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ON THE AUTOMORPHISM GROUP OF AN INTEGRAL GROUP RING, II

BY

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1. Introduction

Let G be a finite group, $Z(G)$ denote the integral group ring of G, and $NA(G)$ denote the group of normalized automorphisms of $Z(G)$. That is, $NA(G)$ denotes the group of ring automorphisms f of $Z(G)$ such that $f(g)$ has augmentation one for all $g \in G$. As remarked in [1] and [5], little generality is lost by studying normalized automorphisms over arbitrary automorphisms of $Z(G)$.

The objective of this paper is to extend the previously known list of metabelian E. R. groups. E. R. *groups* are groups G in which every element of $NA(G)$ has an elementary representation. Here, by saying that f in $NA(G)$ has an *elementary representation*, we mean that f can be written in the form $f = \tau_u \sigma$ where σ lies in the automorphism group of G, denoted by Aut (G), (actually σ extended linearly to $Z(G)$) and τ_u denotes conjugation by a unit u in $Q(G)$ (the group algebra of G over the rational field). In the notation of [5], saying that G is an E. R. group is equivalent to saying that $NA(G) = CP(G)$ Aut (G) where

$$CP(G) = \{\tau_u \mid u \text{ is a unit in } Q(G) \text{ normalizing } Z(G)\}.$$

Metabelian E. R. groups which have been obtained elsewhere include: (1) class ≤ 2 nilpotent groups from [7], and from [1], (2) groups with a cyclic normal subgroup of prime index, (3) groups with at most one nonlinear irreducible character, and (4) groups G in which $|G'| = 2$ or 3. In [6], it is shown that the symmetric groups are E. R. groups.

In Section 3, we will obtain a sufficient condition for a group which is a product of an abelian normal subgroup and an abelian subgroup to be an E. R. group. Using this result, we will show that groups containing a cyclic normal subgroup with an abelian supplement are E. R. groups, thereby generalizing (2) and, it turns out, (4). We will also see that groups G in which G/Z is metacyclic, Z the center of G, are E. R. groups. In Section 4, we will obtain some additional metabelian p-groups which are E. R. groups and consider a related problem on when the complement for Aut (G) in $NA(G)$ obtained in [5] for metabelian p-groups is contained in $CP(G)$.

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2. Preliminary notions

We have that $NA(G)$ acts as a permutation group on the class sums of G and the kernel of this permutation representation is $CP(G)$ (see [1], [5], or [7]). To be explicit, if $f \in NA(G)$ and $\bar{C}_g$ denotes the class sum of $g \in G$, $f(\bar{C}_g) = \bar{C}_x$ for some x in G. Further, the subgroup of $NA(G)$ fixing every class sum of G is in fact $CP(G)$.

The reader should now note that a normalized automorphism f will have an elementary representation provided there is a $\sigma \in \operatorname{Aut}(G)$ which has the same action on the class sums of G as f.

Note that if $f(\bar{C}_g) = \bar{C}_x$, $|C_g| = |C_x|$ where C_g denotes the conjugacy class of $g \in G$. Another result we will need concerning the action of $NA(G)$ on class sums is the following from [1] (see also [4, Proposition 2]).

LEMMA 2.1. *Let $f \in NA(G)$ and suppose $f(\bar{C}_g) = \bar{C}_x$ where $g, x \in G$. Then for every integer n, $f(\bar{C}_{g^n}) = \bar{C}_{x^n}$ and $|g| = |x|$.*

$NA(G)$ also acts as a permutation group on the characters of G. That is, if χ is a character of G and $f \in NA(G)$, we can define a new character χ^f by setting $\chi^f(g) = \chi(f(g))$, $g \in G$. For our needs, it will suffice to assume that our representations are defined over the complex field. Some elementary facts are as follows:

