EQUIVARIANT MAPS AND HAAR SYSTEMS

BY

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1. Introduction

Let X denote a compact space and $\mathbf{C}(X)$ the function space of complex-valued maps on X with the uniform norm. If $H = \{f_1, \ldots, f_n\}$ is a linearly independent finite set of functions, then H is a Haar system (of length n) on X [10] if and only if for any choice of n distinct points $x_1, \ldots, x_n$ of X,

$$\det(f_i(x_j)) \neq 0. \tag{1}$$

If we let $F_n(X)$ denote the nth configuration space of X [6], i.e.,

$$F_n(X) = \{(x_1, \ldots, x_n) \mid x_i \in X, x_i \neq x_j \text{ for } i \neq j\}, \tag{2}$$

then (1) gives rise to a function $\phi: F_n(X) \to S^1$ with the property that

$$\phi(x_{i_1}, \ldots, x_{i_n}) = \operatorname{sgn}(i_1, \ldots, i_n)\phi(x_1, \ldots, x_n). \tag{3}$$

In the language of equivariant maps, (3) says the following:

If Σ^n is the full symmetric group, then Σ^n acts (freely) on the configuration space $F_n(X)$ by permuting coordinates. Σ^n also acts on any sphere S^k, using the usual homomorphism $\Sigma^n \to \mathbf{Z}_2$ and the free action of $\mathbf{Z}_2$ on S^k via the antipodal map. Then, we see that the previous comments amount to saying that the existence of a Haar system on X implies the existence of an equivariant map

$$\phi: F_n(X) \to S^1.$$

Schoenberg and Yang [10] showed that if X is a finite polyhedron, then X admits a Haar system of length n for some $n \geq 2$ if and only if X imbeds in the plane. This result was extended to Peano continua by Overdeck [9]. In both papers, a key result (whose proof is due to Loewner) is that S^2 does not admit a Haar system of length $n \geq 2$. The following theorem is the content of [5]:

THEOREM 1. *There is no equivariant map $\phi: F_n(S^k) \to S^{k-1}$, $n \geq 2$, $k \geq 2$, unless $k = 3, 7$, and in these cases only if $n = 3$.*

This then gives a purely topological result which implies the nonexistence of Haar systems on S^2. The proof of Theorem 1 in the special case of S^2 involved the existence of elements of finite order in the fundamental group $\pi_1(F_n(S^2)/\Sigma^n)$, namely the Dirac Braid [4]. Our first objective (Section 2) is to give an alternate

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proof of this special case which happens to work for arbitrary orientable compact 2-manifolds. This is interesting because the braid groups $\pi_1(F_n(M^2)/\Sigma^n)$ of compact manifolds, other than the sphere and projective plane, have no elements of finite order [6]. In Section 3 we prove the topological analogue of the Schoenberg-Yang result for finite polyhedra of dimension ≥ 2, namely:

THEOREM 2. *If X is a finite polyhedron,* $\dim X \geq 2$, *then X admits an equivariant map*

$$\phi\colon F_n(X) \to S^1, \quad n \geq 2, \tag{4}$$

if and only if X imbeds in the plane $\mathbf{R}^2$.

In Section 4, we exhibit explicit examples of equivariant maps

$$F_3(S^2) \to S^2, \quad F_3(S^7) \to S^6,$$

which are the exceptional cases of Theorem 1. Finally, in Section 5 we introduce the concept of configuration index of a space which allows a simpler reformulation of results of this kind.

