THE KERNEL OF THE LOOP SUSPENSION MAP

BY

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Let X be a 1-connected H space such that the Hopf algebra $H^*(X; Z_p)$ has finite type. In this paper we characterize elements of the kernel of the loop map

$$\sigma: QH^q(X; Z_p) \to PH^{q-1}(\Omega X; Z_p)$$

both in terms of restricted types of Massey products and, of more interest, in terms of elementary stable cohomology operations. Basically the main result, Theorem B, states that if $\sigma x = 0$, then either $x \in \beta_k \mathscr{P}^I(u)$ or $x \in \beta_k \mathscr{P}^J \psi_r(v)$, where β_k is the p^kth order Bockstein, $\mathscr{P}^I$ and $\mathscr{P}^J$ are particularly simple primary operations, ψ_r is a specific secondary cohomology operation, and u and v are indecomposable cohomology classes of $H^*(X; Z_p)$. One of the applications is a characterization of differentials in certain spectral sequences in terms of these stable cohomology operations.

Section 1

Let $H_*(X)$ and $H^*(X)$ denote mod p singular homology and cohomology theories for a fixed prime p. If $\pi: PX \to X$ is the standard path space fibration with fiber the loop space ΩX, then the loop suspension map is the composite $\sigma = \delta^{-1}\pi^* j^{*-1}$:

$$H^q(X) \underset{\sim}{\xleftarrow{j^{*-1}}} H^q(X, x_0) \xrightarrow{\pi^*} H^q(PX, \Omega X) \underset{\sim}{\xleftarrow{\delta^{-1}}} \tilde{H}^{q-1}(\Omega X) \quad \text{for } q \geq 1.$$

This map was first introduced by Eilenberg and MacLane in their study of the relation between $K(\pi, n)$ and $K(\pi, n-1)$ [5]. It was generalized by Serre [23] to arbitrary fibrations. G. W. Whitehead [29] showed that σ annihilates decomposables of $H^*(X)$ and that $\operatorname{Im} \sigma \subset PH^*(\Omega X)$, the submodule of primitives in the Hopf algebra $H^*(\Omega X)$. Thus σ extends to a homomorphism:

$$\sigma: QH^q(X) \to PH^{q-1}(\Omega X).$$

The fact that σ annihilates decomposables was generalized to the statement $\sigma\langle u_1, \ldots, u_n\rangle = \{0\}$ for all Massey products [11]. Conversely J. P. May showed that every element of Ker σ belongs to some canonically defined matric Massey product (MMP) (see [19] or [8]).

If X is an H space, then $H^*(X)$ is a commutative Hopf algebra. Thus there are only a few multiplicative relations in $H^*(X)$. This means that there are only

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a few MMP's which can be defined. In fact Browder [3], Clark [4], and others have shown that σ is a monomorphism unless $q \equiv 2 \pmod{p}$ (compare with Theorem 2.4).

Our first major theorem combines these results to characterize the kernel of σ in terms of Massey products for a 1-connected H space X of finite type. In Section 2 we will define a p^k-fold MMP $\mu(u, p^k)$ of dimension $2p^k m + 2$ for $u \in H^{2m+1}(X)$ and a $2p^k$-fold MMP $\mu(p^r, v, p^k)$ of dimension $2p^k(p^r s - 1) + 2$ for $v \in H^{2s}(X)$ with $v^{p^r} = 0$. If defined, then the p^k-fold Massey product $\langle u, \ldots, u\rangle$ is a subset of $\mu(u, p^k)$. Similarly, if defined, then the $2p^k$-fold Massey product $\langle v, v^{p^r-1}, \ldots, v, v^{p^r-1}\rangle$ is a subset of $\mu(p^r, v, p^k)$. In fact, it is probable that equality holds as subsets of $QH^*(X)$ whenever either side is defined.

THEOREM A. *Let X be a* 1-*connected H space of finite type. Then* Ker σ *is generated by MMP's of the form $\mu(u, p^k)$ and $\mu(p^r, v, p^k)$ described above and in Definition* 2.6.

This theorem is essentially a translation of the results of Clark [4] into the language of May [8], [19]. The proof and precise definitions will be given in Section 2.

Since MMP's are difficult to compute in general, this result at first glance does not appear to be especially useful. Theorem B, the main result of the paper, identifies these MMP's with certain well-defined stable cohomology operations.

Let β_k denote the p^kth order Bockstein [2]. That is, β_1 is the usual Bockstein associated with the sequence $Z_p \to Z_{p^2} \to Z_p$ and β_k is defined from Ker β_{k-1} to $H^*(X)/\text{Im }\beta_{k-1}$. Thus β_k detects p^k torsion in $H^*(X; Z)$.

In Section 3, a secondary operation ψ_r will be defined using the Adem relation

$$\mathscr{P}^{p^r s-1}(\mathscr{P}^{p^{r-1}s} \cdots \mathscr{P}^s) = 0$$

$$(Sq^{2^r t-2}(Sq^{2^{r-1}t} \cdots Sq^t) = 0 \text{ for } p = 2 \text{ and } r > 1).$$

In particular ψ_r will be defined on classes of height p^r ($r > 1$ if $p = 2$).

THEOREM B. *Let X be a* 1-*connected H space of finite type and let $x \in$* Ker σ. *Assume p is an odd prime (resp. $p = 2$). Then either there is an indecomposable class $u \in H^{2m+1}(X)$ (resp. $u \in H^{m+1}(X)$) such that $\beta_k\mathscr{P}^I u$ (resp. $\beta_k Sq^I u$) for $I = (p^{k-1}m, \ldots, m)$ is defined and contains x or else there is an indecomposable class $v \in H^{2s}(X)$ of height p^r (resp. $v \in H^s(X)$ of height $2^r > 2$) such that $\beta_k\mathscr{P}^J\psi_r(v)$ (resp. $\beta_k Sq^J\psi_r(v)$) for $J = (p^{k-1}(p^r s - 1), \ldots, p(p^r s - 1))$ is defined and contains x.*

The Milnor basis element $\mathscr{P}_k(m)$ dual to ξ_k^m differs from $\mathscr{P}^I$ in the theorem by terms of higher excess [14]. Thus $\mathscr{P}^I$ may be replaced by $\mathscr{P}_k(m)$ and similarly $\mathscr{P}^J$ may be replaced by $\mathscr{P}_{k-1}(p(p^r s - 1))$ in the statement of Theorem B.

The proof of Theorem B will occupy most of Sections 3–5. In Section 3 the operation ψ_r is studied and related to the transpotence. As a special case of Theorem B, $\beta\psi_r(v)$ is related to $\langle v, v^{p^r-1}, \ldots, v, v^{p^r-1}\rangle \subset \mu(p^r, v, p)$. In

Section 4, the universal example for $\mu(p^r, v, p^k)$ is constructed. This is a Postnikov system which is a generalization of those studied in [13] and [16]. The study of this Postnikov system is continued using Eilenberg-Moore spectral sequences, and the key k invariant is explicitly identified, which completes the induction step in the definition of the universal example. In Section 5 this k invariant class is shown to represent the stable operation of Theorem B and so this theorem will follow immediately from the general theory of universal examples.

