ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. I

BY

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1. Introduction

Let E be an arbitrary nonempty set of (positive rational) prime numbers. For each positive integer n, let $\omega(n; E)$ denote the number of distinct primes in E which divide n, and let $\Omega(n; E)$ be the total number of primes in E which divide n, counted according to multiplicity. Thus if $n = p_1^{a_1} \cdots p_r^{a_r} n'$, where $r \geq 0$, $p_1, \ldots, p_r$ are distinct primes in E, each a_j is a positive integer, and n' has no prime factor in E, then $\omega(n; E) = r$ and $\Omega(n; E) = a_1 + \cdots + a_r$. When E is the set of all primes, we write $\omega(n; E) = \omega(n)$, $\Omega(n; E) = \Omega(n)$. The main objective of this paper and a subsequent one [31] is to derive some very accurate information about the distribution of values of these functions.

In the study of the sizes of $\omega(n; E)$ and $\Omega(n; E)$, an important role is played by the function

$$E(x) = \sum_{p \leq x,\, p \in E} p^{-1}. \tag{1.1}$$

For example, it is easy to show by the method of [20, Section 22.10] that $\omega(n; E)$ has average order $E(n)$, and the same is true of $\Omega(n; E)$ if $E(n)$ tends to infinity with n. Furthermore, a general theorem of Turán [38] shows in particular that if $E(x) \to +\infty$ as $x \to +\infty$, then each of $\omega(n; E)$ and $\Omega(n; E)$ has normal order $E(n)$. Turán's proof yields a quantitative version of this result which can be stated as follows for $\omega(n; E)$: if $\alpha = \alpha(x) > 0$ and $E(x) \geq 1$, then

$$\operatorname{card}\{n\colon n \leq x \text{ and } |\omega(n; E) - E(x)| \geq \alpha E(x)\} \leq c_1 x/\alpha^2 E(x). \tag{1.2}$$

(Throughout this paper, card B denotes the number of members of the set B. For $i = 1, 2, \ldots, c_i(\delta, \varepsilon, \ldots)$ means a positive number depending only on $\delta, \varepsilon, \ldots$, while c_i means a positive absolute constant.) In particular,

$$\operatorname{card}\{n\colon n \leq x \text{ and } |\omega(n; E) - E(x)| \geq \alpha E(x)\} = o(x) \tag{1.3}$$

if $E(x) \to +\infty$ and $\alpha E(x)^{1/2} \to +\infty$ as $x \to +\infty$. The result (1.3) was first proved by Hardy and Ramanujan [19] (this paper is reprinted in [33, pp. 262–275]) for the special case in which E is the set of all primes (in this case, it is well known that $E(x) = \log\log x + O(1)$ for $x \geq 2$, and in fact, $E(x)$ can be replaced by $\log\log x$ in (1.3)).

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Using a difficult probabilistic method, Elliott [12, Theorem 6] recently obtained a very general theorem of which a special case is

(1.4) $$\operatorname{card}\{n: n \leq x \text{ and } |\omega(n; E) - E(x)| \geq \alpha E(x)\} \leq c_2 x \exp\{-c_3\alpha^2 E(x)\}$$

for $x \geq 1$, $0 < \alpha \leq 1$. Although this estimate is far stronger than Turán's inequality (1.2) when $\alpha^2 E(x)$ is large, it is not immediately clear just how precise (1.4) is, and it does not even seem easy to calculate the positive absolute constants c_2 and c_3 by Elliott's method.

In this paper, we shall derive some new and very precise improvements of (1.2) and (1.4). First we state a result in which the set E is arbitrary.

(1.5) THEOREM. *Suppose that $g(n) = \omega(n; E)$ (for all n) or $g(n) = \Omega(n; E)$ (for all n). For any real numbers x, α, define*

(1.6) $$A(x, \alpha; E, g) = \operatorname{card}\{n: n \leq x \text{ and } |g(n) - E(x)| \geq \alpha E(x)\}.$$

If $0 < \alpha \leq \beta < 1$ and $E(x) > 0$, then

(1.7) $$A(x, \alpha; E, g) \leq c_4(\beta)\alpha^{-1} x E(x)^{-1/2} e^{Q(\alpha)E(x)},$$

where

(1.8) $$Q(\alpha) = \alpha - (1 + \alpha)\log(1 + \alpha).$$

The inequality (1.7) is best possible in the following sense: If $E(z) \to +\infty$ as $z \to +\infty$, and if $0 < \beta < 1$, then there is a number $c_5(\beta, E)$ such that whenever $x \geq c_5(\beta, E)$ and $E(x)^{-1/2} \leq \alpha \leq \beta$, we have

(1.9) $$A(x, \alpha; E, g) \geq c_6(\beta)\alpha^{-1} x E(x)^{-1/2} e^{Q(\alpha)E(x)}.$$

It should be noted that

(1.10) $$-\alpha^2/2 < Q(\alpha) < (-0.386)\alpha^2 \quad \text{for } 0 < \alpha < 1.$$

For larger values of α, our results are less precise than Theorem (1.5). For example, our methods show that

(1.11) $$A(x, \alpha; E, \omega) \leq c_7 x e^{Q(\alpha)E(x)} \quad \text{for } x \geq 1, \alpha \geq 0,$$

and similar but weaker upper bounds can be obtained for $A(x, \alpha; E, \Omega)$ in some cases. However, we have been unable to get lower bounds for either $A(x, \alpha; E, \omega)$ or $A(x, \alpha; E, \Omega)$ when $\alpha \geq 1$. For the proofs of Theorem (1.5) and related inequalities, see Section 5. The proofs depend on some beautiful theorems of Halász [14], [15] and on certain elementary inequalities obtained in Sections 3 and 4 below.

In Section 6, we consider analogues of Theorem (1.5) for a set E which consists of the primes in various arithmetic progressions with the same modulus. Here we mention only a special case of the results of Section 6. Suppose that k, l are integers with $k \geq 1$ and $(k, l) = 1$. Then for each real $x \geq 2$,

(1.12) $$\sum_{p \leq x,\, p \equiv l \pmod{k}} p^{-1} = \phi(k)^{-1}\log\log x + O(1),$$

where ϕ is Euler's function and the implied constant is *absolute*. This estimation of the error term in (1.12) is new and best possible. Using (1.12), we can derive the following result:

(1.13) THEOREM. *Let k, l be integers with $k \geq 1$, $(k, l) = 1$. Let E be the set of all primes p satisfying $p \equiv l \pmod{k}$, and let $g(n) = \omega(n; E)$ (for all n) or $g(n) = \Omega(n; E)$ (for all n). Write* $\log\log x = \log_2 x$, *and for any real x, α with $x \geq 3$, let*

$$(1.14)\quad A_1(x, \alpha; E, g) = \operatorname{card}\{n: n \leq x \text{ and } |g(n) - \phi(k)^{-1}\log_2 x| \geq \alpha\phi(k)^{-1}\log_2 x\},$$

where ϕ is Euler's function. If $x \geq 3$ and $0 < \alpha \leq \beta < 1$, then

$$(1.15)\quad A_1(x, \alpha; E, g) \leq c_8(\beta)\alpha^{-1}x\{\phi(k)/\log_2 x\}^{1/2}(\log x)^{Q(\alpha)/\phi(k)},$$

where $Q(\alpha)$ is defined by (1.8). *Furthermore, if $0 < \beta < 1$, then there is a number $c_9(\beta, k)$ such that whenever $x \geq c_9(\beta, k)$ and $\{\phi(k)/\log_2 x\}^{1/2} \leq \alpha \leq \beta$, we have*

$$(1.16)\quad A_1(x, \alpha; E, g) \geq c_{10}(\beta)\alpha^{-1}x\{\phi(k)/\log_2 x\}^{1/2}(\log x)^{Q(\alpha)/\phi(k)}.$$

For the proofs of more general results, see Section 6.

It is interesting to compare Theorems (1.5) and (1.13) with a result of Kubilius. In [23, Theorem 9.2], he derives a theorem from which it follows that if $\alpha = \alpha(x) = o(1)$ and $\alpha(x)(\log_2 x)^{1/2} \to +\infty$ as $x \to +\infty$, then

$$(1.17)\quad \operatorname{card}\{n: n \leq x \text{ and } \omega(n) \leq (1 - \alpha)\log_2 x\} \sim (2\pi)^{-1/2}\alpha^{-1}x(\log_2 x)^{-1/2}(\log x)^{Q(-\alpha)}$$

and

$$(1.18)\quad \operatorname{card}\{n: n \leq x \text{ and } \omega(n) \geq (1 + \alpha)\log_2 x\} \sim (2\pi)^{-1/2}\alpha^{-1}x(\log_2 x)^{-1/2}(\log x)^{Q(\alpha)}$$

as $x \to +\infty$, where $\sim$ denotes asymptotic equivalence. Our results are of almost the same precision, do not require the hypothesis $\alpha = o(1)$, and apply to more general functions. (For a different kind of generalization, see Kubilius [24]. See also [23, p. 168] for further remarks.) It should be added that the proof of Theorem (1.5) is basically much simpler than Kubilius's proof of (1.17) and (1.18).

