A NOTE ON MODULES WITH WAISTS

BY

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In [2], a class of indecomposable modules was introduced, namely modules which contain a waist. Recall that a module M contains a waist if there is a proper nonzero submodule N of M which either contains or is contained in each submodule of M. The purpose of this note is to extend Theorems 2.2 and 2.4 of [2] to arbitrary Artin algebras, not just those with radical square zero. Recall also that an Artin algebra Λ is an Artin ring which is a finitely generated module over its center, C, which is a commutative Artin ring. The most important class of Artin algebras is the class of rings which are finite dimensional over a field. If Λ is an Artin algebra with center C then there is a duality, D, between the categories of left and right finitely generated Λ-modules. This duality is given by $D(X) = \operatorname{Hom}_C(X, E)$ where X is finitely generated left or right Λ-module and E is the C-injective envelope of $C/\operatorname{rad}(C)$. Finally, we also correct a misprint in [2].

We fix some notation for this paper. Unless otherwise stated, all modules will be finitely generated left modules. Let Λ be an Artin algebra with radical $\mathbf{r}$ and let X be a Λ-module. Then the top of X, denoted top (X), is $X/\mathbf{r}X$. The Loewy length of X, denoted $ll(X)$, is smallest integer n such that $\mathbf{r}^n X = 0$. Finally, the lower Loewy series for X,

$$0 \subset_{\neq} S^0(X) \subset S^1(X) \subset \cdots \subset S^t(X) = X$$

is defined by $S^0(X) = \operatorname{soc}(X)$, the socle of X, and

$$S^i(X) = \pi_i^{-1}(\operatorname{soc}(X/S^{i-1}(X)))$$

where $\pi_i\colon X \to X/S^{i-1}(X)$ is the canonical surjection.

THEOREM 1. *Let Λ be an Artin algebra. If Λ is of finite representation type then every module containing a waist has either a simple top or simple socle.*

Proof. Assume that Λ is of finite representation type and let X be a module containing a waist. By [2, Proposition 1.4], X is finitely generated. Assume that both the top of X and the socle of X are not simple.

By [3, Theorem 2.2], it follows that $ll(X) \geq 3$. We proceed by induction on $ll(X)$. We first show that top $(\mathbf{r}X)$ is simple. If top $(\mathbf{r}X)$ is not simple then either $\mathbf{r}X$ or $X/\mathbf{r}^2X$ has a waist contradicting the induction hypothesis. By [2, Proposition 1.3] $\mathbf{r}X$ is a waist in X.

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Next, it follows that $\mathbf{r}^nX/\mathbf{r}^{n+1}X$ is nonsimple for some $n > 1$. If not, each $\mathbf{r}^jX/\mathbf{r}^{j+1}X$ is simple and hence $\operatorname{soc}(X) = \mathbf{r}^mX/\mathbf{r}^{m+1}X$, where $ll(X) = m + 1$, a contradiction. Let n be the smallest integer greater than 1 such that $\mathbf{r}^nX/\mathbf{r}^{n+1}X$ is nonsimple. Since $X/\mathbf{r}^{n+1}X$ contains a waist [2, Theorem 1.2], by induction on $ll(X)$, we conclude that $n + 1 = ll(X)$.

Next, if $0 \subset_{\neq} S^0(X) \subset S^1(X) \subset \cdots \subset S^n(X)$ is the lower Loewy series for X, then $S^i(X) = \mathbf{r}^{n-i}X$. This follows immediately from the fact that $\operatorname{top}(\mathbf{r}^iX)$ is simple for $1 < i < n$.

Now let $S_1 \oplus S_2$ be a summand of $\mathbf{r}^nX = \operatorname{soc}(X)$ and $T_1 \oplus T_2$ be a summand of $\operatorname{top}(X)$, where S_1, S_2, T_1, T_2 are simple Λ-modules.

Consider the ring $\Gamma = \Lambda/\mathbf{r} \ltimes \mathbf{r}^n/\mathbf{r}^{n+1}$, the trivial extension of $\Lambda/\mathbf{r}$ by $\mathbf{r}^n/\mathbf{r}^{n+1}$ (see [1], [3]). Then Γ is a radical square zero Artin algebra. We will show that $S_1 \oplus S_2$ is a summand of $\mathbf{r}^*P^*(T_i)$, $i = 1, 2$ and $T_1 \oplus T_2$ is a summand $E^*(S_i)/S_i$, $i = 1, 2$ where $\mathbf{r}^* = \operatorname{rad}(\Gamma)$, $P^*(T_i)$ is the left Γ-projective cover of T_i and $E^*(S_i)$ is the left Γ-injective envelope of S_i. Once this is shown, it follows by the arguments given in [2, Theorem 2.2] (which use the results of W. Müller [5]) that Γ is of infinite representation type. This though, contradicts the assumption that Λ is of finite representation type [1, Proposition 4.8].

