

SCATTERED SPACES II

BY

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Dedicated to Prof. M. Venktaraman who was more than a teacher to us.

Introduction

In this paper we study the images of scattered spaces under various maps. In this process we solve two problems raised by two sets of mathematicians. J. J. Schäffer raised the following problem as problem no. 45 in the problem book of department of mathematics of Carnegie-Mellon University: Can a regular, countably compact, scattered space be mapped by a continuous function onto $[0, 1]$?

The same problem was implied in the remark of P. Nyikos and J. J. Schäffer on page 228 in their paper [9]. It was known earlier [8], [10], [12] that if X is a compact, T_2 scattered space then X cannot be mapped continuously onto $[0, 1]$. Now if X is a countably compact, completely regular space and $f: X \rightarrow [0, 1]$ is a continuous function on X then the range $f(X)$ is closed in $[0, 1]$. Thus the Stone-Čech compactification βX of X can be mapped continuously onto $[0, 1]$ if and only if X itself can be done so. So, attacking the above mentioned problem of Schäffer from this approach, S. P. Franklin raised the problem: If X is a completely regular, scattered, countably compact space, then is βX scattered?

It was conjectured that the answer to the above problem of Franklin is yes and hence the answer to Schäffer's problem above is no.

However, we shall construct a completely regular, scattered, countably compact space X which can be mapped onto $[0, 1]$ by a continuous function. Thus we disprove the above conjecture. Our space X is even locally compact, T_2 , first countable and locally countable and consequently sequentially compact, too. Thus characterizing completely regular scattered spaces that can be mapped continuously onto $[0, 1]$ is nontrivial.

In a related way M. E. Rudin [11] included the following problem in the lecture notes that she gave in a topology conference in Wyoming in August, 1974: Should the image of a scattered space by a closed continuous map be scattered?

The same question was raised by Telgarsky in [14] and [15]. Our space X obviously solves the above problem of Telgarsky in the negative. If we drop the condition that X be completely regular and countably compact, then we get the following surprising result: Every topological space is a closed continuous

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image of a scattered T_1 space X . We show that T_1 cannot be replaced by T_3 in this last statement. A T_2 sequential space is shown to be a closed continuous image of a scattered T_2 sequential space.

A part of the results mentioned here was announced by M. Jayachandran and M. Rajagopalan in [3]. However, their proofs were incorrect and needed some changes which are given here. Between the time of submission of this paper and its publication the paper [20] appeared. So some proofs of theorems in Section 2 of this paper are omitted.

1. The category of scattered spaces

In this section we study the closed continuous images of scattered spaces. We do not assume that the spaces are even Hausdorff in this section.

DEFINITION 1.1. A topological space X is said to be scattered if every subset $A \subset X$ which is not empty, has an isolated point relative to A . In this case we say that X has a scattered topology.

DEFINITION 1.2. Let X be a topological space. We put $X^0 = X$ and

$$X^1 = \{x \mid x \in X \text{ and } x \text{ is not isolated in } X\}.$$

If α is any ordinal number and if X^β is already defined for all ordinals $\beta < \alpha$, then we put

$$X^\alpha = (X^\gamma)^1 \quad \text{if } \alpha = \gamma + 1 \text{ and } \gamma \text{ is an ordinal,}$$

$$\bigcap_{\beta < \alpha} X^\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Note 1.3. X^0, X^1, X^2, \dots is a decreasing transfinite sequence of subsets of X . Then there is either an ordinal δ so that $X^\delta = \emptyset$ or $X^\delta = X^{\delta+1}$.

Note 1.4. A topological space X is scattered if and only if there is an ordinal δ so that $X^\delta = \emptyset$.

DEFINITION 1.5. Let X be a topological space. We say that X has a derived length if there is an ordinal δ so that $X^\delta = \emptyset$. In this case the least ordinal γ so that $X^\gamma = \emptyset$ is called the derived length of X .