Our theorems lead to new information about the distribution of values of divisor functions. If m, n are any integers with $m \geq 2$, $n \geq 1$, define $d_m(n)$ to be the number of ordered m-tuples $(t_1, \ldots, t_m)$ of positive integers such that $t_1 \cdots t_m = n$. (Thus $d_2(n) = d(n)$ is the number of distinct positive divisors of n.) It is possible to show by the method in [20, Section 18.1] that for fixed m, the maximum order of $d_m(n)$ is about $m^{(\log n)/\log_2 n}$. On the other hand, the average order of $d_m(n)$ is the much smaller quantity $(\log n)^{m-1}/(m - 1)!$. (The latter assertion is classical and can be proved rather simply by induction on m. It is also a special case of a result due to Selberg [36, Theorem 1].) However,

$d_m(n)$ is usually considerably smaller than even its average order. To see this, we observe that the inequalities

$$(1.19) \qquad m^{\omega(n)} \leq d_m(n) \leq m^{\Omega(n)} \quad (m \geq 2, n \geq 1)$$

follow easily from the obvious formula $d_{m+1}(n) = \sum_{t \mid n} d_m(t)$ by induction on m, and a combination of (1.19) with the results of Section 6 (when $k = l = 1$) yields:

(1.20) THEOREM. *Let $m \geq 2$. For any real x, α with $x \geq 3$, let*

$$D(x, \alpha, m) = \text{card}\,\{n: n \leq x, \text{ and either } d_m(n) \leq (\log x)^{(1-\alpha)\log m} \text{ or } d_m(n) \geq (\log x)^{(1+\alpha)\log m}\}.$$

If $x \geq 3$ and $0 < \alpha \leq \beta < 1$, then

$$(1.21) \qquad D(x, \alpha, m) \leq c_{11}(\beta)\alpha^{-1}x\,(\log_2 x)^{-1/2}\,(\log x)^{Q(\alpha)}.$$

Furthermore, if $0 < \beta < 1$, then there is a number $c_{12}(\beta)$ such that whenever $x \geq c_{12}(\beta)$ and $(\log_2 x)^{-1/2} \leq \alpha \leq \beta$, we have

$$(1.22) \qquad D(x, \alpha, m) \geq c_{13}(\beta)\alpha^{-1}x\,(\log_2 x)^{-1/2}\,(\log x)^{Q(\alpha)}.$$

A problem posed by Dr. John Steinig is to estimate the number of $n \leq x$ for which $d(n)$ $(= d_2(n))$ is as large as its average order $\log n$. From (1.19) and the results of Section 6, it is easy to deduce that for $x \geq 3$, this number is bounded by positive constant multiples of $x\,(\log_2 x)^{-1/2}\,(\log x)^{\delta}$, where

$$\delta = (\log 2)^{-1}(1 - \log 2 + \log\log 2) \approx -0.086.$$

It is possible to give specific numerical inequalities for a few of the quantities we have considered. For example, if $x \geq \exp\exp 6$ and $0 \leq \alpha \leq 1$, then

$$(1.23) \quad \text{card}\,\{n: n \leq x \text{ and } |\omega(n) - \log_2 x| \geq \alpha \log_2 x\} < 3x\,(\log x)^{Q(\alpha)}.$$

Also, if $x \geq \exp\exp 8$ and $0 \leq \alpha \leq 0.6$, then

$$(1.24) \quad \text{card}\,\{n: n \leq x \text{ and } |\Omega(n) - \log_2 x| \geq \alpha \log_2 x\} < 5x\,(\log x)^{Q(\alpha)}.$$

We shall not present the proofs here, since they involve some rather tedious calculations. We merely remark that the proofs depend on a recent paper of Hall [17] and on some inequalities due to Rosser [34] and Rosser and Schoenfeld [35, Theorems 5, 8, 12]. When x is quite large and α is not too close to 0, (1.23) and (1.24) are worthwhile for computations.

In Section 7, we conclude with a few remarks about the limitations of the methods used here and mention some unsolved problems.

The early history of this subject is interesting. The pioneers in the field were Hardy and Ramanujan [19], who proved (1.3) for the case in which E is the set of all primes. (They established also the same theorem for $\Omega(n)$.) Their proof, although elementary, was somewhat involved and difficult to generalize (see further remarks in Section 3 below). In 1934, Turán [37] (see also [20, Section

22.11]) presented a new proof of their theorem along quite different lines. Turán's proof is a model of beautiful simplicity, and he later showed in [38] that it could be extended to prove similar results for more general additive functions $\psi(n)$ in place of $\omega(n)$ or $\Omega(n)$, and also for functions of the forms $\omega(|f(n)|)$ and $\Omega(|f(n)|)$, where f is a polynomial with integral coefficients. Furthermore, the deduction of (1.2) is very similar to the proof of Chebyshev's inequality in the theory of probability, and together with the Erdös-Kac theorem [13], it could be regarded as having provided the initial inspiration for the field of probabilistic number theory. For all of these reasons, Turán's method has been very influential, and the original method of Hardy and Ramanujan has fallen into disuse. Surprisingly, it seems to have gone generally unnoticed that for the particular functions $\omega(n)$ and $\Omega(n)$, the Hardy-Ramanujan proof gives results which are much sharper than those of Turán. In fact, by reading [19] carefully and filling in a number of details, one can see that Hardy and Ramanujan essentially proved (1.15) when $k = l = 1$ and $g(n) = \omega(n)$, and their result for $\Omega(n)$, though weaker, was also much superior to Turán's in a quantitative sense. However, they never stated an estimate like (1.15), being content with the qualitative result (1.3) (with E the set of all primes). As they were apparently unable to get good lower bounds like (1.16), they could not have known how precise their work was.

Many years later, in discussing his joint work with Ramanujan on this subject [18, Chapter III], Hardy remarked that their method was "in some ways more suggestive" than Turán's, but he said nothing about the remarkable quantitative difference between the results.

Here we shall use the original method of Hardy and Ramanujan, suitably augmented by recent results of Halász [14], [15] and some simple lemmas given below. In a later paper [31], we shall obtain further theorems on the distribution of $\omega(n; E)$ and $\Omega(n; E)$. These will involve certain quantitative improvements of the Erdös-Kac theorem.

Much of this work was done while I held a visiting research position in the Mathematics Department of the University of Geneva. Special thanks are due to Dr. John Steinig for arranging and facilitating my very pleasant visit. I would also like to thank Dr. Steinig and Professor P. D. T. A. Elliott for valuable and stimulating conversations about some of the problems considered here.

2. Notation and a lemma

The symbols k, l, m, n always represent integers, with k and n being positive. The letter p always denotes a prime, while v, x, y, z, α, β, δ, ε are real numbers. $[x]$ means the largest integer $\leq x$, and $\log_2 x$ means $\log\log x$. ϕ always denotes Euler's function. Empty sums mean 0, empty products 1. The notation $x_1 \cdots x_m/y_1 \cdots y_n$ is sometimes used instead of $(x_1 \cdots x_m)(y_1 \cdots y_n)^{-1}$.

The notation $O_{\delta,\varepsilon}\ldots$ indicates an implied constant depending at most on $\delta, \varepsilon, \ldots$, while O without subscripts implies an absolute constant. A similar convention holds for the positive constants $c_i(\delta, \varepsilon, \dots)$, c_i (see the remark after (1.2)). We shall also occasionally use the notations $\ll$, $\gg$, which always imply *absolute* constants in this paper. Thus $A \ll B$ is equivalent to $A = O(B)$.

Throughout the paper, E denotes a nonempty set of primes, to be regarded as quite arbitrary unless further assumptions are stated. $E(x)$ is always defined by (1.1).

The function $Q(\alpha)$ is defined throughout as in the following lemma.

(2.1) LEMMA. *Define*

$$Q(\alpha) = \alpha - (1 + \alpha)\log(1 + \alpha) \quad \textit{for } \alpha > -1, \qquad Q(-1) = -1 = \lim_{\alpha \to -1+} Q(\alpha), \tag{2.2}$$

so that

$$Q(\alpha) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\alpha^n}{(n-1)n} \quad \textit{for } |\alpha| \le 1. \tag{2.3}$$

$Q(\alpha)$ is strictly increasing on $[-1, 0]$ and strictly decreasing on $[0, +\infty)$ (thus $Q(\alpha) < 0$ for $\alpha \neq 0$). If we define

$$h(\alpha) = \alpha^{-2}Q(\alpha) \quad \textit{for } \alpha \ge -1, \alpha \neq 0, \qquad h(0) = -1/2 = \lim_{\alpha \to 0} h(\alpha), \tag{2.4}$$

then $h(\alpha)$ is strictly increasing on $[-1, +\infty)$. Hence

$$-\alpha^2 < Q(\alpha) < -\alpha^2/2 \quad \textit{for } -1 < \alpha < 0, \tag{2.5}$$

$$-\alpha^2/2 < Q(\alpha) < (1 - 2\log 2)\alpha^2 < (-0.386)\alpha^2 \quad \textit{for } 0 < \alpha < 1. \tag{2.6}$$

Also,

$$Q(-\alpha) < Q(\alpha) - \alpha^3/3 \quad \textit{for } 0 < \alpha \le 1. \tag{2.7}$$

Proof. To prove the statement about $h(\alpha)$, define

$$H(\alpha) = -2\alpha + (\alpha + 2)\log(1 + \alpha) \quad \text{for } \alpha > -1$$

and note that

$$H(\alpha) = \alpha^3 h'(\alpha) \quad \text{for } \alpha > -1, \alpha \neq 0. \tag{2.8}$$

Computing $H'(\alpha)$ and $H''(\alpha)$, we find that $H'(\alpha)$ has its minimum at $\alpha = 0$, so $H'(\alpha) > 0$ for $\alpha \neq 0$. Hence $H(\alpha)$ is strictly increasing, so by (2.8), $h'(\alpha) > 0$ for $\alpha > -1$, $\alpha \neq 0$. Since $h(\alpha)$ is continuous on $[-1, +\infty)$, it is strictly increasing on the same interval. (2.5) and (2.6) follow immediately. Finally, (2.7) follows from (2.3). Q.E.D.