2. The case of two manifolds

If X is any space and $F_n(X)$ is its nth configuration space, we let $B_n(X)$ denote the orbit space $F_n(X)/\Sigma^n$. Then, if $\phi\colon F_n(X) \to S^1$ is an equivariant map, we have an induced diagram

$$\begin{array}{ccc} F_n(X) & \xrightarrow{\phi} & S^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B_n(X) & \xrightarrow{\bar{\Phi}} & S^1 \end{array} \tag{5}$$

where p is an $(n!)$-fold covering map, q a 2-fold covering map, and $\bar{\Phi}$ is induced by ϕ. Then (5) in turn gives rise to a commutative diagram

$$\begin{array}{ccc} 1 & & 0 \\ \downarrow & & \downarrow \\ \pi_1(F_n(X)) & \xrightarrow{\phi_\#} & \pi_1(S^1) \\ {\scriptstyle p_\#}\downarrow & & \downarrow{\scriptstyle q_\#} \\ \pi_1(B_n(X)) & \xrightarrow{\bar{\Phi}_\#} & \pi_1(S^1) \\ \downarrow & & \downarrow \\ \Sigma^n & \longrightarrow & \mathbf{Z}_2 \\ \downarrow & & \downarrow \\ 1 & & 0 \end{array} \tag{6}$$

As usual, $\bar{\Phi}_\#$ factors through singular homology

(7)
$$\begin{tikzcd} \pi_1(B_n(X)) \arrow[rr, "\bar{\Phi}_\#"] \arrow[dr, "\gamma"'] & & \pi_1(S^1) \\ & H_1(B_n(X)) \arrow[ur, "\bar{\Phi}_*"'] & \end{tikzcd}$$

where γ makes $\pi_1(B_n(X))$ abelian. Now, if M^2 is any 2-manifold, we may take n distinct points in a coordinate patch and consider the element σ_i in $\pi_1(B_n(M^2))$ determined by the path (see [7]) in Figure 1.

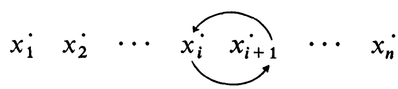

Figure 1. σ_i

Just as in the $\mathbf{R}^2$-case studied by Artin [1], these elements satisfy the relations

(8) $$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i - j| \geq 2,$$

(9) $$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

Furthermore, the element

$$u = \sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_2\sigma_1$$

is represented by the path [7] in Figure 2.

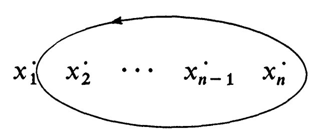

Figure 2. $\sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_2\sigma_1$

Now, we provide a simple proof of the following result.

THEOREM 2.1. *If M^2 is a compact orientable 2-manifold, then M^2 does not admit an equivariant map $\phi: F_n(M^2) \to S^1$, $n \geq 2$.*

Proof. We employ the notation introduced above. First, observe that (9) implies that

(10) $$\bar{\Phi}_\#(\sigma_i) = \bar{\Phi}_\#(\sigma_{i+1}), \quad i = 1, \ldots, n-2,$$

so that

(11) $$\bar{\Phi}_\#(\sigma_1) = \bar{\Phi}_\#(\sigma_i), \quad i = 1, \ldots, n-1.$$

Since the loop in Figure 2 bounds a 2-chain in the "exterior" of this loop, it follows easily that $\gamma(u) = 0$ in $H_1(B_n(X))$ and hence using (7),

$$\bar{\phi}_{\#}(u) = 2(n-1)\bar{\phi}_{\#}(\sigma_1) = 0. \tag{12}$$

But, notice that σ_1 is associated with an odd permutation and using (6)

$$\bar{\phi}_{\#}(\sigma_1) \neq 0. \tag{13}$$

This implies that $\pi_1(S^1)$, written additively, has elements of finite order which contradicts the fact that $\pi_1(S^1) = \mathbf{Z}$.

3. The case of finite polyhedra

We now make use of Theorem 2.1 when M^2 is S^2 and *Wu*'s imbedding classes to prove the following result.

THEOREM 3.1. *If X is a finite polyhedron,* $\dim X \geq 2$, *then there exists an equivariant map ϕ: $F_n(X) \to S^1$ for some $n \geq 2$ if and only if X imbeds in* $\mathbf{R}^2$.