LEMMA 2.2. *Let $f \in NA(G)$.*

(i) *χ is irreducible if and only if χ^f is irreducible.*
(ii) *If $f(\bar{C}_g) = \bar{C}_x$ where $g, x \in G$, then $\chi^f(g) = \chi(x)$.*
(iii) *χ is faithful if and only if χ^f is faithful.*
(iv) *Let $h \in NA(G)$ and $g \in G$. Then $f(\bar{C}_g) = h(\bar{C}_g)$ if and only if $\chi^f(g) = \chi^h(g)$ for every irreducible character χ of G.*

Proof. (i) is clear, and (ii) follows since

$$\chi^f(g) = \chi^f(\bar{C}_g)/|C_g| = \chi(\bar{C}_x)/|C_x| = \chi(x).$$

To see (iii), note that for $g \in G$, $\chi^f(g) = \chi(1)$ if and only if $\chi(x) = \chi(1)$ where $f(\bar{C}_g) = \bar{C}_x$.

(iv) follows easily from (ii).

3. Metabelian groups

For $N \lhd G$, let $\Delta(N)$ denote the kernel of the natural map from $Z(G)$ to $Z(G/N)$. It follows from [5] that the subgroup

$$\omega(G, G') = \{f \in NA(G) \mid f(g) \equiv g \bmod \Delta(G')\Delta(G) \text{ for all } g \in G\}$$

of $NA(G)$ is a complement for $\operatorname{Aut}(G)$ in $NA(G)$ when G is metabelian. Thus, in order to show that a metabelian group G is an E. R. group, it suffices to show every element of $\omega(G, G')$ has an elementary representation.

We will also need the following in which (i) is a restatement of Lemma 2.1 of [5] and holds for any group, (ii) follows immediately from Lemma 5.4 of [5], and part (iii) follows directly from (ii).

LEMMA 3.1. *Let* $N \lhd G$.

(i) *For* $f \in NA(G), f(\Delta(N)) = \Delta(N)$ *if and only if* $f(\bar{C}_n)$ *is the class sum of an element of* N *for all* $n \in N$.

(ii) *If* G *is metabelian and* $f \in \omega(G, G')$, *then* $f(\Delta(N)) = \Delta(N)$.

(iii) *If* G *is metabelian and* $f \in \omega(G, G')$, f *induces an automorphism on* $Z(\bar{G})$ *lying in* $\omega(\bar{G}, \bar{G}')$ *where* $\bar{G} = G/N$.

We next obtain some results concerning the action of $\omega(G, G')$ on class sums. The notation Z_n will be used to denote the nth term of the upper central series G starting with $Z_1 = Z$, Z the center of G.

LEMMA 3.2. *Let* G *be a metabelian group and let* $f \in \omega(G, G')$. *Then* $f(\bar{C}_g) = \bar{C}_g$ *for all* $g \in Z_2$.

Proof. If $g \in Z$, it follows that $f(g) \in Z$ from the action of $NA(G)$ on class sums. Now, it is well known that if N is an abelian normal subgroup and $g_1 \equiv g_2 \bmod \Delta(N)\Delta(G)$, $g_1, g_2 \in G$, then $g_1 = g_2$ (see [9] or [8, Corollary 4]). Hence, since $f(g) \equiv g \bmod \Delta(G')\Delta(G)$, $f(g) = g$.

Next, suppose $g \in Z_2 - Z$ and let χ be an irreducible character of G. It suffices to show $\chi^f(g) = \chi(g)$. If χ is not faithful, we may apply induction on $|G|$ by Lemma 3.1 and conclude $\chi^f(g) = \chi(g)$. If χ is faithful, $\chi^f(g) = \chi(g)$ since faithful characters are zero on $Z_2 - Z$.

One might note that the above lemma yields $\omega(G, G') \subseteq CP(G)$ when G is a class ≤ 2 nilpotent group and hence, the result of [7], that class ≤ 2 nilpotent groups are E. R. groups.

Let G_n denote the nth term of the lower central series of G starting with $G_0 = G$.

LEMMA 3.3. *Let* G *be a metabelian group*, $f \in \omega(G, G')$, *and* $g \in G$. *Then there exists* $x \in gG_2$ *such that* $f(\bar{C}_g) = \bar{C}_x$.