As a corollary of this method, the differentials in various EMSS's are computed. In particular a claim of Moore and Smith [20] about higher Kudo transgression elements is generalized and proved. The mod p analogue of a collapse theorem of Munkholm [21] is proven and other partial collapse theorems are given. The paper ends with a conjecture about the homology analogue of this theorem and higher order Dyer Lashof operations.

Section 2

Throughout the rest of the paper, we will assume that X is a 1-connected H space with $H^*(X)$ of finite type. Then $C^*(X)$ is an associative DGA algebra over Z_p. Let $\bar{B}C^*(X)$ denote the reduced bar construction. A typical generator will be denoted by $[a_1 | \cdots | a_n]$ as usual. $\bar{B}C^*(X)$ is a bigraded coalgebra with bidegree $(-n, \sum \deg a_i)$ and thus we have an associated second quadrant Eilenberg-Moore spectral sequence (EMSS)

$$E_2 \approx \mathrm{Tor}_{H^*(X)}(Z_p, Z_p) \tag{2.1}$$

converging to $H^*(\Omega X)$ as coalgebras (see [20]). If $[a_1 | \cdots | a_k]$ is an $r-1$ cycle, we will denote its class in $E_r^{-k,*}$ by the same symbol.

Since ΩX is an associative homotopy commutative H space, $C^*(\Omega X)$ has the structure of a DGA Hopf algebra over Z_p, and we can form the reduced cobar construction $\bar{\mathscr{F}}C^*(\Omega X)$. A typical generator will be denoted by $[\alpha_1 | \cdots | \alpha_n]$. There is an EMSS with

$$\mathscr{E}_2 \approx \mathrm{Cotor}_{H^*(\Omega X)}(Z_p, Z_p) \tag{2.2}$$

which converges to $H^*(X)$ as algebras [27]. We will abbreviate $\mathrm{Tor}_A(Z_p, Z_p)$ and $\mathrm{Cotor}_C (Z_p, Z_p)$ by Tor_A and Cotor_C respectively.

We also have a homology EMSS

$$E^2 \approx \mathrm{Tor}^{H_*(\Omega X)} \Rightarrow H_*(X).$$

There are many duality relations among these spectral sequences (see [17]). These will be used to identify differentials. The map σ can be identified as the edge homomorphism in these EMSS's.

Proposition 2.3. *The homomorphism*

$$QH^q(X) \approx \mathrm{Tor}^{1,q}_{H^*(X)} \twoheadrightarrow E_\infty^{-1,q} \rightarrowtail PH^{q-1}(\Omega X)$$

denoted by $x \mapsto [x]$ *and the homomorphism*

$$QH^q(X) \twoheadrightarrow \mathscr{E}_\infty^{1,q-1} \rightarrowtail \operatorname{Cotor}^{1,q-1}_{H^*(\Omega X)} \approx PH^{q-1}(\Omega X)$$

denoted by

$$\begin{aligned}[][\alpha_1 \mid \cdots \mid \alpha_n] &\mapsto 0 \qquad \text{if } n > 1\\ &\mapsto [\alpha_1] \quad \text{if } n = 1\end{aligned}$$

each correspond to the loop map.

Proof. See [20] and [17].

This result implies that Ker σ corresponds to the image of differentials

$$d_{r-1}\colon E_{r-1}^{-r,*} \to E_{r-1}^{-1,*}.$$

This fact has been used by several authors (e.g., [4], [6], [19], and [20]) to study Ker σ. The idea is that since $H^*(X)$ is a commutative Hopf algebra, $\operatorname{Tor}_{H^*(X)}$ is an easily describable commutative Hopf algebra. In particular the indecomposables are identifiable and thus the differentials being derivations are algebraically determined.

THEOREM 2.4. *For* $r \geq 2$, $d_r\colon E_r^{-(r+1),*} \to E_r^{-1,*}$ *is* 0 *unless* $r = p^k - 1$ *or* $r = 2p^k - 1$. *Furthermore* d_{p^k-1} *is algebraically determined on* $E_r^{*,*}$ *by its action on elements* $\gamma_{p^k}[u] = [u \mid \cdots \mid u]$ *where* $u \in QH^{2m+1}(X)$, *and* d_{2p^k-1} *is algebraically determined by its action on elements of the form*

$$\gamma_{p^k}[v \mid v^{p^r-1}] = [v \mid v^{p^r-1} \mid \cdots \mid v \mid v^{p^r-1}]$$

where $v \in QH^{2s}(X)$ *has height* p^r. *Thus* $\sigma\colon QH^q(X) \to PH^{q-1}(\Omega X)$ *is a monomorphism unless* $q = 2mp^k + 2$ *or* $q = 2p^k(p^r s - 1) + 2$.

Proof. The homology analogue of this theorem is essentially Theorem 4.1 of [4] (see also [6]). For completeness and to fix notation we sketch the argument. By Borel's structure theorem for commutative Hopf algebras, $H^*(X)$ splits as algebras into a tensor product of monogenic algebras. The functor Tor commutes with this splitting. Moreover as bigraded algebras we have Hopf algebra isomorphisms

$$\operatorname{Tor}_{Z_p[x]} \approx E([x]), \quad \operatorname{Tor}_{Z_p[x]/(x^{p^r})} \approx E([x]) \otimes \Gamma([x \mid x^{p^r-1}])$$

and

$$\operatorname{Tor}_{E(y)} \approx \Gamma([y]).$$

Thus the indecomposables of $\operatorname{Tor}_{H^*(X)}$ of filtration degrees greater than 2 are of the form $\gamma_{p^k}[y]$ and $\gamma_{p^k}[x \mid x^{p^r-1}]$. Since the differentials are derivations in $E_r^{*,*}$ the first part follows from general algebra. The remainder of the theorem follows by recalling that Ker σ can be identified with the image of the differentials by Proposition 1.3.

May identifies the boundaries in $E_r^{-1,*}$ in terms of matric Massey products. We state a version of this theorem.

THEOREM 2.5 (May). *If the Massey product* $\langle u_1, \ldots, u_k \rangle$ *is defined and contains* $x \in QH^*(X)$, *then* $[x] \in E_2^{-1,*}$ *and* $[u_1 \mid \cdots \mid u_k] \in E_2^{-k,*}$ *live to* E_{k-1} *and* $d_{k-1}[u_1 \mid \cdots \mid u_k] = [x]$.

Conversely if $[x]$ *and* $[u_1 \mid \cdots \mid u_k]$ *live to* E_{k-1} *and* $d_{k-1}[u_1 \mid \cdots \mid u_k] = [x]$ *then there is a related canonically defined MMP which contains* x *and thus contains* $\langle u_1, \ldots, u_k \rangle$ *if the latter is defined.*

Proof. The proof of this theorem can be found in the unpublished manuscript of Peter May [19] (see also [8]). The first part is also proven in [17] as Theorem 2.3. Since the canonically defined MMP is a somewhat esoteric cohomology operation, the reader will lose little by considering the canonically defined MMP containing x to be the set of all $y \in H^*(X)$ such that $[x] = [y]$ in $E_{k-1}^{-1,*}$. With this convention the rest of the theorem is a triviality.