LEMMA 1.6. Let X be a topological space. Then we have the following:

- (i) X has a derived length if and only if it is scattered.
- (ii) Every topology which is finer than a scattered topology is scattered.
- (iii) Every subspace of a scattered space is scattered and has a smaller derived length.
- (iv) X^α is closed for each ordinal α .

DEFINITION 1.7. A topological space X is said to be a P -space if any countable intersection of open sets in X is again open.

LEMMA 1.8. *Let X be a scattered T_2 space, Y a topological space, and let $\phi: X \rightarrow Y$ be a closed continuous map. Let ϕ satisfy the following condition:*

(A) *The set $\{\alpha \mid X^\alpha \text{ meets } \phi^{-1}(y)\}$ contains a largest element for every y in Y .*

Then $Y^\alpha \subset \phi(X)$ for every ordinal number α . Consequently Y is also scattered and $\delta(Y) \leq \delta(X^\alpha)$.

Proof. We use induction on $\delta(X) = \alpha$. When $\alpha = 0$ this is obvious.

Now let α be any ordinal number. Suppose, as induction hypothesis, that the assertion is true whenever $\delta(X) < \alpha$.

Case 1. $\alpha = \beta + 1$ for some β . To show $Y^\alpha \subset \phi(X^\alpha)$, let $y \in Y^\alpha$. Consider the set $F = X^\beta \setminus \phi^{-1}(y)$. If $X^{\beta+1} \cap \phi^{-1}(y)$ is empty, then no point of $\phi^{-1}(y)$ is a limit point of X^β and hence F is closed. Since ϕ is a closed map, $\phi(F)$ is closed in Y . But it follows from our induction hypothesis that $\phi(F) \supset Y^\beta \setminus \{y\}$. This implies that y is not in Y^α . This contradiction implies that $Y^\alpha \subset \phi(X^\alpha)$.

Case 2. Let α be a limit ordinal and let $y \in Y^\alpha$. Consider the set $\phi^{-1}(y)$. Let $\alpha_0 = \sup \{\beta \mid X^\beta \text{ meets } \phi^{-1}(y)\}$. By condition (A) assumed, we have that X^{α_0} meets $\phi^{-1}(y)$. If $\alpha_0 \geq \alpha$, it follows that X^α meets $\phi^{-1}(y)$, i.e., $y \in \phi(X^\alpha)$.

If $\alpha_0 < \alpha$, then the complement of $\phi^{-1}\phi(X^{\alpha_0+1})$ is an open neighborhood W of the set $\phi^{-1}(y)$ in X . Let f be the restriction of ϕ to W . Since W is the full inverse of some subset, f is also a closed map. Now W is a scattered Hausdorff space of derived length $\leq \alpha_0 + 1$ and f is a closed continuous map from W onto $Y \setminus \phi(X^{\alpha_0+1})$. Hence by our induction hypothesis, $Z = Y \setminus \phi(X^{\alpha_0+1})$ is scattered and $\delta(Z) \leq \delta(W) \leq \alpha_0 + 1$. But since Z is a neighborhood of y in Y , we have for every ordinal number β , that $y \in Z^\beta$ iff $y \in Y^\beta$. It follows that $y \notin Y^{\alpha_0+1}$. This contradiction shows that α_0 cannot be strictly less than α . Thus $Y^\alpha \subset \phi(X^\alpha)$.

THEOREM 1.9. *Let $\phi: X \rightarrow Y$ be a closed continuous surjection from a Hausdorff scattered space X to a topological space Y . Then each of the following conditions implies that Y is also scattered and $\delta(Y) \leq \delta(X)$:*

- (a) $\delta(X)$ is finite.
- (b) ϕ is perfect (i.e., in addition to ϕ being closed $\phi^{-1}(y)$ is compact for all y in Y).
- (c) X is compact and Y is T_1 .

Proof. We show that any one of the above three conditions implies condition (A) of Lemma 1.8.