Proof. Suppose $f(\bar{C}_g) = \bar{C}_y$, $y \in G$. Since G/G_2 is nilpotent of class ≤ 2,

$$f(\bar{C}_g) = \bar{C}_y \equiv \bar{C}_g \mod \Delta(G_2).$$

Thus, for some x in the conjugacy class of y, $x \equiv g \bmod \Delta(G_2)$. Hence $x \in gG_2$ and $f(\bar{C}_g) = \bar{C}_x$.

Finally, we develop a lemma on faithful characters. To begin with, suppose A is a normal subgroup of G, let χ be an irreducible character of G, and let M denote an irreducible $C(G)$-module (C the complex field) affording χ. From the results of section 50 of [2], we have that if M_1 is a homogeneous component of M viewed as a $C(A)$-module and if $A^* = \{g \in G \mid gM_1 = M_1\}$, then M_1 is an irreducible $C(A^*)$-module and $M_1^G = M$. Let $C_G(A)$ denote the centralizer of A in G.

LEMMA 3.4. *Suppose A and G/A are abelian. If χ is faithful, then $A^* \subseteq C_G(A)$.*

Proof. Let Γ denote the irreducible representation of A^* afforded by M_1. Since M_1 is a direct sum of isomorphic $C(A)$-modules and since A is abelian, $\Gamma(a)$ is a scalar matrix for all $a \in A$. Hence if $g \in A^*$ and $a \in A$, $\Gamma(ga) = \Gamma(ag)$. Moreover, $\Gamma^G(ga) = \Gamma^G(ag)$ since $A^* \lhd G$, whence $A^* \subseteq C_G(A)$ since Γ^G is faithful.

THEOREM 3.5. *Suppose $G = AB$ where A is an abelian normal subgroup of G and B is an abelian subgroup of G. In addition, suppose that for any $f \in \omega(G, G')$ we can find a $\sigma \in \mathrm{Aut}\,(G)$ such that $f(\bar{C}_a) = \sigma(\bar{C}_a)$ for all $a \in A$ and $\sigma(b) = b$ for all $b \in B$. Then $f(\bar{C}_g) = \sigma(\bar{C}_g)$ for all $g \in G$ and hence G is an E. R. group.*

Proof. It suffices to show $\chi^f = \chi^\sigma$ for every irreducible character χ of G.

If χ is not faithful, let M be a minimal normal subgroup of G contained in $\ker \chi$. Set $\bar{G} = G/M$. Since f induces an automorphism on $Z(\bar{G})$ lying in $\omega(\bar{G}, \bar{G}')$, we would be able to conclude $\chi^f = \chi^\sigma$ by induction on $|G|$ provided σ also induces an automorphism on $\bar{G}$. To show this, we show $\sigma(M) = M$.

Suppose $M \subseteq A$. From Lemma 3.1, $\sigma(\bar{C}_m) = f(\bar{C}_m)$ is a class sum of an element of M for all $m \in M$. Hence $\sigma(M) = M$.

Now, suppose $M \nsubseteq A$. Then $M \cap A = 1$ and it follows that $[M, G] = 1$. Thus $M \subseteq Z$. If $ab \in M$ where $a \in A$, $b \in B$, then $a \in Z$ and hence $\sigma(a) = f(a) = a$ by Lemma 3.2. Therefore $\sigma(ab) = ab$, and again $\sigma(M) = M$.

Thus, we may assume χ is faithful. Let A^* be as in the setting of Lemma 3.4. If $g \in G - A^*$, $\chi(g) = 0$ since χ is induced from the normal subgroup A^*. Similarly, $\chi^f(g) = 0$, since if $f(\bar{C}_g) = \bar{C}_x$ where $x \in gG_2$, $x \notin A^*$ and hence $\chi^f(g) = \chi(x) = 0$. We also have $\chi^\sigma(g) = 0$. For by Lemma 3.1, when $a \in A$, $\sigma(\bar{C}_a) = f(\bar{C}_a)$ is the class sum of an element of A and hence $\sigma(A) = A$. Thus, σ fixes the cosets of A in G, whence $\sigma(g) \notin A^*$. We now have $\chi^f(g) = \chi^\sigma(g) = 0$ for $g \in G - A^*$.

