A CLASS OF DIFFERENTIAL INEQUALITIES IMPLYING BOUNDEDNESS

BY

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Let B denote the class of bounded functions $w(z) = w_1 z + w_2 z^2 + \cdots$ regular in the unit disc U for which $|w(z)| < 1$. If $g(z) \in B$, then by using the Schwarz lemma we can show that the function $w(z)$ defined by $w(z) = z^{-1/2} \int_0^z g(t) t^{-1/2}\, dt$ is also in B. Writing this result in terms of derivatives we have

$$(1) \qquad |\tfrac{1}{2}w(z) + zw'(z)| < 1,\ z \in U \Rightarrow |w(z)| < 1,\ z \in U.$$

All of the inequalities considered in this paper hold uniformly in the unit disc U, and in what follows we will omit the condition $z \in U$. Furthermore, if we let $h(u, v) = \frac{1}{2}u + v$ we can write (1) as

$$(2) \qquad |h(w(z), zw'(z))| < 1 \Rightarrow |w(z)| < 1.$$

In this note we will show that (2) holds for functions $h(u, v)$ satisfying the following definition.

DEFINITION 1. Let H be the set of complex functions $h(u, v)$ satisfying:

(i) $h(u, v)$ is continuous in a domain D of $\mathbf{C} \times \mathbf{C}$,
(ii) $(0, 0) \in D$ and $|h(0, 0)| < 1$,
(iii) $|h(e^{i\theta}, ke^{i\theta})| \geq 1$ when $(e^{i\theta}, ke^{i\theta}) \in D$, θ is real and $k \geq 1$.

Examples. It is easy to check that each of the following functions is in H:

$$h_1(u, v) = \alpha u + v \text{ where } \alpha \text{ is complex with } \operatorname{Re} \alpha \geq 0, \text{ and } D = \mathbf{C} \times \mathbf{C},$$

$$h_2(u, v) = u^2 + u + v \quad \text{and} \quad D = \mathbf{C} \times \mathbf{C},$$

$$h_3(u, v) = \tfrac{1}{3}(|u| + |v| + 1) \quad \text{and} \quad D = \mathbf{C} \times \mathbf{C},$$

$$h_4(u, v) = 2\, v/(1 - u) \quad \text{and} \quad D = [\mathbf{C} - \{1\}] \times \mathbf{C},$$

$$h_5(u, v) = ue^{|v|} \quad \text{and} \quad D = \mathbf{C} \times \mathbf{C},$$

$$h_6(u, v) = u^m v^n \text{ where } m \text{ and } n \text{ are non-negative integers, and } D = \mathbf{C} \times \mathbf{C}.$$

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The class H is closed with respect to multiplication, and if g, $h \in H$ with either $g(0, 0) = 0$ or $h(0, 0) = 0$ then $g + h \in H$. In addition, if $h \in H$ with $h(0, 0) = 0$, and if α is any complex number such that $|\alpha| \geq 1$ then $\alpha h \in H$.

DEFINITION 2. Let $h \in H$ with corresponding domain D. We denote by $B(h)$ those functions $w(z) = w_1 z + w_2 z^2 + \cdots$ which are regular in U and which satisfy

(i) $(w(z), zw'(z)) \in D$, and
(ii) $|h(w(z), zw'(z))| < 1$,

when $z \in U$.

The set $B(h)$ is not empty since for any $h \in H$ it is true that $w(z) = w_1 z \in B(h)$ for $|w_1|$ sufficiently small (depending on h).

THEOREM 1. *For any $h \in H$, $B(h) \subset B$.*

In other words, the theorem states that if $h \in H$, with corresponding domain D, and if $w(z) = w_1 z + w_2 z^2 + \cdots$ is regular in U and $(w(z), zw'(z)) \in D$ then (2) holds.

Proof. Let $w(z) \in B(h)$ and suppose that $z_0 = r_0 e^{i\varphi_0}$ is a point of U such that $\max_{|z| \leq r_0} |w(z)| = |w(z_0)| = 1$. At such a point, by using a result of I. S. Jack [1, Lemma 1], we must have $z_0 w'(z_0) = kw(z_0)$, where $k \geq 1$. Setting $w(z_0) = e^{i\theta_0}$, where θ_0 is a real number, we have $z_0 w'(z_0) = ke^{i\theta_0}$ and thus $h(w(z_0), z_0 w'(z_0)) = h(e^{i\theta_0}, ke^{i\theta_0})$. Since $h \in H$ this implies that $|h(w(z_0), z_0 w'(z_0))| \geq 1$ which is a contradiction of the fact that $w(z) \in B(h)$. Hence $|w(z)| < 1$ for all $z \in U$, and thus $w(z) \in B$.

If we apply the theorem to h_1, we obtain

$$|\alpha w(z) + zw'(z)| < 1 \Rightarrow |w(z)| < 1,$$

where α is a complex number such that $\operatorname{Re} \alpha \geq 0$. In the special case $\alpha = \frac{1}{2}$ we obtain (1). Applying the theorem to $h_2, h_3, \ldots, h_6$ we obtain respectively

$$|w^2(z) + w(z) + zw'(z)| < 1 \Rightarrow |w(z)| < 1,$$

$$|w(z)| + |zw'(z)| < 2 \Rightarrow |w(z)| < 1,$$

$$w(z) \neq 1 \quad \text{and} \quad 2|zw'(z)|/|1 - w(z)| < 1 \Rightarrow |w(z)| < 1,$$

$$|w(z)|e^{|zw'(z)|} < 1 \Rightarrow |w(z)| < 1,$$

$$|w(z)|^m |zw'(z)|^n < 1 \Rightarrow |w(z)| < 1,$$

where m and n are non-negative integers.

Theorem 1, moreover, can be used to show that certain first order differential equations have bounded solutions. The proof of the following theorem follows immediately from Theorem 1.

THEOREM 2. *Let* $h \in H$ *and* $b(z)$ *be a regular function in* U *with* $|b(z)| < 1$. *If the differential equation*

$$h(w(z), zw'(z)) = b(z) \quad (w(0) = 0)$$

has a solution $w(z)$ regular *in* U *then* $|w(z)| < 1$.

An interesting application of this theorem was suggested to the author by Professor Zeev Nehari and is presented in the following corollary. It is related to a result of M. S. Robertson [2, Theorem 1].

COROLLARY 2.1. *Let* $zp(z)$ *be regular in* U *with* $|zp(z)| < 1$. *Let* $v(z)$, $z \in U$, *be the unique solution of*

$$v''(z) + p(z)v(z) = 0 \tag{3}$$

with $v(0) = 0$ *and* $v'(0) = 1$. *Then*

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < 1, \tag{4}$$

and $v(z)$ *is a starlike conformal map of the unit disc.*

Proof. If we set

$$w(z) = \frac{zv'(z)}{v(z)} - 1$$

for $z \in U$, then $w(z)$ is regular in U, $w(0) = 0$ and (3) becomes

$$w^2(z) + w(z) + zw'(z) = -z^2p(z),$$

or equivalently

$$h_2(w(z), zw'(z)) = -z^2p(z),$$

where $h_2 = u^2 + u + v$. Since $h_2 \in H$ and $|-z^2p(z)| < 1$ we can use Theorem 2 to obtain $|w(z)| < 1$, and combining this with (5) we obtain (4). In particular this implies that Re $zv'(z)/v(z) > 0$ and thus $v(z)$ is a starlike conformal map of the unit disc.

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