots \times T_{\alpha_n}$ belongs to $\mathscr{S}$ and thus it is ergodic. (This is standard; see [6, Theorem 6.2].) It follows by a standard Fourier series argument that $\{\alpha_0, \ldots, \alpha_n, 1\}$ is independent over the rationals. It would be interesting to find a simple direct proof of this chain of reasoning.

PROPOSITION 1.6. *For all $n > 0$ there exist $(n+1)$-tuples $(\alpha_0, \ldots, \alpha_n)$ of irrational numbers satisfying the conditions of Theorem 1.6.*

Proof. Let $\alpha_0 = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots$, where the a_i are all multiples of $n!$ and are chosen so that $\alpha_0 - p_{i0}/q_{i0} = o(1/q_{i0}^{n+2})$ where p_{i0}/q_{i0} denotes the ith convergent to α_0. It follows from our choice of a_i that for $2 \leq l \leq n$, i even and r an integer, l does not divide $rp_i + q_i$. To see this note that

$$rp_i + q_i = ra_ip_{i-1} + rp_{i-2} + a_iq_{i-1} + q_{i-2}$$

so if l divides $rp_i + q_i$ then l divides $rp_{i-2} + q_{i-2}$ (since l divides a_i) and thus eventually l divides $rp_0 + q_0 = 1$.

Now for $1 \leq r \leq n$ set $\alpha_r = \alpha_0/(r\alpha_0 + 1)$, $p_{ir} = p_{i0}$ and $q_{ir} = rp_{i0} + q_{i0}$. Then

$$\alpha_r - \frac{p_{ir}}{q_{ir}} = \frac{\alpha_0 - (p_{i0}/q_{i0})}{(r\alpha_0 + 1)(r(p_{i0}/q_{i0}) + 1)} = o\left(\frac{1}{q_{i0}^{n+2}}\right).$$

Since $q_{ir} \asymp q_{i0}$ as $i \to \infty$ we can write this as

$$\alpha_r - \frac{p_{ir}}{q_{ir}} = o\left(\frac{1}{q_{ir}\prod_{j=0}^{n} q_{ij}}\right).$$

Note that p_{ir} and q_{ir} are co-prime since p_{i0} and q_{i0} are. Finally to see that $q_{i0}, \ldots, q_{in}$ are pairwise co-prime for even i, suppose that a prime l divides $rp_{i0} + q_{i0}$ and $r'p_{i0} + q_{i0}$, $n \geq r > r' \geq 0$. Then l divides $(r - r')p_{i0}$ also. Now l cannot divide p_{i0} (otherwise it would divide q_{i0} also) so l must divide

$r - r'$ and in particular $l \leq n$. But l divides $rp_{i0} + q_{i0}$ and as we have already seen this is impossible unless $l = 1$.

The $(n + 1)$-tuples constructed above consist of equivalent numbers in the sense defined in [7, Section 10.11]. It would be interesting to see if n-tuples of non-equivalent numbers could be constructed satisfying the conditions of Theorem 1.6. It should be pointed out, however, that for $n \geq 1$, the set of $(n + 1)$-tuples satisfying the conditions of Theorem 1.6 has product Lebesgue measure zero since these conditions force at least one of the α_i to be approximable to order $o(1/q^{n+2})$.

Section 2

Theorems 1.3 and 1.4 leave open the question of whether $T_\alpha \in \mathscr{S}$ for badly approximable α. Another approach to the problem would be to try to give an explicit stacking construction which yields a transformation isomorphic to T_α or at least one which has $e^{2\pi i\alpha}$ as an eigenvalue. This attempt is worthwhile even for T_α to which Theorem 1.3 does apply as that theorem says nothing about how to explicitly construct T_α by the stacking method (even though such a construction must exist). Theorem 2.1 is the result of this attempt. Proposition 2.2 guarantees that Theorem 2.1 applies to all badly approximable α (i.e., those to which Theorem 1.3 does not apply). Thus from 1.3, 2.1 and 2.2 we know that every T_α is a factor of an $\mathscr{S}$-transformation and that for almost all α, T_α is actually an $\mathscr{S}$-transformation.