(a) If $\delta(X) < \omega$, then the set $A = \{\alpha \mid X^\alpha \text{ meets } \phi^{-1}(y)\}$ is finite and hence contains a largest element for all $y \in Y$.

(b) If ϕ is a perfect map and if $y \in Y$, then $\phi^{-1}(y)$ is compact. Hence as α increases in A , the sets $X^\alpha \cap \phi^{-1}(Y)$ form a well-ordered decreasing family of

nonempty closed subsets in the compact space $\phi^{-1}(y)$. Hence they all have a common point x . If α_0 is the supremum of the set

$$A = \{\alpha \mid X^\alpha \text{ meets } f^{-1}(Y)\},$$

it follows that $x \in \phi^{-1}(y) \cap (\bigcup_{\alpha \in A} X^\alpha) = \phi^{-1}(Y) \cap X^{\alpha_0}$ and hence that α_0 belongs to A .

(c) If X is compact and Y is T_1 , then ϕ has to be perfect and hence the result follows from (b).

Remark 1.10. Neither condition (a) nor condition (b) can be weakened in the above theorem. It will follow from the next proposition that for each infinite ordinal number α , there is a Hausdorff scattered space X with $\delta(X) = \alpha$ and a closed continuous surjection from X onto a nonscattered space.

Secondly, to see how far condition (b) can be weakened, we split the definition of perfectness of ϕ into two parts:

- (i) ϕ is closed
- (ii) $\phi^{-1}(y)$ is compact for every y in Y .

We have shown that (i) and (ii) together imply that ϕ preserves scatteredness. A later proposition shows that (i) above is no good. To see that (ii) alone is also not sufficient, refer to [4], where an example of a perfect T_2 space is given and exhibited as a finite-to-one quotient of a well-ordered space.

Thirdly, some natural weakenings of condition (c) also affect the conclusion. In the example just cited, the domain space is locally compact, σ -compact and metrizable.

PROPOSITION 1.11. *Let Y be a space constructed by the c -process described in [6]. Let each base space of Y be a scattered T_2 P -space with a nonisolated base point. Then Y is a perfect Hausdorff P -space, obtainable as a closed continuous image of a scattered space.*

Proof. For each positive integer n let

$$Y_n = \{y \in Y \mid \text{the level of } y \leq n\}.$$

Let X be the topological sum $\sum_{n=1}^{\infty} Y_n$. Let $\phi: X \rightarrow Y$ be the map defined so that $\phi|_{Y_n}$ is the inclusion of Y_n in Y . Then it can be shown that X is scattered Hausdorff. Clearly the map ϕ is continuous and onto. We shall prove that ϕ is closed.

Let $F \subset X$ be closed. To show that $\phi(F)$ is closed, it suffices to show that $\phi(F) \cap Y_n$ is closed in Y_n for every $n = 1, 2, \dots$. Now $\phi(F) \cap Y_n = Y_n \cap (\bigcup_{m=1}^{\infty} \phi(F \cap Y_m))$ is a countable union of closed subsets of Y_n so it is a closed set, since Y_n must be a P -space.

Remark 1.12. (a) Let R denote the set of all real numbers with discrete topology. Take any free filter on R , stable under the formation of countable

intersections. Declare this, adding $\{1\}$ to each of its members, as the neighborhood filter at 1 and declare all the other points to be isolated. Call this space R_1 . Then R_1 is a T_2 P -space with a unique accumulation point. If we take each base space to be a copy of this R_1 with the unique accumulation point as base point, then the c -process gives us a space Y satisfying the hypothesis of the above proposition.

(b) When Y is constructed as above, and X is constructed as in the proof of Proposition 1.11, we can show that $\delta(X) = \omega$. Thus a closed continuous image of a T_2 scattered space X with $\delta(X) = \omega$, need not be scattered.

(c) Note that the above space X satisfies all the separation axioms. It is zero dimensional and completely normal.