Finally, suppose $g \in A^*$. If $g \in A$, $\chi^f(g) = \chi^\sigma(g)$ by our hypothesis. If $g \in A^* - A$, write $g = ab$, $a \in A$, $b \in B$. Since $A^* \subseteq C_G(A)$, $b \in Z$. Hence we again have $\chi^f(g) = \chi^\sigma(g)$ since

$$f(\bar{C}_{ab}) = f(b\bar{C}_a) = bf(\bar{C}_a) = b\sigma(\bar{C}_a) = \sigma(\bar{C}_{ba}).$$

We next apply Theorem 3.5 to obtain some E. R. groups.

THEOREM 3.6. *Suppose $G = AB$ where A is a cyclic normal subgroup of G and B is an abelian subgroup of G. Then G is an E. R. group.*

Proof. Let $f \in \omega(G, G')$. We will construct $\sigma \in \mathrm{Aut}\,(G)$ satisfying the hypothesis of Theorem 3.5.

Write $A = \langle a \rangle$. By Lemma 3.3, $f(\bar{C}_a) = \bar{C}_{a^s}$ for some positive integer s where $a^s \in aG_2$. Also, $(s, |a|) = 1$ by Lemma 2.1. For $g \in G$, write $g = ba^i$, $b \in B$. Define σ by setting $\sigma(ba^i) = ba^{is}$. Assuming that σ is in fact an automorphism of G, σ will satisfy the hypothesis of Theorem 3.5 since σ has the same action as f on the class sums of A by Lemma 2.1 and σ certainly fixes the elements of B.

To see that $\sigma \in \operatorname{Aut}(G)$, we will only show that σ is well defined. The reader can easily check that σ is 1-1 (use the fact that $(s, |a|) = 1$) and that σ is a homomorphism.

First note that if $a^k \in Z$, $f(a^k) = a^{ks} = a^k$. Hence, if $b_1 a^i = b_2 a^j$ where b_1, $b_2 \in B$, $b_1^{-1} b_2 = a^{i-j} \in Z$, whence $a^{i-j} = a^{(i-j)s}$. It now follows that $\sigma(b_1 a^i)\sigma(b_2 a^j)^{-1} = 1$, so that σ is well defined.

As a corollary to Theorem 3.6, we can generalize Brown's result [1] that groups in which $|G'| = 2$ or 3 are E. R. groups.

COROLLARY 3.7. *If $|G'| = p$, p a prime, then G is an E. R. group.*

Proof. Let P denote the p-Sylow subgroup of G and let K be a p'-Hall subgroup of G. Setting $H = N_G(K) \cap P$, $N_G(K)$ the normalizer of K in G, we have $[K, H] = 1$. Further, since K is a p'-Hall subgroup of KG', $G = N_G(K)KG'$ and hence $G = KHG'$.

If $G' \subseteq H$, $K \subseteq Z$ and G is a class 2 nilpotent group which is an E. R. group. If $G' \nsubseteq H$, $G' \cap H = 1$ and we may apply Theorem 3.6 with $A = G'$ and $B = KH$.

We will next show that groups in which the central quotient is metacyclic are E. R. groups. Concerning such groups we first note the following result.

LEMMA 3.8. *Suppose G contains a normal subgroup A containing Z such that G/A and A/Z are both cyclic. If $x \in G$ generates G/A and $a \in A$ generates A/Z, then $G' = \langle [a, x] \rangle$.*

Proof. First note that the mapping $g \to [g, x]$, $g \in A$, is a homomorphism of A onto G' [3, Aufgabe 2, S. 259] with kernel Z. Hence $|G'| = |A/Z|$. Also, since a and $[a, x]$ commute, $[a, x]^n = [a^n, x]$ for every positive integer n. Therefore, $|[a, x]| = |A/Z|$ and hence $\langle [a, x] \rangle = G'$.