DEFINITION 2.6. Set $\mu(u, p^k)$ (resp. $\mu(p^r, u, p^k)$) to be the canonically defined MMP corresponding to the differential

$$d_{2p^k-1}[u \mid \cdots \mid u] = [x] \quad (\text{resp. } d_{2p^k-1}[v \mid v^{p^r-1} \mid \cdots |v|v^{p^r-1}] = [x]).$$

Theorem A is now an immediate consequence of this definition and Theorems 2.4 and 2.5.

Section 3

The Adem relation $\mathscr{P}^{sp^r-1}(\mathscr{P}^{sp^{r-1}} \cdots \mathscr{P}^s) = 0$ gives rise to a stable secondary cohomology operation ψ_r for $r \geq 1$. If $x \in H^{2s}(X)$ and $x^{p^r} = 0$, then $\psi_r(x)$ is defined in $H^q(X)/\mathscr{P}^{sp^r-1}H^*(X)$ where $q = 2sp^{r+1} - 2p + 1$.

The secondary operation can be defined using a universal example (P, v, w). Here P is the total space of the fibration

$$K(Z_p, 2sp^r - 1) \xrightarrow{i} P \xrightarrow{\pi} K(Z_p, 2s)$$

induced by a stable k invariant λ: $K(Z_p, 2s) \to K(Z_p, 2sp^r)$ satisfying $\lambda^*\iota_r = (\iota_0)^{p^r}$ where ι_0 and ι_r are the appropriate fundamental classes. The class v is $\pi^*(\iota_0) \in H^{2s}(P)$, and w is some primitive class in $H^q(P)$ satisfying $i^*w = \mathscr{P}^{sp^r-1}\sigma\iota_r$. Lemma 3.6.4 in [1] asserts the existence of such a triple. In fact Adams shows that $\psi_r(x)$ is the set of all classes $f^*w \in H^q(X)$ as f: $X \to P$ ranges over maps satisfying $f^*v = x$. Also ψ_r is additive and natural. Moreover Theorem 3.6.2 in [1] states that any other secondary operation associated with the Adem relation differs from ψ_r by a stable primary cohomology operation. To see this, one shows that $i^*(w' - w) = 0$ implies that there is a primitive $y \in H^q(K(Z_p, 2s))$ such that $\pi^*y = w' - w$. But such a class y can be written as $\theta\iota$ for some primary stable operation θ by the structure of $PH^*(K(Z_p, n))$.

If we assume that $p = 2$, then the Adem relation to consider is

$$Sq^{t2^r-2}(Sq^{t2^{r-1}} \cdots Sq^t) = 0 \quad \text{where } r \geq 2$$

since $Sq^{4n-2}Sq^{2n}Sq^n = Sq^{4n-1}Sq^{2n-1}Sq^n = 0$. Thus if $x \in H^t(X)$ and $x^{2^r} = 0$ for $r \geq 2$, then $\psi_r(x)$ is defined in $H^q(X)/Sq^{t2^r-2}H^*(X)$ for $q = t2^{r+1} - 1$ and is determined up to a stable primary operation.

Another secondary operation, the transpotence ϕ_r, is also defined on classes $x \in H^{2s}(X)$ satisfying $x^{p^r} = 0$ for $r \geq 1$. If $p = 2$ then $x \in H^t(X)$ must satisfy $x^{2^r} = 0$ for $r \geq 2$ in order for $\phi_r(x)$ to be defined. If it is defined then

$$\phi_r(x) \in PH^{2sp^r-2}(\Omega X)/\sigma H^*(X).$$

See [7] for details.

The space P above is also a universal example for ϕ_r in the following sense. Since $\sigma(\iota_0)^{p^r} = 0$, the looped k invariant $\Omega\lambda$ is null homotopic. Thus there is a (noncanonical) homotopy equivalence

$$\xi\colon \Omega P \to K(Z_p, 2s - 1) \times K(Z_p, 2sp^r - 2) \tag{3.1}$$

such that $p_2\xi(\Omega i) \simeq 1$ and $p_1\xi \simeq \Omega\pi$. Set $\alpha = (p_2\xi)^*(\sigma^2\iota_r) \in H^{2sp^r-2}(\Omega P)$, so $(\Omega i)^*\alpha = \sigma^2\iota_r$.

Then $\phi_r(x)$ can be characterized as the set of classes $(\Omega f)^*\alpha \in H^{2sp^r-2}(\Omega X)$ as $f\colon X \to P$ ranges over maps satisfying $f^*v = x$. See ([9] and [17].) A different choice of ξ, and thus α, will not change the coset ϕ_r since $\alpha - \alpha' = \sigma(\pi^*x)$ for some $x \in H^*(K(Z_p, 2s))$ if α' is another primitive satisfying $(\Omega i)^*\alpha' = \sigma^2\iota$.

The fact that both ψ_r and ϕ_r have the same universal example suggests some kind of relation between them. In fact the main theorem of this section is the following Peterson-Stein type formula.

THEOREM 3.2. *Assume that $x \in H^{2s}(X)$ satisfies $x^{p^r} = 0$ for $r \geq 1$ and p odd, or $x \in H^t(X)$ satisfies $x^{p^r} = 0$ for $r \geq 2$ and $p = 2$. Then the secondary operation ψ_r can be chosen so that*

$$\sigma\psi_r(x) = [\phi_r(x)]^p$$

in $PH^{2sp^{r+1}-2p}(\Omega X)/\sigma\mathcal{P}^{sp^r-1}H^*(X)$. Thus the indeterminacy consists of classes of the form $(\sigma y)^p$ for $y \in H^{2sp^r-1}(X)$.

Proof. Let (P, v, w) be the universal example for ψ_r. The theorem will follow if we prove it for $x = v \in H^{2s}(P)$.

$$(\Omega i)^*\sigma w = \sigma\mathcal{P}^{sp^r-1}\sigma\iota_r = (\sigma^2\iota_r)^p = (\Omega i)^*\alpha^p.$$

The homotopy equivalence ξ induces an algebra isomorphism

$$H^*(\Omega P) \approx H^*(K(Z_p, 2s - 1)) \otimes H^*(K(Z_p, 2sp^r - 2)) \tag{3.3}$$

and thus an exact sequence

$$QH^{q-1}(K(Z_p, 2s - 1)) \to QH^{q-1}(\Omega P) \to QH^{q-1}(K(Z_p, 2sp^r - 1)),$$

where $q - 1 = 2sp^{r+1} - 2p$. Since $\alpha^p - \sigma w$ is primitive, by Milnor-Moore it is either indecomposable or the pth power of a primitive. Since σ is an isomorphism in dimension $2sp^r - 2$, in the latter case $\alpha^p - \sigma w$ is of the form

$(\sigma y)^p$ for some $y \in QH^{2sp^r-1}(P)$ and thus in the indeterminacy subgroup. In the former case $(\Omega i)^*(\alpha^p - \sigma w) = 0$ implies that $\alpha^p - \sigma w = (\Omega\pi)^*\sigma z$ for some

$$z \in PH^q(K(Z_p, 2s)) \underset{\sim}{\xrightarrow{\sigma}} PH^{q-1}(KZ_p, 2s - 1)).$$

In this case replace the original universal example by $(P, v, w + (\Omega\pi)^*z)$ and the theorem will follow. Note that the class $(\Omega\pi)^*z$ corresponds to a stable primary cohomology operation.