Next we consider the question: What are all the closed continuous images of scattered spaces (in the presence or absence of Hausdorffness condition)? Surprisingly, we prove:

THEOREM 1.13. *Every topological space is a closed continuous image of a scattered space.*

Discussion. Let X be any topological space. Our job is to construct a scattered space Y and a closed continuous map from Y onto X . To motivate the steps of our proof, first we consider a big set Y and a map $\phi: Y \rightarrow X$. We shall show that a scattered topology on Y can be given, making ϕ to be closed and continuous, provided some mild conditions are satisfied.

For this, we first observe that on any set provided with a well-ordering, the left rays (i.e., the subsets which contain with each of its elements all elements smaller than it) form a scattered topology. Hence we shall give a topology to Y , bigger than one such topology so that scatteredness is assured. (Every subset has its least element relatively isolated.) Secondly, there is the smallest topology on Y making ϕ continuous. We have to choose our topology bigger than this also.

Thirdly, the finer the topology on Y , the less is the chance for ϕ to be closed. Hence we take the smallest topology with the above two properties.

There is some hope that when this topology is given for Y , we have the required properties. We show that this indeed is the case when the set Y , the well-order on Y and the map ϕ are suitably chosen.

Proof of Theorem 1.13. We take a well-ordered set P , of the order type of the initial ordinal of an infinite cardinal whose cofinality type is bigger than $|X|$. We let the set Y be the cartesian product $P \times X$. We fix some well-order in the set X and give the lexicographical order to Y . We take ϕ to be the projection map from Y onto X . The topology τ on Y is the join $\tau_1 \vee \tau_2$ of the two topologies

$$\tau_1 = \{\phi^{-1}(V) \mid V \text{ is open in } X\}, \quad \tau_2 = \{L \mid L \text{ is a left ray in } Y\}.$$

The proof of the theorem will be complete, if we show that ϕ is a closed map. Further, let $A \subset Y$ be closed and be such that $X = \overline{\phi(A)}$. Then for each point of $\phi(A)$, choose exactly one preimage in A and thus form a subset B of A such that $|B| \leq |X|$ and $\phi(B) = \phi(A)$. Clearly, B is not cofinal in Y . But for each x in X , the subset $P \times \{x\}$ is cofinal in Y . Therefore there exists an element y of Y such that y is greater than every element of B and such that $\phi(y) = x$. We claim that y belongs to the closure of B . If W is any basic neighborhood of y , then $W = \phi^{-1}(V) \cap L$ where V is a neighborhood of x in X and L is some left ray containing y . Since x is in the closure of $\phi(B)$, the set $V \cap \phi(B)$ is nonempty. Therefore there is an element b in $B \cap \phi^{-1}(V)$. Since y belongs to L and $b < y$ (because b is in B), b belongs to L also. Thus b is common to W and B . This proves that y is in the closure of B and hence in the closure of the bigger set A . But A is closed. Therefore y belongs to A and hence x (which is same as $\phi(y)$) belongs to $\phi(A)$. Thus $\phi(A)$ is closed.

Remark 1.14. The above topological space Y is necessarily non-Hausdorff, whatever X may be (so long as X is infinite).

In the light of the above theory, it is natural to ask: Is every space, a closed continuous image of a scattered space satisfying stronger separation axioms? We can give a negative answer to this question as follows:

LEMMA 1.15. βN is not a closed continuous image of any regular scattered T_2 -space.

Proof. If possible, let X be a regular scattered T_2 space and $\Psi: X \rightarrow \beta N$ a continuous closed map from X onto βN . Let $\Psi^{-1}(\{n\}) = A_n$ and let $a_n \in A_n$ for all $n \in N$. Let

$$Y = \overline{\{a_1, a_2, \dots, a_n, \dots\}}$$

in X . Then Ψ maps Y onto βN and a_n is isolated in Y for all $n \in N$ and $\Psi|_Y$ is a closed continuous map. Y is also a regular T_2 scattered space. Moreover