3. Some preliminary results

In order to prove (1.3) when E is the set of all primes, Hardy and Ramanujan began by defining

$$N(m, x) = \text{card}\,\{n\colon n \le x \text{ and } \omega(n) = m\} \quad \text{for } m = 0, 1, 2, \ldots .$$

Assuming that $x \ge 3$, replacing $E(x)$ by its close approximant $\log_2 x$, and writing

$$y = (1 - \alpha) \log_2 x, \quad z = (1 + \alpha) \log_2 x$$

(where $\alpha > 0$), they observed that

$$\text{(3.1)} \quad \text{card}\,\{n\colon n \le x \text{ and } |\omega(n) - \log_2 x| \ge \alpha \log_2 x\} = \sum_{0 \le m \le y} N(m, x) + \sum_{m \ge z} N(m, x).$$

They were able to show by an elementary inductive method that

$$\text{(3.2)} \quad N(m, x) \le \frac{c_{14} x \{\log_2 x + c_{15}\}^{m-1}}{(m-1)! \log x} \quad \text{for } x \ge 2,\ m = 1, 2, \ldots .$$

Combining (3.2) with (3.1) and using some simple upper estimates for partial sums of the exponential series, they arrived at the desired result. Their proof of the corresponding theorem for $\Omega(n)$ was similar but more difficult, since the obvious analogue of (3.2) for $\Omega(n)$ does not hold without some restriction on the size of m, so that (3.2) must be replaced by a more complicated inequality.

In attempting to use this method to deal with the functions $\omega(n; E)$ and $\Omega(n; E)$ (where E is arbitrary), one finds that the principal difficulty is to obtain an appropriate extension of (3.2), since the simple inductive proof of Hardy and Ramanujan does not generalize. For values of m which are not too large, this problem was solved very successfully by Halász [14], [15]. What we need here is the following result from [15]:

(3.3) LEMMA. *Suppose that either $g(n) = \omega(n; E)$ (for all n) or $g(n) = \Omega(n; E)$ (for all n). Define*

$$\text{(3.4)} \quad N(m, x; E, g) = \text{card}\,\{n\colon n \le x \text{ and } g(n) = m\}$$

for $m = 0, 1, 2, \ldots$. Let $0 < \delta \le 2$. If $x \ge 1$ and $0 \le m \le (2 - \delta)E(x)$, then

$$\text{(3.5)} \quad N(m, x; E, g) \le c_{16}(\delta) x \frac{E^m(x)}{m!} e^{-E(x)}.$$

Suppose also that $E(z) \to +\infty$ as $z \to +\infty$ and that x is sufficiently large (i.e., $x \ge c_{17}(\delta, E)$). Then for $0 \le m \le (2 - \delta)E(x)$, we have

$$\text{(3.6)} \quad N(m, x; E, g) + N(m + 1, x; E, g) \ge c_{18}(\delta) x \frac{E^m(x)}{m!} e^{-E(x)}.$$

In [15], the proof of Lemma (3.3) is given only for $g(n) = \Omega(n; E)$. It is virtually elementary but very ingenious and depends in part on estimations involving the function

$$F(z, \sigma) = \sum_{n=1}^{\infty} z^{\Omega(n; E)} n^{-\sigma} = \prod_{p \in E} (1 - zp^{-\sigma})^{-1} \prod_{p \notin E} (1 - p^{-\sigma})^{-1},$$

the series and the products being absolutely convergent for $\sigma > 1$ and any complex z with $|z| < 2$. A similar proof can be given for the case $g(n) = \omega(n; E)$ by making use of the function

$$G(z, \sigma) = \sum_{n=1}^{\infty} z^{\omega(n; E)} n^{-\sigma} = \prod_{p \in E} \left(1 + \frac{z}{p^{\sigma} - 1}\right) \prod_{p \notin E} (1 - p^{-\sigma})^{-1},$$

the series and products being absolutely convergent for $\sigma > 1$ and any finite complex z. One calculates the function $H(z, \sigma)$ determined by

$$G(z, \sigma) = H(z, \sigma)F(z, \sigma) \quad \text{for } \sigma > 1, |z| < 2,$$

and shows that $H(z, \sigma)$ has certain simple bounds for $0 < |z| \leq 2 - \delta, \sigma \geq 1$. It follows that the behavior of $G(z, \sigma)$ is very similar to that of $F(z, \sigma)$ for $\sigma > 1$, $|z| < 2$. Hence one can take advantage of Halász's analysis of $F(z, \sigma)$ in order to estimate $G(z, \sigma)$ and the integral involving it which arises as in [15]. The rest of the proof proceeds as in [15] with only minor changes.

In [14], Halász derived an asymptotic formula for $N(m, x; E, g)$ which is valid if $E(x) \to +\infty$ and $m = E(x) + o(E(x))$ as $x \to +\infty$. Lemma (3.3) and the results of [14] are remarkable for their precision, their uniformity in m, and their almost complete lack of assumptions about E. Asymptotic formulas for $N(m, x; E, g)$ had previously been obtained by several authors, but only under more restrictive conditions. For example, Landau proved such formulas when m is fixed and E is the set of all primes (see [26, pp. 203–213] or [20, Section 22.18]), and he even derived asymptotic expansions in this case [25]. (Incidentally, his work shows that (3.2) is sharp except for the constants c_{14} and c_{15}.) His work was considerably generalized in a series of papers by Delange (see [6, Theorems 12, 14, 16, 34, 36], [7], [8], [9], [10, pp. 130, 132, 136–146]), but Delange's work depended on certain assumptions about the distribution of the primes in E, and his formulas were not asserted to hold uniformly in m. Wirsing [40] obtained an asymptotic formula for the case $m = 0$ (again under an assumption about the distribution of E), and Selberg [36] gave certain asymptotic formulas which hold when E is the set of all primes, the results being uniform in m if m is not too large. Wintner [39] and Delange [11] gave asymptotic formulas which hold for any fixed m and for other rather general integral-valued additive functions in place of $\omega(n; E)$ or $\Omega(n; E)$. For further discussion and references to related work of Sathe, Erdös, Pillai, Kubilius, Wirsing, and Delange, see [14].

The success of Hardy and Ramanujan's work is due largely to the precision of their estimate (3.2) and the fact that it holds uniformly for all m, so that it can be substituted in (3.1). (3.5) and (3.6) are of comparable precision but are asserted to hold only for $m \leq (2 - \delta)E(x)$, and some such restriction on the size of m seems to be an essential feature of Halász's proof. Furthermore, it is easy to see that no inequality of the form $N(m, x; E, \Omega) \ll xe^{-E(x)}E^m(x)/m!$ could be true for all x and all m, for if $m = [(\log x)/\log p_1]$ (where p_1 is the smallest member of E), Stirling's formula shows that the right-hand side of the inequality tends to 0 as $x \to +\infty$, whereas clearly $N(m, x; E, \Omega) \geq 1$.

Thus we need a substitute for Lemma (3.3) when m is large. More precisely, we need to show that $\omega(n; E)$ and $\Omega(n; E)$ rarely take large values. There are several ways of seeing this. One of the simplest and most effective was suggested to the author by Professor Elliott, who observed that from the elementary inequality

$$\sum_{n \leq x} 2^{\omega(n)} \ll x \log x \quad (x \geq 2), \tag{3.7}$$

it can be deduced that $\omega(n)$ is seldom very large. (As Elliott pointed out, a similar idea was used in a different context by Hooley [21, Lemmas 6, 7], who dealt with the function $\Omega(n)$.) In order to exploit this idea, we need a generalization of (3.7), and here we can use the following result of Halász [14, Theorem 2]:

(3.8) LEMMA. *Let $f(n)$ be a complex-valued completely multiplicative function. Define $\theta_p = \arg f(p)$. Suppose there exist fixed real numbers $\delta > 0$ and θ_0 such that for all p, we have $\delta \leq |f(p)| \leq 2 - \delta$ and $|e^{i\theta_p} - e^{i\theta_0}| \geq \delta$. Then for $x \geq 1$,*

$$\left|\sum_{n \leq x} f(n)\right| \leq c_{19}(\delta)x \exp\left\{\sum_{p \leq x} \frac{|f(p)| - 1}{p} - c_{20}(\delta) \sum_{p \leq x} \frac{|f(p)| - \operatorname{Re} f(p)}{p}\right\}.$$