Before we proceed to the proof, we will recall some pertinent facts. Given X, we have a diagram

$$\begin{array}{ccc} F_2(X) & \longrightarrow & S^\infty \\ {\scriptstyle p}\downarrow & & \downarrow \\ B_2(X) & \xrightarrow{f} & \mathbf{R}P^\infty \end{array} \tag{14}$$

where f classifies the covering map p. Then the *Wu* (mod 2) imbedding classes $\Phi^i(X)$ are given by

$$\Phi^i(X) = f^*(x^i) \in H^i(B_2(X); \mathbf{Z}_2) \tag{15}$$

where x is the nonzero element of $H^1(\mathbf{R}P^\infty; \mathbf{Z}_2)$. For example, it is known that in the following cases $\Phi^2(X) \neq 0$ (see [11] and [12]):

Case 1. $X = K_1$ or K_2 the "Kuratowski graphs" [8],
Case 2. $X = S^2$,
Case 3. $X = L$, a 2-disc with a "feeler" emanating from its center.

It is also more or less classical that a 2-complex X imbeds in S^2 if and only if X fails to contain copies of K_1, K_2, or L (see [2] and [8]). Putting all these facts together, we obtain the more or less known result:

PROPOSITION 3.2. *A finite 2-complex X is imbeddable in the plane* $\mathbf{R}^2$ *if and only if* $\Phi^2(X) = 0$.

Proof of Theorem 3.1. First we observe that because of Theorem 2.1, we may assume that $\dim X = 2$. Furthermore, since equivariant maps $F_n(\mathbf{R}^2) \to S^1$ abound, we consider only the "only if" part and proceed by induction on n.

For $n = 2$, the existence of an equivariant map $\phi\colon F_2(X) \to S^1$ gives rise to a diagram

$$\begin{array}{ccccc} F_2(X) & \to & S^1 & \to & S^\infty \\ \downarrow & & \downarrow & & \downarrow \\ B_2(X) & \to & S^1 & \to & \mathbf{R}P^\infty \end{array} \tag{16}$$

which forces $\Phi^2(X) = 0$. Thus in this case, X imbeds in the plane $\mathbf{R}^2$. Proceeding by induction, suppose $\phi\colon F_n(X) \to S^1$ is a given equivariant map, $n > 2$. Choose a point p in the interior of a 2-simplex and a small open disc neighborhood N of p so that $X - \bar{N}$ remains connected, where $\bar{N}$ is the closure of N. Then we can define an equivariant map

$$\psi(x_1, \ldots, x_{n-1}) = \phi(x_1, \ldots, x_{n-1}, p). \tag{17}$$

Thus, by induction, $X - N$ imbeds in $\mathbf{R}^2 \subset S^2$. Since $X - \bar{N}$ is connected, the imbedding takes $X - \bar{N}$ into one component of $S^2 - C$, where C is the simple closed curve corresponding to the boundary of N. Thus the imbedding extends easily to X by mapping N to the other component of $S^2 - C$. Now, that we have X imbedded in S^2 and we know that since X is not all of S^2 (because of Theorem 2.1), we have X is imbedded in $\mathbf{R}^2$.

Remark. It is not difficult to see that Theorem 3.1 remains valid for finite CW-complexes.

4. Some special cases

We consider now the exceptional cases $k = 3, 7$ of Theorem 1. We handle the case $k = 3$ and remark that the case $k = 7$ follows in a completely analogous fashion.

We are looking for an equivariant map $\phi\colon F_3(S^3) \to S^2$. Consider the maps

$$S^2 \xleftarrow{\gamma} SO(3) \xleftarrow{\beta} SO(4) \xrightarrow{\alpha} G^+_{2,2} \tag{18}$$

where $G^+_{k,n}$ represents the oriented k-planes in $(n + k)$-space and $SO(n)$ the special orthogonal group in $\mathbf{R}^n$. The maps α, β, and γ are given explicitly as follows.