$$\Psi^{-1}(\{n\}) \cap Y = \{a_n\}$$

for all $n \in N$ since A_n is open and closed in X for all $n \in N$. Let b be an isolated point of $Y - \{a_1, a_2, \dots, a_n, \dots\}$. Then, by the regularity of Y , we have that there is a closed neighborhood V of b relative to Y so that

$$V = \{b\} \cup (V \cap \{a_1, a_2, \dots, a_n, \dots\}).$$

Since V is closed in Y , $\Psi(V)$ is closed in βN . Since $\Psi(V)$ is further countable it follows that $\Psi(V)$ is finite. Since Ψ is 1-1 on V it follows that V is finite. Then b has to be isolated in Y which is not true. Thus βN cannot be a closed continuous image of a T_2 regular scattered space.

In the case of sequential Hausdorff spaces we have a positive result:

THEOREM 1.16. *Let X be a Hausdorff sequential space. Then there exists a Hausdorff sequential scattered space Y and a closed continuous map ϕ from Y onto X .*

Proof. The set Y and the map ϕ are chosen as in the proof of Theorem 1.13, but the topology on Y is changed (to a finer one). Also in choosing Y , we shall insist that $|Y| = |Y|^{\aleph_0}$.

We shall now define a suitable topology on Y and show that it satisfies our requirements. If X is discrete, we allow Y also to be discrete and we are through in this trivial case. Hence we assume that X is not discrete. In that case there is at least one convergent sequence in X with distinct terms that are distinct from its limit. If we choose one preimage under ϕ of each of its terms, they constitute a subset A of Y with the following conditions:

- (a) A is countably infinite.
- (b) ϕ is one-to-one on A .
- (c) $\phi(A)$ is the set of a convergent sequence in X , without its limit.

Let F be the family of all those subsets A of Y satisfying the above three conditions. We have just now seen that F is nonempty. As an easy application of Zorn's lemma, we choose a maximal subfamily F_1 of F such that whenever A and B are distinct members of F_1 , we have that $A \cap B$ is finite. We observe the following for later use. If A is a member of F then there is a member B of F_1 such that $A \cap B$ is infinite.

The topology that we are going to define on Y , is in terms of this F_1 . For each member of F_1 , we shall first associate an element of Y as follows: well-order the members of F_1 in the form $A_1, A_2, \dots, A_\alpha, \dots$ to the order type of the initial ordinal of $|F_1|$. Observe that $|F_1|$ does not exceed $|Y|$.

Suppose we have defined y_β in Y for A_β for each $\beta < \alpha$. To define y_α , we look at $\phi(A_\alpha)$. Since $A_\alpha \in F$, $\phi(A_\alpha)$ is the set of a convergent sequence in X . This convergent sequence has a unique limit in X , say x_α . Consider the set $\phi^{-1}(x_\alpha)$. This is cofinal in Y . On the other hand, the set $\{y_\beta: \beta < \alpha\} \cup A_\alpha$ has a smaller cardinality than that of Y . Hence there are points y in Y with the following properties:

- (i) $\phi(y) = x_\alpha$.
- (ii) $y \neq y_\beta$ for every $\beta < \alpha$.
- (iii) y is greater than all but a finite number of elements of A_α .

Among such elements y , we take the least element and call it y_α .

Just to make it convenient in writing, we put $A_\alpha = \emptyset$ if $|\alpha| \geq |F_1|$.

Now we define a topology on Y as follows: A subset $V \subset Y$ is open if and only if for each ordinal α , either y_α is not in V , or $A_\alpha \setminus V$ is finite. We prove that this topology has all the required properties.

Claim 1. ϕ is continuous.

Let $W \subset X$ be open. To show that $\phi^{-1}(W)$ is open in Y , we show that if $y_\alpha \in \phi^{-1}(W)$, then $A_\alpha \setminus \phi^{-1}(W)$ is finite. If $y_\alpha \in \phi^{-1}(W)$, then $\phi(y_\alpha) \in W$. Also, by the choice of y_α , we have that $\phi(A_\alpha)$ is the set of a sequence in X converging to y_α . Since W is open, this sequence must be eventually in W . Therefore $\phi(A_\alpha) \setminus W$ is finite. Since ϕ is one-to-one on A_α it follows that $A_\alpha \setminus \phi^{-1}(W)$ is finite.