As an immediate corollary, we have (for each $\delta > 0$)

$$\sum_{n \leq x} z^{\omega(n; E)} \leq \sum_{n \leq x} z^{\Omega(n; E)} \leq c_{19}(\delta)xe^{(z-1)E(x)} \quad \text{for } 1 \leq z \leq 2 - \delta. \tag{3.9}$$

(3.9) is sufficient to obtain our main results below. However, the proof of Lemma (3.8) is rather long and difficult, and it is not technically elementary. Since we need only the very special case (3.9), it seems worthwhile to indicate briefly how to give a much simpler proof of it which actually yields somewhat more. For the sum involving $z^{\omega(n; E)}$, this is particularly easy. The basic idea is to find a function $h(n)$ such that for each $n \geq 1$, $z^{\omega(n; E)} = \sum_{d \mid n} h(d)$. The Möbius inversion formula shows that such a function $h(n)$ exists and is determined by

$$h(n) = \sum_{d \mid n} \mu(d) z^{\omega(n/d; E)}.$$

Since h is the "Dirichlet convolution" of two multiplicative functions, it is also multiplicative. It is easy to verify that if a is a positive integer, we have $h(p^a) =$

$z - 1$ if $p \in E$ and $a = 1$, while $h(p^a) = 0$ otherwise. Thus $h(n) \geq 0$ for all n, and

$$\sum_{n \leq x} z^{\omega(n; E)} = \sum_{d \leq x} h(d)[x/d] \leq x \sum_{d \leq x} h(d)d^{-1} \leq x \prod_{p \leq x} \{1 + h(p)p^{-1}\}.$$

This yields:

(3.10) LEMMA. *If* $z \geq 1$ *and* $x \geq 1$, *then* $\sum_{n \leq x} z^{\omega(n; E)} \leq xe^{(z-1)E(x)}$.

In this proof, we estimated a sum of the form $\sum_{n \leq x} f(n)$ (where $f(n)$ is real-valued and multiplicative) by writing $f(n) = \sum_{d \mid n} h(d)$ and using the non-negativity of $h(n)$, which follows (in general) from the assumption that $1 \leq f(p) \leq f(p^2) \leq \cdots$ for each prime p. This idea was used previously by Barban [1]. (The results of this paper are summarized in [2, pp. 98–100]. I am indebted to Professor Elliott for these references.) Although Barban's results do not seem to be applicable to the problems considered here, we can use the same basic idea to obtain:

(3.11) LEMMA. *Let* p_1 *be the smallest member of* E. *If* $x \geq 1$ *and* $1 \leq z < p_1$, *then*

$$\sum_{n \leq x} z^{\Omega(n; E)} < xG_1(x, z)e^{(z-1)E(x)+4z},$$

where

$$G_1(x, z) = \min\left\{1 + \frac{\log x}{\log p_1}, \frac{p_1 - 1}{p_1 - z}\right\}.$$

The proof begins as before by expressing $z^{\Omega(n; E)}$ in the form $\sum_{d \mid n} h(d)$. The assumption $z \geq 1$ implies that $h(n) \geq 0$ for all n. Writing $x_p = (\log x)/\log p - 1$ and using the multiplicativity of h, we get

$$\begin{aligned} \sum_{n \leq x} z^{\Omega(n; E)} &\leq x \sum_{n \leq x} h(n)n^{-1} \\ &\leq x \prod_{p \leq x} \sum_{0 \leq a \leq x_p + 1} h(p^a)p^{-a} \\ &= x \prod_{p \leq x}{}' \left\{1 + (z - 1)p^{-1} \sum_{0 \leq a \leq x_p} (z/p)^a\right\}, \end{aligned} \tag{3.12}$$

where the $'$ means that the product is taken over primes $p \in E$. We may assume $x \geq p_1$ (otherwise the lemma is trivial). In the last product occurring in (3.12), the term corresponding to $p = p_1$ does not exceed $G_1(x, z)$ (since $z < p_1$ by hypothesis). Thus we get

$$\begin{aligned} \sum_{n \leq x} z^{\Omega(n; E)} &\leq xG_1(x, z) \prod_{p_1 < p \leq x}{}' \left\{1 + \frac{z - 1}{p - z}\right\} \\ &\leq xG_1(x, z) \left\{\prod_{p \leq 2z} p\right\} \exp\left\{(z - 1) \sum_{2z < p \leq x}{}' (p - z)^{-1}\right\}. \end{aligned} \tag{3.13}$$

Now if $p > 2z$, then

$$(p - z)^{-1} = p^{-1}\{1 + zp^{-1}(1 - zp^{-1})^{-1}\} < p^{-1}\{1 + 2zp^{-1}\}.$$

Furthermore,

$$\sum_{2z<p\leq x}' p^{-2} < \sum_{n>2z} n^{-2} < (2z)^{-2} + \int_{2z}^{+\infty} t^{-2}\,dt = (2z)^{-2} + (2z)^{-1}.$$

Combining these estimates with (3.13), we get

$$\sum_{n\leq x} z^{\Omega(n;\,E)} \leq xG_1(x, z)\left\{\prod_{p\leq 2z} p\right\} e^{(z-1)E(x)+z}.$$

But by a well-known elementary lemma (see [27, p. 109]), $\prod_{p\leq y} p < 4^y$ for $y \geq 1$, and the proof of Lemma (3.11) is complete.

It is possible to obtain a result somewhat like Lemma (3.11) for the case $z > p_1$, but the proof is more complicated, and we have no application for the result.

Lemmas (3.10) and (3.11) can be used to give certain extensions of (3.5). For example, it follows from Lemma (3.10) and the obvious inequality

$$\sum_{n\leq x} z^{\omega(n;\,E)} \geq z^m N(m, x;\, E, \omega) \quad (m = 0, 1, 2, \ldots)$$

that

(3.14) $$N(m, x\colon E, \omega) \leq xe^{-E(x)}(eE(x)/m)^m \quad \text{for } m \geq E(x).$$

This inequality is only slightly weaker than (3.5) and does not require the assumption $m \leq (2 - \delta)E(x)$. A similar method (using Lemma (3.11)) leads to various upper bounds for $N(m, x;\, E, \Omega)$ when $m \geq E(x)$, but the results and the proofs are more complicated, and we shall not discuss them here. Unfortunately, these methods give no information about possible extensions of (3.6).

When $0 < z < 1$, the method used to prove Lemmas (3.10) and (3.11) will no longer work (since the functions $h(n)$ are sometimes negative), and a completely different method seems to be required. In this case, we can use a recent elementary theorem of Hall [17]. As an immediate corollary of his result, we get

(3.15) $$\sum_{n\leq x} z^{\Omega(n;\,E)} \leq \sum_{n\leq x} z^{\omega(n;\,E)} \ll xe^{(z-1)E(x)} \quad \text{for } x \geq 1,\ 0 < z \leq 1.$$

This goes a bit further than what we could get from Lemma (3.8) and is again much simpler to prove. However, we shall make only a minor application of (3.15) (in the proof of Lemma (6.20)).

It is also possible to establish lower bounds for the sums we have considered. It turns out that if z is not too large, and if $E(x) \to +\infty$ as $x \to +\infty$, then the upper bounds given in Lemmas (3.10) and (3.11) cannot be improved significantly. We suppress the details since we have no application for this fact.

Quite a bit of work has been done previously on the estimation of the sums considered in Lemmas (3.10) and (3.11) (or more general sums). We shall not give a complete survey of this work but refer, for example, to Selberg [36, Theorem 2], Bateman [3], Delange [7], [10, Theorem B], Wirsing [41], [42], and Levin and Fainleib [28], [29], [30]. All of these authors obtained asymptotic formulas or asymptotic expansions for the sums in question (often for complex z), but only under special assumptions about the set E (in particular, about its distribution). Halász [14, Theorems 2, 3] appears to be the only previous author to give results such as (3.9) in which nothing is assumed about E.

4. Partial sums of the exponential series

It is clear from the remarks in Section 3 (cf. especially (3.1) and Lemma (3.3)) that we need to estimate certain partial sums of the Taylor series for e^x. In this section, we shall obtain estimates which are elementary but quite precise. In stating these, we make use of the function $Q(\alpha)$ defined by (2.2).

The first result is known (see [16, p. 149]), but the proof is so simple that we include it for completeness.

(4.1) LEMMA. *Let v, α be real with $v > 0$. Then*

$$\sum_{0 \le m \le (1-\alpha)v} \frac{e^{-v}v^m}{m!} \le e^{Q(-\alpha)v} \quad \text{for } 0 \le \alpha \le 1, \tag{4.2}$$

$$\sum_{m \ge (1+\alpha)v} \frac{e^{-v}v^m}{m!} < e^{Q(\alpha)v} \quad \text{for } \alpha \ge 0. \tag{4.3}$$

Proof. If $0 < w \le v$, then

$$\sum_{0 \le m \le w} \frac{v^m}{m!} = \sum_{0 \le m \le w} \frac{w^m}{m!}\left(\frac{v}{w}\right)^m \le \left(\frac{v}{w}\right)^w \sum_{0 \le m \le w} \frac{w^m}{m!} < \left(\frac{ev}{w}\right)^w.$$

This yields (4.2) for $\alpha < 1$, and (4.2) is trivial for $\alpha = 1$. The proof of (4.3) is similar. Q.E.D.