We first show that $S_1 \oplus S_2$ is a summand of $\mathbf{r}^*P^*(T_i)$, $i = 1, 2$. Fix i and let $T = T_i$ and $U = S_1 \oplus S_2$. Let e be a primitive idempotent in Λ such that $\Lambda e/\mathbf{r}e \cong T$. Then the result follows if U is a summand of $\mathbf{r}^ne/\mathbf{r}^{n+1}e$ since $\mathbf{r}^* = \mathbf{r}^n/\mathbf{r}^{n+1}$. Let $P(T)$ be the Λ-projective cover of T. Then

$$\begin{tikzcd} P(T) \arrow[r, two heads] \arrow[drr, dashed, "f"'] & T \arrow[r, hook] & \operatorname{top}(X) \\ & & X \arrow[u, two heads] \end{tikzcd}$$

induces a map $f\colon P(T) \to X$ making the diagram commute. Since $\mathbf{r}^nX = \operatorname{soc}(X)$ is contained in every waist in X [2, Corollary 1.2] it follows that

$$f(P(T)) \cap \mathbf{r}^nX = \mathbf{r}^nX.$$

Now $f(P(T)) \cap \mathbf{r}^nX = f(\mathbf{r}^nP(T))$. Thus $\mathbf{r}^nP(T)$ maps surjectively onto U and hence U is a summand of

$$\mathbf{r}^nP(T)/\mathbf{r}^{n+1}P(T) = \mathbf{r}^ne/\mathbf{r}^{n+1}e.$$

Let D denote the duality for Λ and D^* denote the duality for Γ. Fix i and let $S = S_i$ and $W = T_1 \oplus T_2$. We now show W is a summand of $E^*(S)/S$. First note that W is a summand of $E^*(S)/S$ if and only if $D^*(W)$ is a summand of $D^*(E^*(S))\mathbf{r}^*$. Thus it suffices to show that $D(W)$ is a summand of

$$D(E(S))\mathbf{r}^n/D(E(S))\mathbf{r}^{n+1},$$

where $E(S)$ is the Λ-injective envelope of S, since $D^*(E^*(S))$ (resp. $D(E(S))$) is the right Γ (resp. Λ)-projective cover of $D^*(S)$ (resp. D(S)). Now $D(W)$ is a summand of $D(E(S))\mathbf{r}^n/D(E(S))\mathbf{r}^{n+1}$ if and only if W is a summand of $S^n(E(S))/S^{n-1}(E(S))$ where

$$0 \subset_{\neq} S^0(E(S)) \subset \cdots \subset S^j(E(S))$$

is the lower Loewy series for $E(S)$.

Now

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & \operatorname{soc}(X) \arrow[r, hook] & X \arrow[dll, dashed] \\ E(S) & & \end{tikzcd}$$

induces a map $g\colon X \to E(S)$. An argument dual to the one given above shows that $g(X/S^{n-1}(X)) = g(X/\mathbf{r}X) \subset E(S)/S^{n-1}(E(S))$. Thus W is a summand of $S^n(E(S))/S^{n-1}(E(S))$. This completes the proof.

COROLLARY 2. *Let Λ be an Artin algebra. The following statements are equivalent.*

(1) *Every left indecomposable Λ-module contains a waist or is simple.*
(2) *Every left indecomposable Λ-module has a simple top or simple socle.*
(1′) *Every right indecomposable Λ-module contains a waist or is simple.*
(2′) *Every right indecomposable Λ-module has a simple top or simple socle.*

Proof. By duality we need only prove (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2). By [2, Corollary 1.6], Λ has finite representation type. Then (2) follows directly from Theorem 1.

(2) $\Rightarrow$ (1). Clear.

We remark that (1) and (2) are equivalent for left Artin rings. The proof is more complicated and will appear in [4]. The validity of Theorem 1 for left Artin rings is an open question.

Finally, page 474 of [2], line 8, should read: "Again using [5], [7], one sees that for every proper factor ring of this ring every indecomposable . . . ."

REFERENCES

1. M. AUSLANDER, *Representation dimension of Artin algebras*, Queen Mary College Math. Notes, 1971.
2. M. AUSLANDER, E. L. GREEN, AND I. REITEN, *Modules with waists*, Illinois J. Math., vol. 19 (1975), pp. 467–478.
3. R. FOSSUM, P. GRIFFITH, AND I. REITEN, *Trivial extensions of Abelian categories*, Lecture Notes in Mathematics, no. 456, Springer-Verlag, 1975.
4. R. GORDON AND E. L. GREEN, *Modules with cores and amalgamations of indecomposable modules*, Mem. Amer. Math. Soc., to appear.
5. W. MÜLLER, *Unzerlegbare Moduln über Artinschen Ringen*, Math. Zeitschr., vol. 137 (1974), pp. 197–226.

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