Claim 2. ϕ is closed.

Let $F \subset Y$ be closed and let (x_n) be a sequence in $\phi(F)$ with distinct terms and converging to x . We claim that $x \in \phi(F)$. For each positive integer n choose an element y_n in F such that $\phi(y_n) = x_n$ and let $A = \{y \in Y \mid y = y_n \text{ for some } n\}$. Then by the observation made just before defining the topology of Y , it follows that $A \cap A_{\alpha_0}$ is infinite for some ordinal number α_0 . Now the element y_{α_0} has to be in F . For if it is in the set $Y \setminus F$, then since $Y \setminus F$ is open, we have $A_{\alpha_0} \setminus (Y \setminus F)$ finite. But $A_{\alpha_0} \setminus (Y \setminus F)$ is the same as $F \cap A_{\alpha_0}$ which contains the infinite set $A \cap A_{\alpha_0}$. This contradiction proves that $y_{\alpha_0} \in F$. Therefore $\phi(y_{\alpha_0}) \in \phi(F)$. Now, $\phi(y_{\alpha_0})$ is the unique limit of the sequence $\phi(A_{\alpha_0})$. But the sequence $\phi(A_{\alpha_0})$ is a subsequence of (x_n) and hence must converge to x . Thus $x = \phi(y_{\alpha_0}) \in \phi(F)$. Thus $\phi(F)$ is closed under limits of sequences with distinct terms. Since X is a T_2 -space, it follows that $\phi(F)$ is sequentially closed. Since X is sequential, $\phi(F)$ is closed in X .

Claim 3. Y is scattered.

It suffices to show that every left ray L is open. Let $y_\alpha \in L$. Then by the definition of y_α we have that y_α is larger than all but a finite number of elements of A_α . This implies that $A_\alpha \setminus L$ is finite. Thus L is open.

Claim 4. Y is sequential.

First we observe that for every ordinal α , the set A_α when arranged as a sequence with distinct terms in any manner, converges to y_α . For, if V is any neighborhood of y_α , by the definition of the topology, V contains all but a finite number of elements of A_α .

Now let W be any sequentially open subset of Y . Let $y_\alpha \in W$. Then by the above observation, the sequence A_α is eventually in W . That is, $A_\alpha \setminus W$ is finite. This shows that W is open.

Claim 5. Y is Hausdorff.

The proof of this is somewhat complicated; the idea underlying this proof is given at the end as a separate remark.

For each y in Y , we define a set S_y by the rule

$$S_y = \begin{cases} A_\alpha & \text{if } y = y_\alpha \\ \{y\} & \text{if there is no } \alpha \text{ such that } y = y_\alpha. \end{cases}$$

Note that a set V is open if and only if $S_y \setminus V$ is finite for each y in V .

Now let p and q be any two distinct elements of Y . To separate them by disjoint neighborhoods, we proceed as follows:

Let FS be the set of all finite sequences of natural numbers. We shall now define two functions p^* and q^* from FS to Y and later prove the following four assertions.

- (1) p is in the range of p^* .
- (2) q is in the range of q^* .
- (3) p^* and q^* have disjoint ranges.
- (4) The ranges of p^* and q^* are open.

We observe first that the set FS is countable. We write its elements in the form of a sequence $0 = w_1, w_2, \dots$ where 0 is the empty sequence (of length 0). This is to be done in a particular way so as to satisfy a condition that we shall soon prescribe.

If w and w' are two elements of FS , we say that w just extends w' if the following are true:

- (i) For some nonnegative integer γ it is true that w' has exactly γ terms and w has exactly $\gamma + 1$ terms.
- (ii) w and w' have the same n th term if n is a positive integer $\leq \gamma$.

