A NOTE ON MODULES WITH WAISTS

BY

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In [2], a class of indecomposable modules was introduced, namely modules which contain a waist. Recall that a module M contains a waist if there is a proper nonzero submodule N of M which either contains or is contained in each submodule of M. The purpose of this note is to extend Theorems 2.2 and 2.4 of [2] to arbitrary Artin algebras, not just those with radical square zero. Recall also that an Artin algebra Λ is an Artin ring which is a finitely generated module over its center, C, which is a commutative Artin ring. The most important class of Artin algebras is the class of rings which are finite dimensional over a field. If Λ is an Artin algebra with center C then there is a duality, D, between the categories of left and right finitely generated Λ -modules. This duality is given by $D(X) = \text{Hom}_{C}(X, E)$ where X is finitely generated left or right Λ -module and E is the C-injective envelope of C/rad(C). Finally, we also correct a misprint in [2].

We fix some notation for this paper. Unless otherwise stated, all modules will be finitely generated left modules. Let Λ be an Artin algebra with radical **r** and let X be a Λ -module. Then the top of X, denoted top (X), is $X/\mathbf{r}X$. The Loewy length of X, denoted ll(X), is smallest integer n such that $\mathbf{r}^n X = 0$. Finally, the lower Loewy series for X,

 $0 \subset J S^0(X) \subset S^1(X) \subset \cdots \subset S^t(X) = X$

is defined by $S^{0}(X) = \text{soc } (X)$, the socle of X, and

$$S^{i}(X) = \pi_{i}^{-1} (\operatorname{soc} (X/S^{i-1}(X)))$$

where $\pi_i: X \to X/S^{i-1}(X)$ is the canonical surjection.

THEOREM 1. Let Λ be an Artin algebra. If Λ is of finite representation type then every module containing a waist has either a simple top or simple socle.

Proof. Assume that Λ is of finite representation type and let X be a module containing a waist. By [2, Proposition 1.4], X is finitely generated. Assume that both the top of X and the socle of X are not simple.

By [3, Theorem 2.2], it follows that $ll(X) \ge 3$. We proceed by induction on ll(X). We first show that top $(\mathbf{r}X)$ is simple. If top $(\mathbf{r}X)$ is not simple then either $\mathbf{r}X$ or X/\mathbf{r}^2X has a waist contradicting the induction hypothesis. By [2, Proposition 1.3] $\mathbf{r}X$ is a waist in X.

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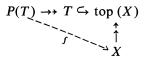
Next, it follows that $\mathbf{r}^n X/\mathbf{r}^{n+1}X$ is nonsimple for some n > 1. If not, each $\mathbf{r}^j X/\mathbf{r}^{j+1}X$ is simple and hence soc $(X) = \mathbf{r}^m X/\mathbf{r}^{m+1}X$, where ll(X) = m + 1, a contradiction. Let *n* be the smallest integer greater than 1 such that $\mathbf{r}^n X/\mathbf{r}^{n+1}X$ is nonsimple. Since $X/\mathbf{r}^{n+1}X$ contains a waist [2, Theorem 1.2], by induction on ll(X), we conclude that n + 1 = ll(X).

Next, if $0 \subset I \leq S^0(X) \subset S^1(X) \subset \cdots \subset S^n(X)$ is the lower Loewy series for X, then $S^i(X) = \mathbf{r}^{n-i}X$. This follows immediately from the fact that top $(\mathbf{r}^i X)$ is simple for 1 < i < n.

Now let $S_1 \oplus S_2$ be a summand of $\mathbf{r}^n X = \operatorname{soc} (X)$ and $T_1 \oplus T_2$ be a summand of top (X), where S_1, S_2, T_1, T_2 are simple Λ -modules.