Remark 3.4. Spanier [25] has constructed a very general theory for higher order operations. In this theory $\psi_r(x)$ corresponds to a Toda type bracket $\langle \mathscr{P}^{sp^r-1}, \mathscr{P}^{sp^r-1}, \ldots, \mathscr{P}^s, x\rangle$ and $\phi_r(x)$ corresponds to the bracket

$$\langle \sigma, \mathscr{P}^{sp^r-1}, \ldots, \mathscr{P}^s, x\rangle$$

(see [25, p. 522]). Then

$$\begin{aligned}\sigma\langle \mathscr{P}^{\cdot}, \mathscr{P}^{\cdot}, \ldots, \mathscr{P}^{\cdot}, x\rangle &= \langle \sigma\mathscr{P}^{\cdot}, \mathscr{P}^{\cdot}, \ldots, \mathscr{P}^{\cdot}, x\rangle \\ &= \mathscr{P}^{\cdot}\langle \sigma, \mathscr{P}^{\cdot}, \ldots, \mathscr{P}^{\cdot}, x\rangle \\ &= (\langle \sigma, \mathscr{P}^{\cdot}, \ldots, \mathscr{P}^{\cdot}, x\rangle)^p\end{aligned}$$

modulo some indeterminacy. Thus a careful reading of Spanier's paper will yield a proof of Theorem 3.2 without using universal examples.

The main theorem of [9] implies that the splitting (3.3) holds as Hopf algebras. Thus the EMSS $\mathscr{E}_2 \simeq \mathrm{Cotor}_{H^*(\Omega P)} \Rightarrow H^*(P)$ has a fairly easy form. In fact $H^*(P)$ can be completely described as a Hopf algebra over the Steenrod algebra. We need only the following:

THEOREM 3.6. *In the EMSS above:*

(1) $d_q = 0$ *unless* $q = p^r - 1$, *and* $d_{p^r-1}[\alpha] = [\sigma v \mid \cdots \mid \sigma v] = [\sigma v]^{p^r}$ *determines that differential, where* $\alpha \in \phi_r(v)$.

(2) $[\alpha^p]$ *represents* $\psi_r(v)$ *in* $H^*(P)$.

(3) $$\frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}[\alpha^i \mid \alpha^{p-i}]$$

represents $\beta\psi_r(v)$ *modulo primitives of the form* π^*z *in* $H^*(P)$.

Proof. The first result uses Theorem 2.2 of [17]. The second is immediate from Theorem 3.2. The last result is a consequence of the proof of Lemma 2 of [9].

Note that $\beta\psi_r(v)$ is a special case of the stable operation mentioned in Theorem B. By the techniques of [11], the restricted $2p$-fold Massey product $\langle v, v^{p^r-1}, v, \ldots, v^{p^r-1}\rangle$ may be defined in $H^*(P)$. Furthermore a complicated cochain argument together with some techniques from [17] can be used to prove that

$$\langle v, \ldots, v^{p^r-1}\rangle = -\beta\psi_r(v) \quad \text{in } H^{p(sp^r-1)+2}(P)/\beta\mathscr{P}^{sp^r-1}H^*(P).$$

We will settle for a much easier statement which implies that P is the universal example for $\mu(p^r, v, p)$.

THEOREM 3.7. *If $x \in H^{2s}(X)$ has height p^r, then $\mu(p^r, x, p)$ is defined and contains $c\beta\psi_r(x)$ for some nonzero constant c.*

Proof. By the theory of universal examples it suffices to prove this for $X = P$ and $x = v$. By Theorems 2.4 and 2.5, we must show that $d_r z = 0$ for $r < 2p - 1$ and $d_{2p-1}z = c[\beta\psi_r(v)]$ where $z = \gamma_p[v \mid v^{p^r-1}]$ in $E_2^{-2p,*}$. But this result has been proved in the appendix of [24].

By [7], the element $[v \mid v^{p^r-1}]$ represents the transpotence in $H^*(\Omega P)$. Thus the divided power coalgebra generated by the transpotence is truncated at height p in $H^*(\Omega P)$.

Section 4

In [13] and [16] a k-stage Postnikov system E_k was constructed for each k. E_k turns out to be the universal example for the MMP $\mu(u, p^k)$. In this section we construct the universal example for $\mu(p^r, v, p^k)$ by splicing E_k onto the two-stage system P of Section 3.

For simplicity of notation we will assume that $p > 2$. The results are similar for $p = 2$. We first record the main properties of E_k, the universal example for $\mu(u, p^k)$.

THEOREM 4.1. *For $k \geq 0$ there is a $k + 1$ stage Postnikov system*

$$\begin{array}{ccc}
K(Z_p, 2mp^k + 1) \xrightarrow{j_k} & E_k & \\
& \downarrow{\scriptstyle \pi} & \\
& E_{k-1} & \xrightarrow{\kappa_{k-1}} K(Z_p, 2mp^k + 2) \\
& \downarrow & \\
& \vdots & \\
& \downarrow & \\
K(Z_p, 2m + 1) \; = & E_0 & \xrightarrow{\kappa_0} K(Z_p, 2mp + 2)
\end{array}$$

satisfying:

(1) $j^*\kappa^*\iota = -\beta\mathscr{P}^{p^i m}\iota'$;
(2) $\sigma\kappa^*\iota = 0$;
(3) *E_k is at least a $(2p - 4)$-fold loop of an H space.*

Here ι and ι' represent appropriate fundamental classes.

Proof. See [16].

We now describe the Postnikov decomposition for the universal example for $\mu(p^r, v, p^k)$.

THEOREM 4.2. *For k, r, and s positive integers and $m = sp^r - 1$, there is a $k + 1$ stage Postnikov system*

$$\begin{array}{ccccc}
K(Z_p, 2mp^{k-1} + 1) & \xrightarrow{j_k} & P_k & & \\
& & \downarrow \pi & & \\
& & P_{k-1} & \xrightarrow{\lambda_{k-1}} & K(Z_p, 2mp^{k-1} + 2) \\
& & \downarrow & & \\
& & \vdots & & \\
& & \downarrow & & \\
K(Z_p, 2m + 1) & \xrightarrow{j_1} & P_1 & \xrightarrow{\lambda_1} & K(Z_p, 2mp + 2) \\
& & \downarrow & & \\
K(Z_p, 2s) & = & P_0 & \xrightarrow{\lambda_0} & K(Z_p, 2p^r s)
\end{array}$$

satisfying:

(1) $\lambda_0^* \iota = \eta^{p^r}, j_i^* \lambda_i^* \iota = -\beta \mathscr{P}^{mp^{i-1}} \iota'$, *for* $i > 0$;
(2) $\sigma \lambda^* \iota = 0$;
(3) P_k *is at least a* $(2p - 4)$*-fold loop of an* H *space.*

Furthermore $P_1 = P$ *is the universal example for* ψ_r *and there are* $(2p - 4)$*-fold loops of* H *maps*

$$f_i\colon E_i \to P_{i+1} \quad \textit{for } i = 0, 1, \ldots, k - 1$$

such that $\pi f_i = f_{i-1}\pi$.