THEOREM 2.1. *Suppose that α is an irrational number and that there exists a sequence p_i/q_i of fractions such that, denoting $|\alpha - p_i/q_i|$ by ε_i:*

(1) $\sum \varepsilon_i q_{i+1} < \infty$;
(2) $\sum q_i/q_{i+1} < \infty$.

Then there is an explicit stacking construction which yields a transformation with $e^{2\pi i\alpha}$ as an eigenvalue.

Proof. We start with a stack S_1 of height q_1. Suppose that the stack S_k with height q_k has already been constructed. S_{k+1} is constructed by cutting S_k into $[q_{k+1}/q_k]$ stacks of equal width, stacking these above each other into a single stack and adding $q_{k+1} - q_k[q_{k+1}/q_k]$ levels on top.

Let us show that the total measure of the space so obtained is finite. If λ_k denotes the measure of S_k and $s_k = [q_{k+1}/q_k]$ then

$$\begin{aligned}\lambda_{k+1} &= \frac{q_{k+1} - q_k s_k}{q_k s_k}\lambda_k + \lambda_k \\ &< \frac{q_k}{q_k s_k}\lambda_k + \lambda_k \\ &\leq \left(\frac{q_k}{q_{k+1} - q_k} + 1\right)\lambda_k.\end{aligned}$$

Our hypotheses imply

$$\sum \frac{q_k}{q_{k+1} - q_k} < \infty$$

so

$$\prod \left(\frac{q_k}{q_{k+1} - q_k} + 1\right) < \infty$$

and thus $\lim_k \lambda_k < \infty$.

Let T denote the transformation defined by this sequence of stacks, X the space on which T acts and μ the normalized measure on X. We now construct a function f such that $Tf = \lambda f$, where $\lambda = e^{2\pi i\alpha}$. Define f_k by setting $f_k = \lambda^i$ on the ith level of S_k (recall that the base is the $0th$ level) and $f_k = 0$ off S_k. We want to show f_k is a Cauchy sequence. Let S_{ki} be the ith stack (in order of appearance in S_{k+1}) into which S_k is cut. We have $f_{k+1} - f_k = \lambda^j(\lambda^{iq_k} - 1)$ on the jth level of S_{ki}. Thus

$$\begin{aligned} |f_{k+1} - f_k| &= |\lambda^{iq_k} - 1| \quad \text{on } S_{ki} \\ &< s_k q_k(2\pi\varepsilon_k) \quad \text{on } S_k \end{aligned}$$

since $i < s_k$. Also $|f_{k+1} - f_k| < 2$ on $S_{k+1} - S_k$ and $|f_{k+1} - f_k| = 0$ off S_{k+1}. Thus

$$\begin{aligned} \|f_{k+1} - f_k\|_1 &\leq 2\pi\varepsilon_k s_k q_k + 2[\mu(S_{k+1}) - \mu(S_k)] \\ &\leq 2\pi\varepsilon_k q_{k+1} + 2[\mu(S_{k+1}) - \mu(S_k)]. \end{aligned}$$

Hence $\sum_k \|f_{k+1} - f_k\|_1 < \infty$ and if we set $f = \lim_k f_k$ it is clear that $Tf = \lambda f$.

COROLLARY. *If α satisfies the conditions of Theorem 2.1 then T_α is a factor of an $\mathscr{S}$-transformation.*

Proof. Retaining the notation of the above proof, let $\mathscr{G}$ denote the σ-algebra $f^{-1}(\mathscr{B})$ where $\mathscr{B}$ denotes the σ-algebra of Borel sets in $\mathbf{C}$. One can show by a straightforward measure theoretic argument that $\mathscr{L}^2(X, \mathscr{G}, \mu)$ is spanned by the functions f^i, $i \in \mathbf{Z}$. Since $Tf^i = \lambda^i f^i$, it follows that $\mathscr{G}$ is T-invariant and that $T \mid_{\mathscr{G}}$ is isomorphic to T_α.