THEOREM 3.9. *If G/Z is metacyclic, then G is an E. R. group.*

Proof. Let A, a, and x be as in the previous lemma. Suppose $f \in \omega(G, G')$ and set $a^x = a^r z_x$ where $z_x \in Z$ and r is a positive integer. Write $f(\bar{C}_a) = \bar{C}_{ay}$ where $y \in G_2$. Then, since $y \in G'$,

$$y = (a^{r-1} z_x)^k = a^{(r-1)k} z_x^k \tag{1}$$

for some positive integer k.

Let $g \in G$ and write $g = x^i a^j z$ where $z \in Z$. We define an automorphism σ of G by setting $\sigma(g) = x^i a^j y^j z$. Assuming that σ is an automorphism of G, we will have σ satisfies the conditions of Theorem 3.5 with $B = \langle x \rangle$. For if $g \in A$, write

$g = a^i z$ where $z \in Z$. Using Lemmas 2.1 and 3.2 we have

$$f(\bar{C}_{a^i z}) = f(z\bar{C}_{a^i}) = z\bar{C}_{a^i y^i} = \sigma(\bar{C}_{a^i z}).$$

To see that σ is well defined, first note that if $a^n \in Z$, $y^n_\cdot = 1$ since $f(a^n) = a^n = a^n y^n$. Thus, if $x^i a^j z_1 = x^m a^n z_2$ where $z_1, z_2 \in Z$, $x^{i-m} z_1 z_2^{-1} = a^{n-j} \in Z$. Hence $y^{n-j} = 1$ and it follows that $\sigma(x^i a^j z_1) = \sigma(x^m a^n z_2)$.

Next, we show σ is 1-1. Suppose $\sigma(x^i a^j z) = x^i a^j y^j z = 1$ where $z \in Z$. Then $a^j y^j = x^{-i} z^{-1} \in Z$. Writing

$$a^j y^j = a^{(1+k(r-1))j} z_x^{kj},$$

we have $a^{(1+k(r-1))j} \in Z$. Also, $(1 + k(r-1), |A/Z|) = 1$, for if $\bar{f}$ denotes the automorphism f induces on $Z(G/Z)$,

$$\bar{f}(\bar{C}_{aZ}) = \bar{C}_{a^{1+k(r-1)}Z}.$$

Hence $1 + k(r-1)$ and $|A/Z|$ are relatively prime by Lemma 2.1. Thus $|A/Z|$ divides j, whence $a^j \in Z$. But we have already seen that this implies $y^j = 1$, and hence $x^i a^j z = 1$.

Finally, we show σ is a homomorphism. Let $x^i a^j z_1$ and $x^m a^n z_2$ be elements of G where $z_1, z_2 \in Z$. Writing $x^{-m} a x^m = a^s z_x^t$ where $s = r^m$ and $t = r^{m-1} + r^{m-2} + \cdots + 1$, we have

$$\begin{aligned}\sigma(x^i a^j z_1 x^m a^n z_2) &= \sigma(x^{i+m} a^{js} z_x^{jt} z_1 a^n z_2)\\ &= x^{i+m} a^{js} y^{js} z_x^{jt} z_1 a^n y^n z_2\\ &= x^i a^j x^m y^{js} x^{-m} z_1 x^m a^n y^n z_2.\end{aligned}$$

Since

$$\sigma(x^i a^j z_1)\sigma(x^m a^n z_2) = x^i a^j y^j z_1 x^m a^n y^n z_2,$$

we need $x^m y^{js} x^{-m} = y^j$ or $x^{-m} y^j x^m = y^{js}$. But this follows, since by (1),

$$\begin{aligned}x^{-m} y^j x^m &= x^{-m} a^{jk(r-1)} z_x^{jk} x^m\\ &= a^{jk(r-1)s} z_x^{jk(r-1)t} z_x^{jk}\\ &= a^{jk(r-1)s} z_x^{jk(s-1)} z_x^{jk}\\ &= y^{js}.\end{aligned}$$

4. Metabelian p-groups

Let us first note the following. If G is a nilpotent group, it suffices to show that each Sylow subgroup of G is an E. R. group in order to establish that G is an E. R. group [5, Corollary 5.3]. Hence, if G is nilpotent, we may assume that G is a p-group.