Lemma (4.1) can be improved for most values of α. To obtain the improvements, we shall use Stirling's formula stated in the form of two inequalities (see [22, p. 529]):

$$n^n e^{-n}(2\pi n)^{1/2} < n! < n^n e^{-n}(2\pi n)^{1/2} e^{1/12n} \quad \text{for } n = 1, 2, \ldots. \tag{4.4}$$

(4.5) LEMMA. *Let v, α be real with $v > 0$, $0 < \alpha < 1$. Then*

$$\sum_{0 \le m \le (1-\alpha)v} \frac{e^{-v}v^m}{m!} < \alpha^{-1}(1-\alpha)^{-1/2}v^{-1/2}e^{Q(-\alpha)v}.$$

Proof. The result is trivial if $(1 - \alpha)v < 1$, since $Q(-\alpha) > Q(-1) = -1$ by Lemma (2.1). Suppose $(1 - \alpha)v \geq 1$, and let $n = [(1 - \alpha)v]$. Then

$$\begin{aligned}\sum_{0 \leq m \leq (1-\alpha)v} \frac{v^m}{m!} &= \frac{v^n}{n!} \sum_{l=0}^{n} \frac{n!}{(n-l)!\, v^l} \\ &\leq \frac{v^n}{n!} \sum_{l=0}^{n} \left(\frac{n}{v}\right)^l \\ &< \frac{v^n}{n!} \sum_{l=0}^{\infty} (1-\alpha)^l \\ &= \frac{v^n}{n!\,\alpha} \\ &< (ev/n)^n \alpha^{-1} (2\pi n)^{-1/2}\end{aligned}$$

by (4.4). But if $v > 0$ is fixed, then $(ev/w)^w$ increases with w for $0 < w \leq v$, so

$$\left(\frac{ev}{n}\right)^n \leq \left(\frac{ev}{(1-\alpha)v}\right)^{(1-\alpha)v} = e^{v + Q(-\alpha)v}. \qquad \text{Q.E.D.}$$

The next result shows that Lemma (4.5) is virtually best possible provided α is not too close to 0 or 1.

(4.6) LEMMA. *Let v, α be real with $v \geq 6$, $v^{-1/2} \leq \alpha \leq 1 - 3v^{-1}$. Set*

$$n = [(1-\alpha)v] - 1, \quad \gamma = \{\alpha(1-\alpha)v\}^{-1}$$

(so $0 < \gamma < 1$). Then

$$\sum_{(1-\gamma)n \leq m \leq n} \frac{e^{-v} v^m}{m!} \gg \alpha^{-1}(1-\alpha)^{3/2} v^{-1/2} e^{Q(-\alpha)v}.$$

Proof. Write $\beta = n(1-\gamma)v^{-1}$, so $0 < \beta < 1$. Then

$$\begin{aligned}\sum_{(1-\gamma)n \leq m \leq n} \frac{v^m}{m!} &= \frac{v^n}{n!} \sum_{0 \leq m \leq \gamma n} \frac{n!}{(n-m)!\, v^m} \\ &\geq \frac{v^n}{n!} \sum_{0 \leq m \leq \gamma n} \left(\frac{n-m+1}{v}\right)^m \\ &\geq \frac{v^n}{n!} \sum_{0 \leq m \leq \gamma n} \beta^m \\ &= \frac{v^n}{n!} \cdot \frac{1 - \beta^{[\gamma n]+1}}{1-\beta}.\end{aligned}$$

Now, $n > (1-\alpha)v - 2 \geq (1-\alpha)v/3$, so

$$\log \beta^{[\gamma n]+1} < \gamma n \log \beta < \gamma n \log(1-\alpha) < -\gamma n \alpha < -\gamma\alpha(1-\alpha)v/3 = -1/3.$$

Also,

$$\begin{aligned}\beta &> (1 - \alpha - 2v^{-1})(1 - \gamma) \\ &> 1 - \alpha - 2v^{-1} - \gamma(1 - \alpha) \\ &> 1 - \alpha - 3(\alpha v)^{-1} \\ &\geq 1 - 4\alpha\end{aligned}$$

since $\alpha \geq v^{-1/2}$. Collecting these results and using (4.4), we obtain

$$\begin{aligned}\sum_{(1-\gamma)n \leq m \leq n} \frac{v^m}{m!} &\geq \frac{v^n}{n!} \cdot \frac{1 - e^{-1/3}}{4\alpha} \\ &\gg \frac{v^{n+2}}{\alpha(n + 2)!} \cdot \frac{(n + 2)(n + 1)}{v^2} \\ &\gg \alpha^{-1}(1 - \alpha)^2 \cdot \frac{v^{n+2}}{(n + 2)!} \\ &\gg \alpha^{-1}(1 - \alpha)^{3/2} v^{-1/2} \left(\frac{ev}{n + 2}\right)^{n+2}.\end{aligned}$$

But

$$(1 - \alpha)v \leq n + 2 \leq v(1 - \alpha + v^{-1}) \leq v(1 - v^{-1/2} + v^{-1}) < v,$$

and since $(ev/w)^w$ increases with w for $0 < w \leq v$, we have

$$\left(\frac{ev}{n + 2}\right)^{n+2} \geq \left(\frac{ev}{(1 - \alpha)v}\right)^{(1-\alpha)v},$$

which completes the proof.

(4.7) LEMMA. *Let v, α be real and positive. Then*

$$\sum_{m \geq (1+\alpha)v} \frac{e^{-v} v^m}{m!} < (2\pi)^{-1/2} \alpha^{-1} (1 + \alpha)^{1/2} v^{-1/2} e^{Q(\alpha)v}.$$

Proof. Let n be the smallest integer such that $n \geq (1 + \alpha)v$. Using (4.4), we get

$$\begin{aligned}\sum_{m=n}^{\infty} \frac{v^m}{m!} &< \frac{v^n}{n!} \sum_{l=0}^{\infty} \left(\frac{v}{n}\right)^l \\ &\leq \frac{v^n}{n!} \sum_{l=0}^{\infty} (1 + \alpha)^{-l} \\ &< (2\pi)^{-1/2} \alpha^{-1} (1 + \alpha)^{1/2} v^{-1/2} (ev/n)^n.\end{aligned}$$

But if $v > 0$ is fixed, then $(ev/w)^w$ is a decreasing function of w for $w \geq v$. Hence

$$\left(\frac{ev}{n}\right)^n \leq \left(\frac{ev}{(1 + \alpha)v}\right)^{(1+\alpha)v} = e^{v + Q(\alpha)v}. \qquad \text{Q.E.D.}$$

We now show that the inequality of Lemma (4.7) is essentially best possible if α is neither too small nor too large.

(4.8) LEMMA. *Let v, α be real with $v \geq 1$, $\alpha \geq v^{-1/2}$. Set*

$$n = [(1 + \alpha)v] + 1, \quad \gamma = (2v\alpha)^{-1}.$$

Then

$$\sum_{n \leq m \leq (1+\gamma)n} \frac{e^{-v}v^m}{m!} \gg \alpha^{-1}(1 + \alpha)^{-1/2}v^{-1/2}e^{Q(\alpha)v}.$$

Proof. Write $\beta = v/n(1 + \gamma)$, so $0 < \beta < 1$. Then

$$\begin{aligned}\sum_{n \leq m \leq (1+\gamma)n} \frac{v^m}{m!} &= \frac{v^n}{n!} \sum_{0 \leq m \leq \gamma n} \frac{n!\, v^m}{(n + m)!} \\ &\geq \frac{v^n}{n!} \sum_{0 \leq m \leq \gamma n} \beta^m \\ &= \frac{v^n}{n!} \cdot \frac{1 - \beta^{[\gamma n]+1}}{1 - \beta}.\end{aligned}$$

Since $\log(1 + \alpha) \geq \alpha(1 + \alpha)^{-1}$, we have

$$\log \beta^{[\gamma n]+1} < \gamma n \log \beta < -\gamma n \log(1 + \alpha) \leq -\gamma v\alpha = -1/2.$$

Furthermore,

$$\begin{aligned}\beta^{-1} &\leq v^{-1}\{1 + (2v\alpha)^{-1}\}\{(1 + \alpha)v + 1\} \\ &< 1 + \alpha + 2v^{-1} + (\alpha v)^{-1} \\ &\leq 1 + 4\alpha\end{aligned}$$

since $v^{-1} \leq \alpha$ and $(\alpha v)^{-1} \leq \alpha$. Thus $(1 - \beta)^{-1} \gg \alpha^{-1}(1 + \alpha)$. Collecting these estimates and using (4.4), we obtain

$$\begin{aligned}\sum_{n \leq m \leq (1+\gamma)n} \frac{v^m}{m!} &\gg \alpha^{-1}(1 + \alpha)\frac{v^n}{n!} \\ &\gg \alpha^{-1} \frac{v^{n-1}}{(n - 1)!} \\ &\gg \alpha^{-1}(1 + \alpha)^{-1/2}v^{-1/2}\left(\frac{ev}{n - 1}\right)^{n-1}.\end{aligned}$$

But $v \leq (1 + \alpha)v - 1 < n - 1 \leq (1 + \alpha)v$, and since $(ev/w)^w$ is a decreasing function of w for $w \geq v$, we get the desired result.