The proof of Theorem 4.2 follows the proof in [16] of Theorem 4.1 closely. We proceed by induction on k. $P_1 = P$ was described in Section 3 and has the required properties. Assume P_{k-1} exists and λ_{k-1} is defined as a $2p - 4$ loop of an H map. Then P_k is induced by λ_{k-1} and satisfies (3). We must construct λ_k to be a $2p - 4$ loop of an H map satisfying conditions (1) and (2).

Since $\sigma(\lambda_i^* \iota) = 0$ for $i \leq k - 1$, the fibrations $\Omega P_{i+1} \to \Omega P_i$ split on the space level. Thus there is a homotopy equivalence

(4.3) $\xi\colon \Omega P_k \to K(Z_p, 2s - 1) \times K(Z_p, 2m) \times \cdots \times K(Z_p, 2mp^{k-1})$.

This splitting together with the techniques and some results of [16] can be used to compute the differentials in the EMSS

$$\mathscr{E}_2 \approx \mathrm{Cotor}_{H^*(\Omega P_k)} \Rightarrow H^*(P_k).$$

Actually the Hopf algebra structure of $H^*(P_k)$ can be computed. As the results are complicated to state (compare with Theorem 5.3 of [16]) and not needed in what follows, they will be omitted.

Using the Hopf algebra structure of $H^*(\Omega P_k)$, we will proceed to identify an element $\sum_{i+j=p^k} [\gamma_i \mid \gamma_j]$ in the EMSS $\mathscr{E}_2$. This element will represent the k invariant λ_k of P_{k+1}, essentially completing the induction step in Theorem 5.2.

Moreover this element will be seen to represent $\mu(p^r, v, p^k)$, at least up to a nonzero constant. In the next section, $\beta_k \mathscr{P}^{p^{k-1}m} \cdots \mathscr{P}^{pm}\psi_r(v)$ will be computed in the EMSS and shown to be represented by $-\sum [\gamma_i \mid \gamma_j]$ (compare with Theorem 3.6). This will complete the proof of Theorem B.

THEOREM 4.5. *Let $\eta, \alpha_0, \ldots, \alpha_{k-1}$ in $H^*(\Omega P_k)$ be the images of the fundamental classes under the splitting* (4.3). *Then the Hopf algebra generated by these classes is isomorphic to $E[\eta] \otimes A_{k-1}$ where A_{k-1} is a truncated bipolynomial Hopf algebra. That is, A_{k-1} is isomorphic to $Z_p[\alpha_0, \alpha_1, \ldots, \alpha_{k-1}]$ as algebras and α_0 generates a divided power coalgebra truncated at height p^k with $\gamma_j = \gamma_j\alpha_0 \in A_{k-1}$, satisfying $\gamma_{p^r} = \alpha_r$ for $r = 0, \ldots, k-1$.*

Proof. As with the universal example P of the previous section,

$$H^*(\Omega P_k) \approx H^*(K(Z_p, 2s-1)) \otimes H^*(\Omega E_k)$$

as Hopf algebras. That is, the twisted H structure induced by the first k invariant cannot be detected in any primary way (see [9]). The theorem now follows immediately from a similar result on $H^*(\Omega E_k)$ (see Section 1 of [16]).

The Hopf algebra A_{k-1} was constructed in [15]. It will be used in the next section.

The element $\sum_{i+j=p^k} [\gamma_i \mid \gamma_j]$ in $\mathscr{E}_2^{2,q,-2}$ represents an indecomposable in $H^q(P_k)$ for $q = 2p^k(p^r s - 1) + 2$. By Milnor-Moore, $PH^q(P_k)$ is isomorphic to $QH^q(P_k)$ since p does not divide q. Thus we can make the following definition.

DEFINITION 4.6. Let $\mu\alpha \in PH^q(P_k)$ be the unique primitive represented by the element $\sum_{i+j=p^k} [\gamma_i \mid \gamma_j]$ in the EMSS

$$\mathscr{E}_2 \approx \mathrm{Cotor}_{H^*(\Omega P_k)} \Rightarrow H^*(P_k).$$

PROPOSITION 4.7. *If $f\colon E_{k-1} \to P_k$ is the map of Theorem* 4.2, *then*

$$f^*(\mu\alpha) \in H^*(E_{k-1})$$

represents the next k invariant

$$\kappa_k\colon E_{k-1} \to K(Z_p, 2mp^{k+1} + 2).$$

Proof. In the spectral sequence $\mathrm{Cotor}_{H^*(\Omega E_{k-1})} \Rightarrow H^*(E_{k-1})$, the element which lived to represent κ_k was

$$\sum_{i+j=p^k} [\gamma_i \Omega f^*\alpha_0 \mid \gamma_j \Omega f^*\alpha_0]$$

where $\Omega f^*\alpha_0 \in H^{2m+1}(\Omega E_{k-1})$ is the lowest dimensional cohomology class. Since $(\Omega f)^*\colon H^*(\Omega P_k) \to H^*(\Omega E_{k-1})$ is an epimorphism, the proposition is immediate by naturality.

THEOREM 4.8. *The k invariant $\lambda_k\colon P_k \to K(Z_p, 2mp^k + 2)$ can be chosen to be a $(2p-4)$-fold loop of an H map such that $(\lambda_k)^*\iota = \mu\alpha$ in $PH^q(P_k)$. Furthermore, $\lambda_k f_k$ is homotopic to $\kappa_{k-1}\colon E_{k-1} \to K(Z_p, 2mp^k + 2)$ as $(2p-4)$-fold loops of an H map.*

Proof. By the induction hypothesis there is an H space P'_k such that $\Omega^{2p-4}P'_k \simeq P_k$. By Theorem 6.1 of [16], it follows that

$$\sigma^{2p-4}: PH^{q+2p-4}(P'_k) \to PH^q(P_k)$$

is an isomorphism. Let $\lambda': P'_k \to K(Z_p, q+2p-4)$ be an H map representing $(\sigma^{2p-4-1})^{-1}\mu\alpha$. Define P'_{k+1} to be the H space fiber space over P'_k induced by λ'. Finally define $P_{k+1} = \Omega^{2p-4}P'_{k+1}$ and $\lambda_k = \Omega^{2p-4}\lambda'_k$. It follows from Proposition 6.2 and Section 6 of [16] that $\lambda'_k f'_k$ is homotopic to κ'_{k-1} where $\Omega^{2p-4}\kappa' = \kappa$. The theorem follows by looping.

The induction step in the proof of Theorem 4.2 is now essentially complete. The fact that $j^*\lambda^*\iota = -\beta\mathscr{P}^{mp^{k-1}}\iota'$ follows from the similar result in Theorem 4.1 and the fact that λf is homotopic to κ.

By Theorems 2.5 and 2.6, the following will establish that $\mu(p^r, v, p^k)$ does indeed live in $H^*(P_k)$ and that $\mu\alpha$ is a representative of this Massey product.

THEOREM 4.9. *In the EMSS*

$$E_2 \approx \mathrm{Tor}_{H^*(P_k)} \Rightarrow H^*(\Omega P_k)$$

the elements $[\mu\alpha] \in E_2^{-1,*}$ *and* $z_k = \gamma_{p^k}[v \mid v^{p^k-1}] \in E_2^{-2p^k,*}$ *both live to* E_{2p^k-1} *and* $d_{2p^k-1}z_k = c[\mu\alpha]$ *for some nonzero constant* c.