We will obtain two results on metabelian p-groups G which are E. R. groups involving the maximal abelian normal subgroups of G.

THEOREM 4.1. *Suppose G is a p-group containing a maximal abelian normal subgroup A which is cyclic. Then G is an E. R. group.*

Theorem 4.1 will follow from Theorem 3.6 by the following lemma.

LEMMA 4.2. *Suppose G is a p-group containing a maximal abelian normal subgroup A which is cyclic. Then there exists an abelian subgroup B of G such that $G = AB$.*

Proof. We may assume G/A is not cyclic. Now, since $C_G(A) = A$, G/A is isomorphically contained in Aut (A) when G acts on A via conjugation. Thus, if p is odd or if $p = 2$ and $|A| \leq 4$, G/A is cyclic since Aut (A) is cyclic. Hence, we have $|A| = 2^m$, $m \geq 3$.

Write $A = \langle a \rangle$. Then Aut $(A) = \langle \alpha \rangle \times \langle \beta \rangle$ where $\alpha(a) = a^5$ and $\beta(a) = a^{-1}$. Further, since G/A is not cyclic, we can choose d and c in G so that d and c generate G/A with τ_d a power of α and τ_c equal to β. Next, we show the existence of $b \in G$ such that b and c generate G/A and $[b, c] = 1$. Of course, once we have b, the proof is complete by taking $B = \langle b, c \rangle$.

Let $[d, c] = a^r$. We first show $2 \mid r$. If $|dA| = 2$, write $d^2 = a^j$. Then $2 \mid j$ by the maximality of A and

$$a^{-j} = (d^2)^c = (d^c)^2 = (da^r)^2.$$

Also, since $|\tau_d| = 2$ and since $|\alpha| = 2^{m-2}$,

$$a^d = a^{1+2^{m-1}}.$$

Using the above two equations,

$$a^{-j} = (da^r)^2 = d^2 a^{r(2+2^{m-1})} = a^{j+r(2+2^{m-1})}$$

and hence $r(2 + 2^{m-1}) \equiv -2j \bmod 2^m$. Since $4 \mid -2j$ and since $4 \nmid (2 + 2^{m-1})$, $2 \mid r$.

Now, suppose $|dA| > 2$. If $2 \nmid r$, $a \in G'$ and the Frattini subgroup of G is $\Phi(G) = \langle d^2, a \rangle$. Thus the center of $\Phi(G)$, which is contained in $C_G(A) = A$, is cyclic. But then $\Phi(G)$ is cyclic [3, Satz 7.8 (c), S. 306], a contradiction.

Finally, set $b = da^{r/2}$. Then b and c generate G/A and $[b, c] = 1$ since $b^c = (da^{r/2})^c = da^r a^{-r/2} = b$.

THEOREM 4.3. *Let G be a p-group where $p > 3$. Suppose that every maximal abelian subgroup of G is generated by at most two elements. Then G is an E. R. group.*

Proof. By Satz 12.4, S. 343 of [3], G is one of the following types of groups:

(i) G is metacyclic,
(ii) $G = \langle x, y, z \mid x^p = y^p = z^{p^n} = [x, z] = [y, z] = 1,\ y^x = yz^{p^{n-1}} \rangle$
(iii) $G = \langle x, y, z \mid x^p = y^p = z^{p^n} = [y, z] = 1,\ y^x = yz^{sp^{n-1}},\ z^x = zy \rangle$ where $s = 1$ or is a quadratic nonresidue mod p.

Groups of type (i) are E. R. groups by Theorem 3.9 and groups of type (ii) are also E. R. groups since they have nilpotence class 2. Hence we may assume G is of type (iii).