(a) $\alpha(A) = [e_1, e_2]$, where $[e_1, e_2]$ is the oriented 2-plane determined by the first two columns of the matrix $A \in SO(4)$.

(b) $\beta(A) = q_R^{-1} \circ A$, where q is the first column of A considered as a quaternion and q_R^{-1} is right multiplication by q^{-1}.

(c) γ evaluates at $(0, 1, 0, 0)$, where $SO(3)$ acts on $0 \times \mathbf{R}^3 \subseteq \mathbf{R}^4$.

LEMMA 4.1. *The map $f = \gamma\beta\alpha^{-1}\colon G^+_{2,2} \to S^2$ is well defined and equivariant, where $\mathbf{Z}_2$ acts on oriented planes by reversing orientations and on S^2 via the antipodal map.*

Proof. A straightforward exercise.

Given three distinct points (x_1, x_2, x_3) on any sphere S^n, they determine an oriented nondegenerate triangle and hence an oriented 2-plane $[x_1, x_2, x_3]$. The resulting map $g\colon F_3(S^n) \to G^+_{2,n-1}$ is clearly equivariant in the sense that an odd permutation of (x_1, x_2, x_3) results in a change in orientation of $[x_1, x_2, x_3]$. Combining this remark with Lemma 4.1 which has an analogue for the case $G^+_{2,6} \to S^6$, we obtain:

THEOREM 4.2. *The map* $\phi = g \circ f\colon F_3(S^k) \to S^{k-1}$, $k = 3, 7$ *is equivariant.*

5. The configuration index

The results in [5] and in the previous section can be reformulated if one introduces the following notion.

DEFINITION 5.1. Given a space X and a positive integer $n \geq 2$, we define the *nth configuration index* $c_n(X)$ of X as the smallest integer k such that $F_n(X)$ admits an equivariant map (in the sense of previous sections) to the k-sphere S^k (compare [3]).

Thus, for example, $c_2(S^k) = k$; $c_3(S^3) = 2$.

THEOREM 5.2. *If a space X is a k-dimensional locally finite complex* ($k < \infty$), *then*

$$c_n(X) \leq nk, \tag{19}$$

so that for such X's, $c_n(X)$ is a well-defined nonnegative integer.

Proof. The proof is not difficult and we content ourselves with a sketch. The covering map $p\colon F_n(X) \to B_n(X)$ induces a homomorphism $\pi_1(B_n(X)) \to \Sigma^n$ which in turn induces the diagram

$$\begin{tikzcd} \pi_1(B_n(X)) \arrow[r, "\gamma"] \arrow[d] \arrow[dr] & H_1(B_n(X)) \arrow[d, "\eta"] \\ \Sigma^n \arrow[r] & \mathbf{Z}_2 \end{tikzcd}$$

where γ makes π_1 abelian. Then,

$$\eta \in \operatorname{Hom}(H_1(B_n(X)), \mathbf{Z}_2) \cong H^1(B_n(X); \mathbf{Z}_2) \cong [B_n(X), \mathbf{R}P^\infty].$$

Choose a map $\bar{\phi}\colon B_n(X) \to \mathbf{R}P^\infty$ corresponding to η under the above identifications. Since $\pi_i(\mathbf{R}P^\infty, \mathbf{R}P^m) = 0$ for $i \leq m$, we may deform $\bar{\phi}$ into $\mathbf{R}P^m$ as long as $nk \leq m$. There is no difficulty lifting $\bar{\phi}\colon B_n(X) \to \mathbf{R}P^m$ to obtain

$$\begin{tikzcd} F_n(X) \arrow[r, "\phi"] \arrow[d] & S^m \arrow[d] \\ B_n(X) \arrow[r, "\bar{\phi}"] & \mathbf{R}P^m \end{tikzcd}$$

and one checks easily that ϕ is equivariant.