Proof. For $k = 1$, this is Theorem 3.7. Since $\mu\alpha$ arises from an element of filtration 2 in $\mathscr{E}_r$, $\sigma\mu\alpha = 0$ by the description of the edge homomorphism (2.3). Thus $[\mu\alpha]$ is in the image of d_r for some r. By [7], $\phi = [v \mid v^{p^r-1}]$ represents a transpotence element which we may assume to be α_0. Thus z_k, the p^k-fold divided power of ϕ, represents $\gamma_{p^k}\alpha_0$ which is 0 in $H^*(\Omega P_k)$. However z_{k-1}, the p^{k-1}-fold divided power of ϕ, represents $\gamma_{p^{k+1}}\alpha_0 = \alpha_{k-1}$ which is not zero in $H^*(\Omega P_k)$. Thus z_k must survive to E_{2p^k-1} and no further. A check of the relevant dimensions will show that z_k must indeed kill $[\mu\alpha]$.

The theorem implies that $(P_k, v, \mu\alpha)$ is the universal example for $\mu(p^r, v, p^k)$ [22]. Similarly the space E_{k-1} of Theorem 4.1 is the universal example for $\mu(u, p^k)$. Thus by Theorem A we have the following reformulation of the kernel of σ.

COROLLARY 4.10. *Assume that* $\sigma x = 0$ *for* $x \in QH^q(X)$. *Then either there is a map* $g: X \to E_{k-1}$ *for some* m *and* k, *such that* $\mu(g^*u, p^k)$ *is defined and contains* x, *or there is a map* $h: X \to P_k$ *for some* s, r, *and* k *such that* $\mu(p^r, h^*v, p^k)$ *is defined and contains* x.

Section 5

In the EMSS $\mathscr{E}_r \Rightarrow H^*(P_k)$, the element $[\alpha_0^p]$ represents $\psi_r(v)$ by Theorem 3.6. Thus $[\alpha_0^{p^k}]$ represents $\mathscr{P}^{p^{k-1}m}\cdots\mathscr{P}^{pm}\psi_r(v)$ with $m = p^r s - 1$. We know that if $x \in \mathrm{Ker}\,\sigma$, then $x \in \mu(u, p^k)$ or $x \in \mu(p^r, v, p^k)$. Also, $\mu\alpha \in \mathscr{E}_r^{2,*}$ represents $\mu(p^r, v, p^k)$. Assume that $x \in \mu(p^r, v, p^k)$. To prove Theorem B, we must show that β_k is defined on the element of $H^*(P_k)$ represented by $[\alpha_0^{p^k}]$ and that $\beta_k[\alpha_0^{p^k}] = -\mu\alpha$.

We must first get an appropriate characterization of higher order Bocksteins. If $n > m$ let $\rho: Z_{p^n} \to Z_{p^m}$ and $\eta: Z_{p^m} \to Z_{p^n}$ be the standard nontrivial maps and also the induced maps on cochains and cohomology. The following proposition is immediate from definitions (see for example [2]).

PROPOSITION 5.1. *Let $w \in H^n(X; Z_p)$. Then $\beta_k w$ is defined in $H^{n+1}(X; Z_p)/\operatorname{Im} \beta_{k-1}$ if and only if there is a cochain $\omega \in C^n(X; Z_{p^{k+1}})$ such that $\delta\omega \equiv 0 \pmod{p^k}$ and $\rho\omega \in C^n(X; Z_p)$ represents w. In this case $\beta_k w$ is represented by a cocycle $\xi \in C^{n+1}(X; Z_p)$ satisfying $\eta\xi = \delta\omega \in C^{n+1}(X; Z_{p^{k+1}})$.*

For our application of Theorem 5.1, we consider $\mathscr{F}C^*(\Omega P_k; R)$ as the cochain complex for $H^*(P_k; R)$. As noted in Theorem 4.5, there is a subHopf algebra A_{k-1} in $H^*(\Omega P_k; Z_p)$. A_{k-1} was constructed in [15] as the mod p reduction of a bicommutative Hopf algebra $B = B_{k-1}$ defined over $Z_{(p)}$, the subring of rationals with denominators prime to p.

The following facts from [15] will be used to evaluate the higher order Bockstein. As an algebra B is isomorphic to a polynomial algebra over $Z_{(p)}$ on generators $X_0, \ldots, X_{k-1}$. The module of primitives of B is generated by the Witt polynomials

$$W_i = X_0^{p^i} + \cdots + p^{i-1}X_{i-1}^p + p^i X_i \quad \text{for } i = 0, \ldots, k-1. \tag{5.2}$$

Set $a_0 = 1$, $a_1 = X_0 = W_0$, and inductively define a_n for $n < p^k$ by

$$a_n = (a_{n-1}W_0 + a_{n-p}W_1 + \cdots + a_{n-p^i}W_i)/n \quad \text{for } p^i \le n < p^{i+1}.$$

Theorem 3 in [15] implies that a_n is indeed a polynomial in B, that is the coefficients do lie in $Z_{(p)}$. Furthermore by Lemma 2 of [15] we have that

$$\Delta a_n = \sum_{i=0}^{n} a_i \otimes a_{n-i} \tag{5.3}$$

where Δ is the coproduct. Thus the divided powers $\gamma_i \alpha_0$ in A_{k-1} are the mod p reductions of the elements a_i.

By Lemma 4 in [15], there is an algebra isomorphism

$$B = Z_{(p)}[X_0, \ldots, X_{k-1}] \approx Z_{(p)}[a_1, a_p, \ldots, a_{p^{k-1}}].$$

By (5.2) and (5.3) and the fact that the Witt polynomials are primitive we have the following result.

PROPOSITION 5.4. *$\tilde{W}_k = W_k - p^k \alpha_{p^k}$ is defined in $B = B_{k-1}$ and*

$$\tilde{W}_k \equiv a_1^{p^k} \pmod{p}, \qquad \bar{\Delta}\tilde{W}_k = -p^k \sum_{i+j=p^k} a_i \otimes a_j$$

where $\bar{\Delta}x = \Delta x - x \otimes 1 - 1 \otimes x$.

We would like to construct $\omega_k \in \mathscr{F}C^*(\Omega P_k; Z_{p^{k+1}})$ of Proposition 5.1 to be the polynomial $[\tilde{W}_k]$ with the fundamental cocycles substituted in place of the

a_{p^i}'s. Unfortunately this argument must be followed with extreme care, since the cocycle product is not commutative. Also even though there is a class $x \in H^{2mp^r}(\Omega P_k; Z_{p^r})$ such that $\rho x = \alpha_0^{p^r}$, this class is not even a pth power. Thus the map $B \to H^*(\Omega P_k; Z_p)$ does not lift to a map $B \to H^*(\Omega P_k; Z_{p^r})$. If we restrict to a suitable subalgebra of B, we can get this important lifting.

DEFINITION 5.5. Let C be a commutative algebra over a ring of characteristic 0. Every element of C will be said to have type 1, and, inductively, if x and y have type n, then $x^p + py$ will have type $n + 1$. The elements of type n form a subalgebra $C_{(n)}$ of C.