5. The distribution of $\omega(n;E)$ and $\Omega(n;E)$ for arbitrary E

Throughout this section, E denotes any nonempty set of primes, and (as always) $E(x)$ is defined by (1.1). Suppose that either $g(n) = \omega(n; E)$ (for all n) or $g(n) = \Omega(n; E)$ (for all n). For any real x, α, we define

(5.1) $\quad L(x, \alpha; E, g) = \text{card}\,\{n: n \leq x \text{ and } g(n) \leq (1 - \alpha)E(x)\},$

(5.2) $$R(x, \alpha; E, g) = \text{card}\,\{n: n \leq x \text{ and } g(n) \geq (1 + \alpha)E(x)\}.$$

If $A(x, \alpha; E, g)$ is defined by (1.6), then clearly

(5.3) $$A(x, \alpha; E, g) \leq L(x, \alpha; E, g) + R(x, \alpha; E, g),$$

with equality if $\alpha > 0$ and $E(x) > 0$. Also,

(5.4) $$A(x, \alpha; E, g) = R(x, \alpha; E, g) \quad \text{if } \alpha > 1 \text{ and } E(x) > 0.$$

These results enable us to estimate $A(x, \alpha; E, g)$ by estimating $L(x, \alpha; E, g)$ and $R(x, \alpha; E, g)$ separately. To carry out this program, we define $N(m, x; E, g)$ by (3.4) and note the following obvious formulas (for any real x, α):

(5.5) $$L(x, \alpha; E, g) = \sum_{0 \leq m \leq (1-\alpha)E(x)} N(m, x; E, g),$$

(5.6) $$R(x, \alpha; E, g) = \sum_{m \geq (1+\alpha)E(x)} N(m, x; E, g).$$

Observing that the inequality $\omega(n; E) \leq \Omega(n; E)$ implies

(5.7) $$L(x, \alpha; E, \Omega) \leq L(x, \alpha; E, \omega),$$

(5.8) $$R(x, \alpha; E, \omega) \leq R(x, \alpha; E, \Omega),$$

we now proceed to estimate these four quantities in various ways.

(5.9) THEOREM. *If $x \geq 1$ and $0 \leq \alpha \leq 1$, then*

$$L(x, \alpha; E, \Omega) \leq L(x, \alpha; E, \omega) \ll xe^{Q(-\alpha)E(x)}.$$

Proof. This is trivial if $E(x) = 0$. Otherwise, simply combine (5.5), (3.5) (with $\delta = 1$), and (4.2). Q.E.D.

(5.10) THEOREM. *If $E(x) > 0$ and $0 < \alpha < 1$, then*

$$L(x, \alpha; E, \Omega) \leq L(x, \alpha; E, \omega) \ll \alpha^{-1}(1 - \alpha)^{-1/2}xE(x)^{-1/2}e^{Q(-\alpha)E(x)}.$$

Proof. Combine (5.5), (3.5) (with $\delta = 1$), and Lemma (4.5). Q.E.D.

The next result shows that Theorem (5.10) is almost best possible.

(5.11) THEOREM. *Suppose that $E(z) \to +\infty$ as $z \to +\infty$. Then there is a number $c_{21}(E)$ such that if $x \geq c_{21}(E)$ and $E(x)^{-1/2} \leq \alpha \leq 1 - 3E(x)^{-1}$, we have*

$$L(x, \alpha; E, \omega) \geq L(x, \alpha; E, \Omega) \gg \alpha^{-1}(1 - \alpha)^{3/2}xE(x)^{-1/2}e^{Q(-\alpha)E(x)}.$$

Proof. Write $E(x) = v$, $n = [(1 - \alpha)v] - 1$. By (5.5),

$$L(x, \alpha; E, \Omega) \geq \sum_{m=0}^{n} N(m, x; E, \Omega), \quad L(x, \alpha; E, \Omega) \geq \sum_{m=0}^{n} N(m + 1, x; E, \Omega),$$

so

$$L(x, \alpha; E, \Omega) \gg \sum_{m=0}^{n} \{N(m, x; E, \Omega) + N(m + 1, x; E, \Omega)\}.$$

The theorem now follows from (3.6) (with $\delta = 1$) and Lemma (4.6).

If $0 \leq \alpha \leq E(x)^{-1/2} = \alpha_0$, and if the other hypotheses of Theorem (5.11) hold, then by Theorem (5.11) and (2.3), we have

$$L(x, \alpha; E, \Omega) \geq L(x, \alpha_0; E, \Omega) \gg x,$$

which is best possible except for the undetermined constant factor. More precise results (when α is small) will be obtained in [31].

(5.12) THEOREM. *Suppose $E(x) > 0$ and $0 < \alpha \leq \beta < 1$. Then*

$$R(x, \alpha; E, \omega) \leq R(x, \alpha; E, \Omega) \leq c_{22}(\beta)\alpha^{-1}xE(x)^{-1/2}e^{Q(\alpha)E(x)}.$$

Proof. Let $\gamma = (\beta + 1)/2$, so $\beta < \gamma < 1$. By (5.6),

$$R(x, \alpha; E, \Omega) = \sum_{(1+\alpha)E(x) \leq m < (1+\gamma)E(x)} N(m, x; E, \Omega) + R(x, \gamma; E, \Omega). \tag{5.13}$$

First we estimate $R(x, \gamma; E, \Omega)$ using Halász's inequality (3.9) (Lemma (3.11) would do just as well). For $\delta > 0$ and $1 \leq z \leq 2 - \delta$, we have

$$z^{(1+\gamma)E(x)}R(x, \gamma; E, \Omega) \leq \sum_{n \leq x} z^{\Omega(n; E)} \leq c_{19}(\delta)xe^{(z-1)E(x)},$$

so

$$R(x, \gamma; E, \Omega) \leq c_{19}(\delta)x \exp\{(z - 1)E(x) - (1 + \gamma)E(x) \log z\}.$$

The right-hand side is minimized by taking $z = 1 + \gamma$, which is permissible since $\gamma < 1$. We get

$$R(x, \gamma; E, \Omega) \leq c_{23}(\beta)xe^{Q(\gamma)E(x)}. \tag{5.14}$$

Combining (5.13), (5.14), (3.5), and Lemma (4.7), we get

$$R(x, \alpha; E, \Omega) \leq c_{24}(\beta)\alpha^{-1}xE(x)^{-1/2}e^{Q(\alpha)E(x)}\{1 + E(x)^{1/2}e^{\{Q(\gamma)-Q(\alpha)\}E(x)}\}.$$

But by Lemma (2.1), Q is strictly decreasing on $[0, +\infty)$, so $Q(\alpha) \geq Q(\beta) > Q(\gamma)$ and

$$e^{\{Q(\alpha)-Q(\gamma)\}E(x)} \geq 1 + \{Q(\beta) - Q(\gamma)\}E(x).$$

If we use Lemma (3.11) instead of (3.9), then the method used to prove (5.14) yields

$$R(x, \alpha; E, \Omega) \leq (p_1 - 1)(p_1 - 1 - \alpha)^{-1}xe^{Q(\alpha)E(x)+4(1+\alpha)} \tag{5.15}$$

for $x \geq 1$ and $0 \leq \alpha < p_1 - 1$ (where p_1 is the smallest member of E). Similarly, Lemma (3.10) yields

$$R(x, \alpha; E, \omega) \leq xe^{Q(\alpha)E(x)} \quad \text{for } x \geq 1, \alpha \geq 0. \tag{5.16}$$

Q.E.D.

(5.17) THEOREM. *Suppose that $E(z) \to +\infty$ as $z \to +\infty$. Let $0 < \beta < 1$. Then there is a number $c_{25}(\beta, E)$ such that if $x \geq c_{25}(\beta, E)$ and $E(x)^{-1/2} \leq \alpha \leq \beta$, we have*

$$R(x, \alpha; E, \Omega) \geq R(x, \alpha; E, \omega) \geq c_{26}(\beta)\alpha^{-1}xE(x)^{-1/2}e^{Q(\alpha)E(x)}.$$

Proof. Write $E(x) = v$, $n = [(1 + \alpha)v] + 1$, $\gamma = (2v\alpha)^{-1}$. By (5.6),

$$R(x, \alpha; E, \omega) \gg \sum_{m=n}^{\infty} \{N(m, x; E, \omega) + N(m + 1, x; E, \omega)\} \gg \sum_{n \leq m \leq (1+\gamma)n} . \tag{5.18}$$

Furthermore, if $c_{25}(\beta, E)$ is sufficiently large and $x \geq c_{25}(\beta, E)$, then

$$\begin{aligned}(1 + \gamma)n &\leq v\{1 + \alpha + v^{-1} + (1 + \alpha)(2\alpha v)^{-1} + (2\alpha v^2)^{-1}\}\\ &\leq v\{1 + \beta + v^{-1} + v^{-1/2} + (2v^{3/2})^{-1}\}\\ &\leq v\{1 + \beta + (1 - \beta)/2\}\\ &= v\{2 - (1 - \beta)/2\}.\end{aligned}$$

The result now follows by combining (5.18), (3.6), and Lemma (4.8).