In terms this configuration index we can summarize results as follows:

(a) $c_n(S^1) = \begin{cases} 0 & \text{for } n = 2k+1 \\ 1 & \text{for } n = 2k. \end{cases}$

(b) $c_2(S^k) = k, \quad k \geq 1.$

(c) $c_3(S^k) = \begin{cases} k & \text{for } k \neq 1, 3, 7 \\ k-1 & \text{for } k = 1, 3, 7. \end{cases}$

(d) $c_n(M^k) \geq k$, for any k-manifold, $k \geq 3$, $n \geq 4$.

(e) $c_2(M^2) = 2$ for any compact orientable 2-manifold,

(f) $c_3(M^2) = 2$ for any compact orientable 2-manifold.

(g) $c_n(X) = 1$ for some $n \geq 2$ implies X imbeds in $\mathbf{R}^2$, whenever X is a finite complex of dim ≥ 2.

Remarks. (a) and (b) are simple exercises. (c) and (d) follow from [5]. (e) and (f) are consequences of imbedding $M^2 \subset S^3$. (g) is Theorem 3.1.

6. An open question

Theorem 3.1 contains a glaring omission, namely the case where X is a 1-complex. There is no difficulty in one direction namely: If X is a 1-complex which imbeds in $\mathbf{R}^2$, then there exist equivariant maps $\phi\colon F_n(X) \to S^1$ for *every* $n \geq 2$. This follows because it is easy to construct equivariant maps $F_n(\mathbf{R}^2) \to S^1$ for every $n \geq 2$. For example, think of $\mathbf{R}^2$ as complex numbers and use the Haar system $1, z, \ldots, z^{n-1}$. As stated, the converse requires embedding X in $\mathbf{R}^2$ under the assumption that an equivariant map $\phi\colon F_n(X) \to S^1$ exists for *some* $n \geq 2$. If $n = 2$, there is no problem since this case forces the Wu invariant $\Phi^2(X) = 0$ and X cannot contain the Kuratowski graphs K_1 and K_2. The difficulty then is isolated as follows.

Question 6.1. If $n > 2$ and there is an equivariant map $\phi\colon F_n(X) \to S^1$, does this force $\Phi^2(X) = 0$ when X is a 1-complex?

There is an alternative way of looking at this question and that is to consider the diagram

$$\begin{tikzcd} F_n(X) \arrow[d, "p"'] \arrow[dr] & \\ B_n(X) & A_n(X) \arrow[l, "q"] \end{tikzcd}$$

where $A_n(X)$ is the orbit space of the action of the alternating group $A_n \subset \Sigma^n$ and q is the associated 2-fold cover. Then, we have a classifying map

$$\begin{tikzcd} A_n(X) \arrow[r] \arrow[d] & S^\infty \arrow[d] \\ B_n(X) \arrow[r, "f_n"] & \mathbf{R}P^\infty \end{tikzcd}$$

and we define generalized mod 2 Wu classes by setting

$$\Phi_n^i(X) = f_n^*(u^i) \in H^i(B_n(X); \mathbf{Z}^2)$$

where u is the nonzero element of $H^1(\mathbf{R}P^\infty; \mathbf{Z}_2)$.

Problem 6.2. Relate $\Phi_n^i(X)$ and $\Phi_2^i(X) = \Phi^i(X)$.

More specifically, a solution to the following problem will allow us to extend Theorem 3.1 to include 1-complexes.

Problem 6.3. If K is one of the two Kuratowski graphs, show that $\Phi_n^2(K) \neq 0$ for every $n \geq 2$.

Added in Proof. A detailed proof of the result, used in Proposition 3.2, that a 2-complex X embeds in S^2 if, and only if, X fails to contain K_1, K_2, or L may be found in S. Mardešić and J. Segal, *On polyhedra embeddable in the* 2-*sphere*, Glasnik Matematički, vol. 1 (21), (1966), pp. 167–175.

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