THEOREM 5.6 (Thomas [28]). *There are primary cohomology operations, the Pontrjagin pth powers*

$$\mathfrak{P}: H^{2n}(X; Z_{p^r}) \to H^{2np}(X; Z_{p^{r+1}})$$

which satisfy

$$\mathfrak{P}(xy) = \mathfrak{P}(x)\mathfrak{P}(y), \qquad \mathfrak{P}(x + y) = \mathfrak{P}(x) + \eta \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i \cdot y^{p-i} + \mathfrak{P}(y)$$

and

$$\rho\mathfrak{P}(x) = x^p \in H^*(X; Z_{p^r}).$$

The algebra map $B \to H^*(\Omega P_k; Z_p)$ extends in a natural way to algebra maps

$$B_{(n)} \to H^*(\Omega P_k; Z_{p^n})$$

using these Pontrjagin powers.

THEOREM 5.7. *Let $\phi: C \to H^*(X; Z_p)$ be an algebra map. Then ϕ induces algebra maps $\phi_n: C_{(n)} \to H^*(X; Z_{p^n})$ by $\phi_1 = \phi$ and, inductively, $\phi_n(x^p) = \mathfrak{P}\phi_{n-1}x$ and $\phi_n(px) = \eta\phi_{n-1}(x)$ for $x \in C_{(n-1)}$. Furthermore if $h: X' \to X$ is a continuous function and $g: C \to C'$ and $\phi': C' \to H^*(X'; Z_p)$ are algebra maps such that $h^*\phi = \phi' g$, then for all n,*

$$h^*\phi_n = \phi_n' g_n: C_{(n)} \to H^*(X'; Z_{p^n}).$$

Proof. It is only necessary to check that ϕ_n is well defined. The only non-trivial part of this is the following.

$$\begin{aligned}\phi_n(x + y)^p &= \mathfrak{P}(x + y)\\ &= \mathfrak{P}(x) + \eta \sum \frac{1}{p}\binom{p}{i} x^i \cdot y^{p-i} + \mathfrak{P}(y)\\ &= \phi_n(x^p) + \phi_n\left(\sum_{i=1}^{p-1}\binom{p}{i} x^i y^{p-i}\right) + \phi_n(y^p)\\ &= \phi_n\left(\sum_{i=0}^{p}\binom{p}{i} x^i y^{p-i}\right).\end{aligned}$$

COROLLARY 5.8. *The following diagram commutes:*

$$\begin{array}{ccc} B_{(n)} & \xrightarrow{\phi_n} & H^*(\Omega P_k; Z_{p^n}) \\ {\scriptstyle\Delta_n}\downarrow & & \downarrow{\scriptstyle m^*} \\ (B \otimes B)_{(n)} & \xrightarrow{\phi_n'} & H^*(\Omega P_k \times \Omega P_k; Z_{p^n}) \end{array}$$

By (5.2) and (5.3), $\tilde{W}_k$ has type $k + 1$. Moreover by Proposition 5.4,

$$\bar{\Delta}\tilde{W}_k = -p^k \sum_{i+j=p^k} a_i \otimes a_j \quad \text{in } (B \otimes B)_{(k)}.$$

Thus $\phi_{k+1}\tilde{W}_k$ is defined in $H^*(\Omega P_k; Z_{p^{k+1}})$ and

$$m^*\phi_{k+1}\tilde{W}_k = -\eta \sum_{i+j=p^k} \phi a_i \times \phi a_j \quad \text{in } H^*(\Omega P_k \times \Omega P_k; Z_{p^{k+1}}).$$

Choose a cocycle representative $\omega \in C^*(\Omega P_k; Z_{p^{k+1}})$ of $\phi_{k+1}\tilde{W}_k$. In the bicomplex $\mathscr{F}C^*(\Omega P_k; Z_{p^{k+1}})$, $[\omega]$ is a d^0 cocycle with d^0 class $\{[\omega]\} = [\phi_{k+1}\tilde{W}_k]$. Thus with respect to the total differential $\delta = d^0 + d^1$,

$$\delta[\omega] = [m^*\phi_{k+1}\tilde{W}_k] = \eta[-\mu\alpha].$$

By Proposition 5.4, $\rho[\omega] = [\alpha_0^{p^k}]$ in $\mathscr{F}C^*(\Omega P_k; Z_p)$ since α_0 is the mod p reduction of a_1. The conditions of Theorem 5.1 are now satisfied and so $\beta_k[\alpha_0^{p^k}]$ is indeed defined and represented by $-\mu\alpha$. As we have noted in the beginning of this section, the proof of Theorem B is at last complete.

Section 6

Moore and Smith studied the EMSS $E_2 \approx \mathrm{Tor}_{H^*(X)} \Rightarrow H^*(\Omega X)$ for 1-connected H spaces in considerable detail. They characterized d_{p-1} in terms of $\beta\mathscr{P}$ and stated that a similar characterization exists for d_{p^k-1} (see [20] and [24]).

Theorem B together with Proposition 2.3 and Theorem 2.4 combines to give the following generalization of the statement of Moore and Smith.

THEOREM 6.1. *If $u \in H^{2m+1}(X)$ and $\gamma_{p^k}[u] \in E_s^{-p^k,*}$ is an s cycle for $s < p^k - 1$, then $\beta_k\mathscr{P}_k(m)u = \beta_k\mathscr{P}^{p^{k-1}m}\cdots\mathscr{P}^m u$ is defined and equals $cd_{p^k-1}\gamma_{p^k}[u]$ for some $c \neq 0$ in Z_p. If $v \in H^{2n}(X)$ has height p^r and if $\gamma_{p^k}[v \mid v^{p^r-1}] \in E_s^{-2p^k,*}$ is an s cycle for $s < 2p^k - 1$, then $\beta_k\mathscr{P}_{k-1}(p^{r+1}s - p)\psi_r(v) = \beta_k\mathscr{P}^{p^{k-1}m}\cdots\mathscr{P}^{pm}\psi_r(v)$ is defined for $m = p^r s - 1$ and equals $cd_{2p^k-1}\gamma_{p^k}[v \mid v^{p^r-1}]$ for some $c \neq 0$ in Z_p.*

This gives us a nice collapse theorem for EMSS's (compare with [21]).

COROLLARY 6.2. *Assume that $H^*(X)$ is a free commutative algebra over Z_p and that $\mathscr{P}^m u = 0$ for all indecomposable $u \in H^{2m+1}(X)$ ($Sq^m u = 0$ for $u \in H^{m+1}(X)$ if $p = 2$). Then $E_2 = E_\infty$, so* $\mathrm{Tor}_{H^*(X)}(Z_p, Z_p) \approx H^*(\Omega X)$.

Proof. The hypothesis implies that $H^*(X)$ is a tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd dimensional generators if $p \neq 2$. If $p = 2$ then $H^*(X)$ is a polynomial algebra. Thus no operation ψ_r can be nontrivial in $H^*(X)$. The corollary is now immediate since the hypothesis insures that all differentials vanish by Theorem 6.1.

L. Smith has proven that if $X = \Omega Y$ where Y is an H space and if $H^*(X)$ is primitively generated then it is free commutative [24].

Since each differential is related to Bocksteins, we can get some relation between higher torsion in X and higher differentials.