If $E(z) \to +\infty$ as $z \to +\infty$, and if $x \geq c_{27}(E)$ and $0 \leq \alpha \leq E(x)^{-1/2} = \alpha_0$, then by Theorem (5.17) and (2.3), we have $R(x, \alpha; E, \omega) \geq R(x, \alpha_0; E, \omega) \gg x$, which is best possible except for the undetermined constant. For more precise results when α is small, see [31].

The results of this section yield a proof of Theorem (1.5), for (1.7) follows from Theorems (5.10) and (5.12), while (1.9) is a consequence of Theorem (5.17). Also, if $0 \leq \alpha \leq 1$, then (1.11) follows from (5.3), Theorem (5.9), (5.16), and the inequality $Q(-\alpha) \leq Q(\alpha)$ (cf. (2.7)). For $\alpha > 1$ and $E(x) > 0$, (1.11) follows from (5.4) and (5.16), and it is trivial if $E(x) = 0$. When $0 \leq \alpha < p_1 - 1$, an upper bound for $A(x, \alpha; E, \Omega)$ can be deduced similarly, using (5.15) instead of (5.16).

6. Prime factors in arithmetic progressions

There seem to be few known types of sets E for which $E(x)$ is unbounded but can be calculated rather accurately in terms of elementary functions of x. In this section, we obtain analogues of the results of Section 5 for one such type of set, which consists of the primes in various arithmetic progressions with the same modulus.

Consider first the case of a single arithmetic progression. Let k, l be integers with $k \geq 1$, $(k, l) = 1$, and let E be the set of all primes $p \equiv l \pmod{k}$. The estimate

$$E(x) = \phi(k)^{-1} \log_2 x + B_{k,l} + O_k((\log x)^{-1}) \quad (x \geq 2) \tag{6.1}$$

was proved by Mertens in 1874 (see Landau [26, pp. 41–43, 449–450]). Here $B_{k,l}$ is a number depending only on k and l. After de la Vallée Poussin proved a strong form of the prime number theorem for arithmetic progressions (which was slightly improved by Landau), it was possible to replace (6.1) by

$$E(x) = \phi(k)^{-1} \log_2 x + B_{k,l} + O_k(\exp\{-c_{28}(\log x)^{1/2}\}) \tag{6.2}$$

for $x \geq 2$ (where c_{28} is a positive absolute constant). Both of these results have the disadvantage of giving no information about the size of the error term as a function of k, nor about the magnitude of the number $B_{k,l}$. It was shown by Bateman, Chowla, and Erdös [4, Lemma 3] that (roughly speaking) a result like (6.2) holds with an implied constant which is absolute (i.e., O_k can be replaced by O) provided that x is sufficiently large compared to k, but only if k does not belong to a small exceptional set. They also did not estimate $B_{k,l}$.

In view of all this, the result (1.12) seems to be of intrinsic interest, and it has applications to the problems discussed in Section 5. It can be generalized as follows:

(6.3) LEMMA. *Let k be a positive integer, and let L be a nonempty set of integers such that*

$$1 \leq l \leq k \text{ and } (k, l) = 1 \quad \text{for each } l \in L. \tag{6.4}$$

Write card $L = \lambda$, *and let*

$$E = \bigcup_{l \in L} \{p\colon p \equiv l \pmod{k}\}.$$

Then for $x \geq 2$,

$$E(x) = \lambda\phi(k)^{-1} \log_2 x + \sum_{p \leq x,\, p \in L} p^{-1} + O(\lambda\phi(k)^{-1} \log(3k)). \tag{6.5}$$

Also,

$$\sum_{p \leq x,\, p \in L} p^{-1} \leq \log_2 (\lambda + 3) + O(1). \tag{6.6}$$

Proof. First, it is clear that

$$E(x) = \sum_{p \leq x,\, p \in L} p^{-1} \quad \text{if } 2 \leq x \leq k. \tag{6.7}$$

Next, let $\pi(x; k, l)$ be the number of primes p such that $p \leq x$ and $p \equiv l \pmod{k}$, and let

$$\pi(x; E) = \sum_{p \leq x,\, p \in E} 1 = \sum_{l \in L} \pi(x; k, l).$$

If $k \leq x$, then the Brun-Titchmarsh inequality (see Prachar [32, p. 44]) yields

$$\pi(x; k, l) \ll \frac{x}{\phi(k) \log(3x/k)},$$

and hence

$$\begin{aligned} 0 &\leq E(x) - E(k) \\ &= \int_k^x t^{-1}\, d\pi(t; E) \\ &\leq x^{-1}\pi(x; E) + \int_k^x t^{-2}\pi(t; E)\, dt \\ &\ll \lambda\phi(k)^{-1} \log_2 (3x/k). \end{aligned}$$

Applying (6.7), we obtain

$$(6.8)\quad E(x) = \sum_{p\le x,\, p\in L} p^{-1} + O(\lambda\phi(k)^{-1}\log(3k)) \quad \text{if } k \le x \le \exp k^2.$$

Now suppose that $x \ge \exp k^2 = B$, say. Then the Siegel-Walfisz theorem (see Prachar [32, p. 144] or Davenport [5, p. 136]) yields the estimate

$$(6.9)\qquad \pi(x;k,l) = \phi(k)^{-1}Li(x) + O(x\exp\{-c_{29}(\log x)^{1/2}\})$$

whenever $(k, l) = 1$, where Li is defined by

$$Li(y) = \int_2^y (\log t)^{-1}\,dt \quad \text{for } y > 1.$$

If we write

$$\pi(t;E) = \lambda\phi(k)^{-1}Li(t) + \Delta(t;E) \quad \text{for } t \ge B,$$

then by (6.9), $\Delta(t; E) \ll \lambda t \exp\{-c_{29}(\log t)^{1/2}\}$, and an easy calculation gives

$$\begin{aligned} E(x) - E(B) &= \int_B^x \lambda\{\phi(k)t\log t\}^{-1}\,dt + \int_B^x t^{-1}\,d\Delta(t;E) \\ &= \lambda\phi(k)^{-1}\log_2 x + O(\lambda\phi(k)^{-1}\log(3k)). \end{aligned}$$

It follows from this and (6.8) that (6.5) holds for $x \ge B$. But for $2 \le x \le B$, we have $\lambda\phi(k)^{-1}\log_2 x = O(\lambda\phi(k)^{-1}\log(3k))$, so by (6.7) and (6.8), (6.5) holds for all $x \ge 2$.

Finally, let P_r denote the rth prime ($P_1 = 2$). Then clearly

$$\sum_{p\le x,\, p\in L} p^{-1} \le \sum_{r=1}^{\lambda} P_r^{-1} = \log_2 P_\lambda + O(1) \le \log_2(\lambda + 3) + O(1),$$

and the proof is complete.

If λ is close to $\phi(k)$, we can get a better result than (6.5). In fact, if we continue to use the notation of Lemma (6.3) and let

$$L_1 = \{l: 1 \le l \le k \text{ and } (k,l) = 1\} - L,$$

then for $x \ge 2$,

$$\begin{aligned} (6.10)\quad E(x) = \lambda\phi(k)^{-1}\log_2 x &- \sum_{p\le x,\, p\in L_1} p^{-1} - \sum_{p\le x,\, p\mid k} p^{-1} \\ &+ O(1 + \{\phi(k) - \lambda\}\phi(k)^{-1}\log(3k)). \end{aligned}$$

This is well known if $\lambda = \phi(k)$, and if $1 \le \lambda \le \phi(k) - 1$, it follows in an obvious way from Lemma (6.3).

It is interesting to observe that the implied constant in (6.5) can actually be computed (although we have made no attempt to do so). For it follows from [5, pp. 127–128] that when $x \ge \exp k^2$, both the number c_{29} and the implied constant in (6.9) are effectively computable. (I owe this remark to Professor H. Halberstam.)

If there is no "exceptional" character χ (mod k) for which the Dirichlet L-function $L(s, \chi)$ has a large real zero, then for each real $u > 0$, we have

$$\pi(x; k, l) = \phi(k)^{-1}Li(x) + O_u(x \exp\{-c_{30}(u)(\log x)^{1/2}\})$$

whenever $(k, l) = 1$ and $x \geq \max\{2, \exp(u \log^2 k)\}$. (See [5, p. 127] or [32, pp. 136–138].) In this case, we can obtain the following improvement of (6.5) for each $x \geq 2$:

$$E(x) = \lambda\phi(k)^{-1} \log_2 x + \sum_{p \leq x,\, p \in L} p^{-1} + O(\lambda\phi(k)^{-1} \log_2(3k)).$$

The proof is the same as before except that we now take $B = \exp \log^4 (3k)$.

We now introduce some notation which will be used throughout the remainder of this section.