Consider the ring $\Gamma = \Lambda/\mathbf{r} \ltimes \mathbf{r}^n/\mathbf{r}^{n+1}$, the trivial extension of Λ/\mathbf{r} by $\mathbf{r}^n/\mathbf{r}^{n+1}$ (see [1], [3]). Then Γ is a radical square zero Artin algebra. We will show that $S_1 \oplus S_2$ is a summand of $\mathbf{r}^*P^*(T_i)$, i = 1, 2 and $T_1 \oplus T_2$ is a summand $E^*(S_i)/S_i$, i = 1, 2 where $\mathbf{r}^* = \operatorname{rad}(\Gamma)$, $P^*(T_i)$ is the left Γ -projective cover of T_i and $E^*(S_i)$ is the left Γ -injective envelope of S_i . Once this is shown, it follows by the arguments given in [2, Theorem 2.2] (which use the results of W. Müller [5]) that Γ is of infinite representation type. This though, contradicts the assumption that Λ is of finite representation type [1, Proposition 4.8].

We first show that $S_1 \oplus S_2$ is a summand of $\mathbf{r}^* P^*(T_i)$, i = 1, 2. Fix *i* and let $T = T_i$ and $U = S_1 \oplus S_2$. Let *e* be a primitive idempotent in Λ such that $\Lambda e/\mathbf{r}e \cong T$. Then the result follows if *U* is a summand of $\mathbf{r}^n e/\mathbf{r}^{n+1}e$ since $\mathbf{r}^* = \mathbf{r}^n/\mathbf{r}^{n+1}$. Let P(T) be the Λ -projective cover of *T*. Then



induces a map $f: P(T) \to X$ making the diagram commute. Since $\mathbf{r}^n X = \text{soc } (X)$ is contained in every waist in X [2, Corollary 1.2] it follows that

$$f(P(T)) \cap \mathbf{r}^n X = \mathbf{r}^n X.$$

Now $f(P(T)) \cap \mathbf{r}^n X = f(\mathbf{r}^n P(T))$. Thus $\mathbf{r}^n P(T)$ maps surjectively onto U and hence U is a summand of

$$\mathbf{r}^n P(T)/\mathbf{r}^{n+1} P(T) = \mathbf{r}^n e/\mathbf{r}^{n+1} e.$$

Let D denote the duality for Λ and D* denote the duality for Γ . Fix *i* and let $S = S_i$ and $W = T_1 \oplus T_2$. We now show W is a summand of $E^*(S)/S$. First note that W is a summand of $E^*(S)/S$ if and only if $D^*(W)$ is a summand of $D^*(E^*(S))\mathbf{r}^*$. Thus it suffices to show that D(W) is a summand of

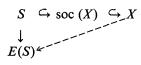
$$D(E(S))\mathbf{r}^n/D(E(S))\mathbf{r}^{n+1},$$

where E(S) is the Λ -injective envelope of S, since $D^*(E^*(S))$ (resp. D(E(S))) is the right Γ (resp. Λ)-projective cover of $D^*(S)$ (resp. D(S)). Now D(W) is a summand of $D(E(S))\mathbf{r}^n/D(E(S))\mathbf{r}^{n+1}$ if and only if W is a summand of $S^n(E(S))/S^{n-1}(E(S))$ where

$$0 \subset {}_{\neq} S^0(E(S)) \subset \cdots \subset S^j(E(S))$$

is the lower Loewy series for E(S).

Now



induces a map $g: X \to E(S)$. An argument dual to the one given above shows that $g(X/S^{n-1}(X)) = g(X/\mathbf{r}X) \subset E(S)/S^{n-1}(E(S))$. Thus W is a summand of $S^n(E(S))/S^{n-1}(E(S))$. This completes the proof.

COROLLARY 2. Let Λ be an Artin algebra. The following statements are equivalent.

- (1) Every left indecomposable Λ -module contains a waist or is simple.
- (2) Every left indecomposable Λ -module has a simple top or simple socle.
- (1') Every right indecomposable Λ -module contains a waist or is simple.
- (2') Every right indecomposable Λ -module has a simple top or simple socle.

Proof. By duality we need only prove $(1) \Leftrightarrow (2)$.

(1) \Rightarrow (2). By [2, Corollary 1.6], Λ has finite representation type. Then (2) follows directly from Theorem 1.

 $(2) \Rightarrow (1)$. Clear.

We remark that (1) and (2) are equivalent for left Artin rings. The proof is more complicated and will appear in [4]. The validity of Theorem 1 for left Artin rings is an open question.

Finally, page 474 of [2], line 8, should read: "Again using [5], [7], one sees that for every proper factor ring of this ring every indecomposable"

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