Let $f \in \omega(G, G')$, set $A = \langle y, z\rangle$, and let $B = \langle x\rangle$. We will construct an automorphism σ of G satisfying Theorem 3.5.

Suppose $f(\bar{C}_z) = \bar{C}_{zc}$ where $c \in G_2 \subseteq Z$. Then $c^p = 1$ since $z^p \in Z$ and since $f(z^p) = z^p = z^p c^p$.

Now, let $z^i y^j \in A$ where $0 \leq i < p^n$, $0 \leq j < p$. We will show

$$f(\bar{C}_{z^i y^j}) = \bar{C}_{z^i c^i y^j}. \tag{2}$$

If $j = 0$, (2) holds by Lemma 2.1. If $j \neq 0$ and if $p \mid i$, $c^i = 0$ and (2) holds since $z^i y^j \in Z_2$. Finally, suppose $j \neq 0$ and $p \nmid i$. Write $i = kp + r$, $0 < r < p$, and choose an integer m so that $mr \equiv j \bmod p$. Then $z^i y^j Z = z^r y^j Z = x^{-m} z^r x^m Z = x^{-m} z^i x^m Z$ and hence for some $v \in Z$, $\bar{C}_{z^i y^j} = \bar{C}_{z^i v} = v\bar{C}_{z^i}$. Thus,

$$f(\bar{C}_{z^i y^j}) = f(v\bar{C}_{z^i}) = v\bar{C}_{z^i c^i} = \bar{C}_{z^i c^i v} = \bar{C}_{z^i c^i y^j}$$

and we again have (2).

Define σ by setting $\sigma(x^i z^j y^k) = x^i z^j c^j y^k$. The reader can check that $\sigma \in \operatorname{Aut}(G)$. In addition, equation (2) shows that Theorem 3.5 applies.

We will conclude by considering the question of when $CP(G)$ contains $\omega(G, G')$ for a metabelian group G. Recall that this is one method of showing class ≤ 2 nilpotent groups are E. R. groups. Moreover, this would be a way of obtaining E. R. groups which avoids the construction of group automorphisms. In fact, we will see the E. R. groups G obtained in Section 3 have $\omega(G, G') \subseteq CP(G)$ provided G is a p-group and p is odd.

Let us return to the setting of Lemma 3.7 and show:

LEMMA 4.4. *Suppose G is a p-group, p odd, and let A be an abelian normal subgroup of G containing Z such that G/A and A/Z are cyclic. Then* $|G'| = |G/A|$.

Proof. If x generates G/A and a generates A/Z we had seen that $\langle [a, x]\rangle = G'$. Hence G is a regular p-group [3, Satz 10.2 (c), S. 322].

Using the fact that $[a, x^{p^n}] = 1$ if and only if $[a, x]^{p^n} = 1$ [3, Satz 10.6 (b), S. 326], it follows that $|G/A| = |\langle [a, x]\rangle| = |G'|$.

THEOREM 4.5. *Suppose G is a p-group, p odd, and either*

(i) *G/Z is metacyclic or*

(ii) *$G = AB$ where A is a cyclic normal subgroup of G and B is an abelian subgroup of G.*

Then $\omega(G, G') \subseteq CP(G)$.

Proof. Suppose (i) holds. Using the notation of the previous lemma, $|C_a| = |G/A| = |G'|$. Thus $C_a = aG'$ and Lemma 3.3 implies $f(\bar{C}_a) = \bar{C}_a$

when $f \in \omega(G, G')$. Moreover, if $a^i z \in A$ where $z \in Z$, then $f(\bar{C}_{a^i z}) = f(z\bar{C}_{a^i}) = z\bar{C}_{a^i} = \bar{C}_{a^i z}$. Hence, we may take $\sigma = 1$ in Theorem 3.5 and conclude $f \in CP(G)$.

(ii) is actually a special case of (i) since B/Z is embedded in Aut (A) when B acts on A by conjugation. Hence $G/Z = (BZ/Z)(AZ/Z)$ is metacyclic since Aut (A) is cyclic.

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