COROLLARY 6.3. *Assume that d_{p^k-1} or d_{2p^k-1} is nontrivial in $E_2 \approx \mathrm{Tor}_{H^*(X)} \Rightarrow H^*(\Omega X)$. Then there is a class $x \in H^q(X; Z)$ for $q \equiv 2 \pmod{2p^k}$ which generates a cyclic group of order p^k.*

This result is an immediate consequence of Theorem 6.1 and the fact that nontrivial classes in Im β_k can be pulled back to classes of order p^k in $H^*(X; Z)$ [2].

The results of Section 7 imply that if the MMP $\mu(u, p^k)$ is defined, then so is the higher order cohomology operation $\beta_k\mathscr{P}_k(m)u$. The converse is not true. To see this one can construct the universal example B for $\beta_k\mathscr{P}_k(m)u$. It is not hard to check that

$$\begin{aligned} \pi_q(B) &= Z_p && \text{if } q = 2m + 1 \\ &= Z_{p^{k-1}} && \text{if } q = 2mp^k + 1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

However, if $u \in H^{2m+1}(B)$ is the generator, $\langle u \rangle^p = -\beta\mathscr{P}^m u \neq 0$. Thus $\mu(u, p^k)$ is not defined if $k > 1$.

Note that $\sigma\mathscr{P}_k(m)u = (\sigma u)^{p^k}$ and $\beta_k(\sigma u)^{p^k} = 0$ in $H^*(\Omega B)/\mathrm{Im}\,\beta_{k-1}$ by Theorem 5.4 in [2]. Thus $\sigma x = 0$ modulo Im β_{k-1} where x represents the next k invariant of B. This, of course, does not imply that $x \in \mathrm{Ker}\,\sigma$.

This suggests that $\mu(u, p^k)$ is a proper subset of $\beta_k\mathscr{P}_k(m)u$. In the next section we conjecture the precise relationship between the Massey products in the cohomology of H spaces and higher order Bocksteins.

Loop operations, the higher order analogues of σ, have been defined and studied in [12]. The primary loop operation $\langle u \rangle_\Omega$ is σu and the secondary loop operation $\langle u, v \rangle_\Omega$ is defined in $H^*(\Omega X)$ if $u \cdot v = 0$ in $H^*(X)$. Furthermore if the Massey product $\langle u_1, \ldots, u_k \rangle$ is defined and contains 0 in $H^*(X)$, then the loop operation $\langle u_1, \ldots, u_k \rangle_\Omega$ is defined in $H^*(\Omega X)$. In the notation of Section 2, $\mu(u, 1)_\Omega = \sigma u$ and $\mu(p^r, v, 1)_\Omega = \langle v, v^{p^r-1} \rangle_\Omega = \phi_r(v)$. This combined with Theorem 9 of [12] gives the following.

THEOREM 6.4. *If $\mu(u, p^k)$ is defined in $H^*(X)$, then $\mu(u, i)_\Omega$ is defined in $H^*(\Omega X)$ and contains the ith divided power $\gamma_i \sigma u$ for $i < p^k$. If $\mu(p^r, v, p^k)$ is defined in $H^*(X)$, then $\mu(p^r, v, i)_\Omega$ for $i < p^k$ is defined in $H^*(\Omega X)$ and contains $\gamma_i \phi_r(v)$ where $\phi_r(v)$ is the transpotence.*

Section 7

Canonically defined n-fold matric Massey products have as indeterminacy the union of all m-fold MMP's of appropriate dimension for $m < n$. Thus it would appear that $\mu(u, p^k)$ would be a much larger subset than $\langle u, \ldots, u\rangle$ (p^k-fold). In [11] it was shown how to define restricted Massey products $\langle u\rangle^n$ which in general are proper subsets of $\langle u, \ldots, u\rangle$ (n-fold).

If $x \cdot y = 0$, then it is sometimes possible to define a proper subset $\langle x, y\rangle^n$ of the $2n$-fold product $\langle x, y, \ldots, x, y\rangle$. It is easy to show that if $\langle u\rangle^{p^k}$ is defined, then so is $\mu(u, p^k)$ and the former is a subset of the latter. A similar relation holds for $\langle v, v^{p^r-1}\rangle^{p^k}$ and $\mu(p^r, v, p^k)$.

Actually it appears that the converse is true in 1-connected H spaces modulo decomposables, that is 2-fold MMP's.

Conjecture 7.1. If $\mu(u, p^k)$ is defined in $H^*(X)$, then $\langle u\rangle^{p^k}$ is defined and $\langle u\rangle^{p^k} \subset -\beta_k \mathscr{P}^{p^{k-1}m} \cdots \mathscr{P}^m u$. If $\mu(p^r, v, p^k)$ is defined, then $\langle v, v^{p^r-1}\rangle^{p^k}$ is defined and is a subset of $-\beta_k \mathscr{P}^{p^{k-1}m} \cdots \mathscr{P}^{pm} \psi_r(v)$ for $m = p^r s - 1$ and $v \in H^{2s}(X)$. Moreover $\langle u\rangle^{p^k}$ and $\langle v, v^{p^r-1}\rangle$ are additive higher order operations of one variable, and operations of these two types generate Ker σ as a Z_p module.

Theorems A and B do not have immediate extensions to coefficient rings other than Z_p. For example with rational coefficients, σ is an isomorphism in each dimension. With Z_{p^k} coefficients the Pontrjagin powers are in the kernel of σ. In general the situation can be quite complicated, as Theorem C in [18] illustrates.

In the two-stage Postnikov system (P, v, w) considered in Section 3, the transpotence element ϕ_r lives in $H^{2sp^r-2}(\Omega P)$. By [17] in the dual homology EMSS

$$E^2 \approx \operatorname{Tor}^{H_*(\Omega P)}(Z_p, Z_p) \Rightarrow H_*(P)$$

we have that $d^{p^r-1}[a \mid \cdots \mid a] = [b]$ for appropriate classes a and b in $H_*(\Omega P)$. By [19] the p^r-fold homology Massey product, or at least the associated canonically defined MMP, is defined and contains b. Kochman has shown that the p-fold homology Massey product is always defined in $H_*(X)$, if X is a three-fold loop space, and equals $-\beta \mathbf{Q}^s a$ where $\mathbf{Q}$ is the Dyer-Lashof operation. Thus if $r = 1$, then $b = -\beta \mathbf{Q}^s a \in H_{2sp-2}(\Omega P)$ [10].

Conjecture 7.2. There is a higher order Dyer-Lashof operation

$$\langle \beta \mathbf{Q}^{p^{r-1}s}, \ldots, \beta \mathbf{Q}^s, a\rangle$$

defined in $H_*(\Omega P)$ which contains b.

In support of this conjecture note that $\beta \mathbf{Q}^{ps} \beta \mathbf{Q}^s = 0$ by the homology Adem relations. However the analogy with cohomology operations is not as good as one might hope. It can easily be shown that $\mathbf{Q}^{ps}\mathbf{Q}^{s}a = 0$ in $H_*(\Omega P)$ and so we cannot hope for the homology analogue of Theorem B, namely $\beta_2\mathbf{Q}^{ps}\mathbf{Q}^{s}a$ cannot be equal to $\langle a, \ldots, a\rangle$ (p^2 times).

It is probable that a Dyer-Lashof analogue of ψ_r can be defined in the Postnikov system E_k of Section 4.

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