THEOREM 2.2. *Let α be an irrational number and suppose that there exists a $c > 0$ such that $|\alpha - p/q| > c/q^2$ for all integers p and q. Then α satisfies the conditions of Theorem 2.1.*

Proof. Let $p(i)/q(i)$ denote the ith convergent to α. Recall that

$$\left|\alpha - \frac{p(i)}{q(i)}\right| = \varepsilon(i) < \frac{1}{q(i)q(i+1)}.$$

Thus our hypotheses imply that there is a $C > 0$ such that

(1) $$q(i+1) < Cq(i).$$

Also, one can show by induction on k that

$$q(i + k) > 2^{(k-1)/2} q(i). \tag{2}$$

Now choose a real number $\theta > 1$ such that for $i \geq 2$

$$\tfrac{1}{2}(i - 1) \log 2 - i \log \theta > 0. \tag{3}$$

Next choose an integer $N > 0$ such that

$$\frac{i}{N} \log C < \frac{i - 1}{2} \log 2 - i \log \theta. \tag{4}$$

Finally define an integer sequence $k(i)$ by the recursion $k(i + 1) = k(i) + [i/N]$. Note that $k(i) \asymp i^2$ so in particular $k(i) > i$ for large i. We will show the conditions of Theorem 2.1 are satisfied by the sequence $p(k(i))/q(k(i))$. Observe that

$$\begin{aligned} \varepsilon(k(i))q(k(i + 1)) &\leq \frac{1}{q(k(i))^2} C^{[i/N]} q(k(i)) && \text{by (1)} \\ &\leq \frac{C^{[i/N]}}{2^{(k(i)-1)/2}} && \text{by (2)} \\ &\leq \frac{C^{[i/N]}}{2^{(i-1)/2}} && \text{for large } i \\ &< \theta^{-i} && \text{by (4).} \end{aligned}$$

Thus $\sum_i \varepsilon(k(i))q(k(i + 1)) < \infty$. As for $\sum_i q(k(i))/q(k(i + 1))$ being finite, this follows easily from (2), so the theorem is proved.

As we have already mentioned Theorem 2.1 is of interest not only for badly approximable α, so we state without proof the following proposition which has as a corollary the fact that the conditions of Theorem 2.1 are satisfied for almost all α. The proof of Proposition 2.3 is similar to that of Theorem 2.2, using a subsequence of the convergents.

PROPOSITION 2.3. *Let α be an irrational number and let $p(i)/q(i)$ denote the ith convergent to α. Then if $\lim_{i\to\infty} q(i)^{1/i}$ exists and is finite, α satisfies the conditions of Theorem* 2.1.

COROLLARY. *For almost all α there is an explicit stacking construction of a transformation having $e^{2\pi i\alpha}$ as an eigenvalue.*

Proof. Follows immediately from Theorem 2.1 and [3, equation 4.18].

It should be pointed out in connection with Theorem 2.1 that it does not seem reasonable to hope that in general the transformation constructed will actually be isomorphic to T_α. One can construct examples of α and $p(i)/q(i)$ satisfying the conditions of Theorem 2.1 such that the transformation T has eigenvalues

$e^{2\pi i\alpha/n}$ for an infinity of integers n. It would be interesting to know, however, whether T can have any continuous spectrum. It would also be very useful to know whether a factor of an $\mathscr{S}$-transformation must be an $\mathscr{S}$-transformation. This is closely related to the question: does every transformation with simple spectrum belong to $\mathscr{S}$? A negative answer to the first question implies a negative answer to the second since factors of transformations with simple spectrum certainly have simple spectrum.

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