(6.11) k, L, λ, E satisfy the hypotheses of Lemma (6.3),

$$v = \lambda\phi(k)^{-1} \log_2 x \quad (x \geq 3), \tag{6.12}$$

$$w = \sum_{p \leq x,\, p \in L} p^{-1}, \tag{6.13}$$

$$y = \lambda\phi(k)^{-1} \log(3k). \tag{6.14}$$

If $g(n) = \omega(n; E)$ (for all n) or $g(n) = \Omega(n; E)$ (for all n), then for any real x, α with $x \geq 3$, we define

$$L_1(x, \alpha; E, g) = \text{card}\{n: n \leq x \text{ and } g(n) \leq (1 - \alpha)v\}, \tag{6.15}$$

$$R_1(x, \alpha; E, g) = \text{card}\{n: n \leq x \text{ and } g(n) \geq (1 + \alpha)v\}, \tag{6.16}$$

$$A_1(x, \alpha; E, g) = \text{card}\{n: n \leq x \text{ and } |g(n) - v| \geq \alpha v\}. \tag{6.17}$$

With these definitions, the obvious analogues of (5.3) and (5.4) hold. If we continue to define $N(m, x; E, g)$ by (3.4) (subject to (6.11)), then (5.5) and (5.6) must be replaced by

$$L_1(x, \alpha; E, g) = \sum_{0 \leq m \leq (1-\alpha)v} N(m, x; E, g), \tag{6.18}$$

$$R_1(x, \alpha; E, g) = \sum_{m \geq (1+\alpha)v} N(m, x; E, g). \tag{6.19}$$

Now, it is possible to derive theorems on the sizes of $L_1(x, \alpha; E, g)$ and $R_1(x, \alpha; E, g)$ directly from the corresponding theorems in Section 5, using (6.5). However, this process is awkward because $E(x)$ is not exactly

$$\lambda\phi(k)^{-1} \log_2 x\ (= v),$$

and the results are in some cases not quite as strong as one would wish. Thus it seems preferable to follow the same path as in Section 5, using (6.18), (6.19), the lemmas of Section 4, and the following analogue of Lemma (3.3):

(6.20) LEMMA. *Let* $g(n) = \omega(n; E)$ (*for all* n) *or* $g(n) = \Omega(n; E)$ (*for all* n), *subject to* (6.11). *Let* $0 < \beta \le \delta \le 2$. *If* $x \ge 3$ *and*

$$0 \le m \le (2 - \delta)v, \tag{6.21}$$

then

$$N(m, x; E, g) \le c_{31}(\beta)x \frac{v^m}{m!} e^{-v+(1-\delta)w+c_{32}y}. \tag{6.22}$$

Furthermore, if $x \ge c_{33}(\beta, k)$ *and* (6.21) *holds, then*

$$N(m, x; E, g) + N(m + 1, x; E, g) \ge c_{34}(\beta)x \frac{v^m}{m!} e^{-v-w-c_{35}y}. \tag{6.23}$$

Proof. By Lemma (6.3),

$$v + w - c_{36}y \le E(x) \le v + w + c_{36}y \quad \text{for } x \ge 3, \tag{6.24}$$

where we may assume $c_{36} \ge 1$. We first note that (6.22) is easy to prove if the hypothesis that $x \ge 3$ is replaced by

$$x \ge \exp\exp\{6c_{36}\beta^{-1}\log(3k)\}. \tag{6.25}$$

For by (6.24) and (6.25), $E(x) \ge v - c_{36}y \ge 4c_{36}\beta^{-1}y$, so $v \le E(x)(1 + \beta/4)$, and hence (6.21) implies $m \le (2 - \beta/2)E(x)$. Thus (3.5) yields

$$N(m, x; E, g) \le c_{16}(\beta/2)x \frac{E^m(x)}{m!} e^{-E(x)},$$

and (6.22) follows easily from this with the use of (6.24) and (6.21).

If $c_{33}(\beta, k)$ is sufficiently large, then the inequality $x \ge c_{33}(\beta, k)$ implies (6.25) and $x \ge c_{17}(\beta/2, E)$ (cf. Lemma (3.3)), and (6.23) follows easily from (6.21), (3.6), and (6.24).

It remains to be shown that (6.22) holds under the assumptions (6.21) and

$$3 \le x < \exp\exp\{6c_{36}\beta^{-1}\log(3k)\}. \tag{6.26}$$

If $m = 0$, then (6.22) follows directly from (3.5) and (6.24). Assume from now on that $m \ge 1$. It may not be true that $m \le (2 - \varepsilon)E(x)$ for some $\varepsilon > 0$, so we can no longer use Lemma (3.3). However, if $z > 0$, then

$$\sum_{n \le x} z^{g(n)} = \sum_{m=0}^{\infty} z^m N(m, x; E, g),$$

so by (3.15) and (3.9) (we could use Lemmas (3.10) and (3.11) in place of (3.9)),

$$N(m, x; E, g) \le c_{37}(\beta)x \exp\{(z - 1)E(x) - m\log z\}$$

for $0 < z \le 2 - \beta$. Using (6.24) and taking $z = m/v$ to get an approximate minimum, we obtain

$$N(m, x; E, g) \le c_{38}(\beta)m^{1/2}x \frac{v^m}{m!} e^{-v+(1-\delta)w+c_{39}y}$$

by Stirling's formula. Since (6.21) and (6.26) imply $m < 12c_{36}\beta^{-1}y$, we get (6.22). Q.E.D.

Using Lemma (6.20) instead of Lemma (3.3), we can prove the next four theorems in the same way as the corresponding theorems in Section 5. The introduction of the parameter β in Lemma (6.20) makes possible the uniform estimates given in Theorems 6.27 and 6.28, since we can take $\beta = 1, \delta = 1 + \alpha$ in the proofs.

(6.27) THEOREM. *If $x \geq 3$ and $0 < \alpha < 1$, then*

$$L_1(x, \alpha; E, \Omega) \leq L_1(x, \alpha; E, \omega) \ll \alpha^{-1}(1 - \alpha)^{-1/2}xv^{-1/2}e^{Q(-\alpha)v - \alpha w + c_{32}y}.$$

(6.28) THEOREM. *There is a number $c_{40}(k)$ such that if $x \geq c_{40}(k)$ and $v^{-1/2} \leq \alpha \leq 1 - 3v^{-1}$, then*

$$L_1(x, \alpha; E, \omega) \geq L_1(x, \alpha; E, \Omega) \gg \alpha^{-1}(1 - \alpha)^{3/2}xv^{-1/2}e^{Q(-\alpha)v - w - c_{35}y}.$$

(6.29) THEOREM. *Suppose $x \geq 3$ and $0 < \alpha \leq \beta < 1$. Then*

$$R_1(x, \alpha; E, \omega) \leq R_1(x, \alpha; E, \Omega) \leq c_{41}(\beta)\alpha^{-1}xv^{-1/2}e^{Q(\alpha)v + w + c_{42}y}.$$

(6.30) THEOREM. *Let $0 < \beta < 1$. Then there is a number $c_{43}(\beta, k)$ such that if $x \geq c_{43}(\beta, k)$ and $v^{-1/2} \leq \alpha \leq \beta$, we have*

$$R_1(x, \alpha; E, \Omega) \geq R_1(x, \alpha; E, \omega) \geq c_{44}(\beta)\alpha^{-1}xv^{-1/2}e^{Q(\alpha)v - w - c_{35}y}.$$

Finally, we note that these results yield some obvious corollaries about the functions $A_1(x, \alpha; E, g)$ defined by (6.17), including a generalization of Theorem (1.13). We shall not state these here.

7. Concluding remarks

It should be pointed out that the methods used in this paper depend heavily on the properties of the particular functions $\omega(n; E)$ and $\Omega(n; E)$. It appears difficult to extend these methods so as to deal with more general additive functions, or even to use similar ideas to deal with problems on the distribution of $\omega(|f(n)|; E)$ and $\Omega(|f(n)|; E)$, where f is a polynomial with integral coefficients. (Turán [38] was apparently the first to attack the latter problem (when E is the set of all primes), and much work has since been done on it; we shall not attempt to discuss it here. See Kubilius [23, Theorems 3.3, 3.4, 4.6, and 4.7], where further references are listed. For more recent work and some additional references, see Elliott [12, Section 6], where a generalization of (1.4) is proved.) In addition, we have nothing to offer on the problem of joint distribution of several functions of the type $\omega(n; E)$ or $\Omega(n; E)$, nor on the problem of the distribution of sums of such functions. (See Kubilius [23] for discussion of certain problems of these types.)

Concerning Theorems (1.5), (1.13), and (1.20), and the corresponding results in Sections 5 and 6, it is natural to ask whether the stated inequalities can be replaced by asymptotic formulas. It turns out that the answer is affirmative if $\alpha = o(1)$ as $x \to +\infty$. This was shown by Kubilius [23, Theorem 9.2] for the function $\omega(n)$ (cf. (1.17) and (1.18) above), and we shall generalize his results (by a method quite different from his) in a forthcoming paper [31]. If $\alpha \neq o(1)$ as $x \to +\infty$, the problem remains open.

Finally, it should be emphasized that although our results on the distribution of $\omega(n; E)$ and $\Omega(n; E)$ are quite sharp in relatively small neighborhoods of $E(x)$, the same methods yield only weaker upper bounds for the frequency of relatively large values of these functions. Thus, for example, we have little information about the precision of (5.15) and (5.16) when $\alpha \geq 1$, since we have obtained no lower bounds in this case. This seems to be a difficult problem, and it may also be difficult to get sharp results on the frequency of large values of $d_m(n)$.

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