If w just extends w' , then we want w to come later than w' in the above sequence. In other words, if n and m are two positive integers with $n < m$, then w_n cannot just extend w_m . It is possible to arrange the elements of FS in a sequence satisfying this condition. For example, after putting 0 as the first term, we may first write those elements $(a_1, a_2, \dots, a_\gamma) = w$ of FS whose sum $a_1 + a_2 + \dots + a_\gamma$ is 1, then those whose sum is 2, and so on.

We define p^* and q^* recursively by induction. For each positive integer n , we let $A_n = \{w \in FS \mid \text{either } w = w_n \text{ or } w \text{ just extends } w_n\}$. We observe that $FS = (\bigcup_{n=1}^{\infty} A_n)$. We can easily prove that for each n , the set $A_{n+1} \setminus \{w_{n+1}\}$ is disjoint with $\bigcup_{i=1}^n A_i$. Our inductive method is to define p^* and q^* on A_n at the n th stage.

We describe the first stage now. We let $p^*(0) = p$ and $q^*(0) = q$. We look at the set S_p . If S_p is a singleton, then we let p^* be constant on all sequences of length ≤ 1 (taking the value p). If S_p is not a singleton, then we fix a bijection from N to $S_p \setminus \{p, q\}$ and map the sequence a_1 (of length 1) to that element of $S_p \setminus \{p, q\}$ that corresponds to the integer a_1 in this bijection. Now we look at the set $S_q^* = (S_q \setminus S_p) \setminus \{p, q\}$. If S_q is a singleton, we let q^* be constant (with value q) on all sequences of length ≤ 1 . If S_q^* is not a singleton, then we observe that S_q^* is countably infinite. We fix a bijection from N to S_q^* and map the sequence a_1 of length 1, under q^* , to that point of S_q^* which corresponds to the integer a_1 in this bijection. Thus p^* and q^* have been defined on A_1 so that $p^*(A_1) \subset S_p$ and $q^*(A_1) \subset S_q$. Also p^* and q^* are 1-1 on A_1 .

Suppose now as induction hypothesis that we have defined $p^*(A_\gamma)$ and $q^*(A_\gamma)$

for every $\gamma \leq n$ such that

- (i) $p^*(A_\gamma) \subset \bigcup_{i \leq \gamma} S_{p^*(w_i)}$ for $\gamma \leq n$,
- (ii) $q^*(A_\gamma) \subset \bigcup_{i \leq \gamma} S_{q^*(w_i)}$ for $\gamma \leq n$,
- (iii) p^* and q^* are one-to-one in the part hitherto defined, and
- (iv) no point is hitherto in the range of both these functions.

Then we define p^* and q^* on A_{n+1} as follows. First we observe that $w_{n+1} \in A_\gamma$ for some $\gamma \leq n$ and hence $p^*(w_{n+1})$ and $q^*(w_{n+1})$ have already been defined. Suppose $w \in A_{n+1} \setminus \{w_{n+1}\}$. Then w just extends w_{n+1} . Look at the set $S_{p^*(w_{n+1})}$. If it is a singleton, define $p^*(w) = p^*(w_{n+1})$. If it is not a singleton, let

$$S_{n+1,p}^* = S_{p^*(w_{n+1})} \left[\left(\bigcup_{i=1}^n p^*(A_i) \right) \cup \left(\bigcup_{i=1}^n q^*(A_i) \right) \right]$$

The set inside the bracket in the right side is, by our induction hypothesis, contained in

$$\bigcup_{i=1}^n (S_{p^*(w_i)} \cup S_{q^*(w_i)}).$$

On the other hand $S_{p^*(w_{n+1})}$ meets $S_{p^*(w_i)} \cup S_{q^*(w_i)}$ in a finite set for each i . (Because these are distinct members of F or finite.) Hence $S_{n+1,p}^* = S_{p^*(w_{n+1})} \setminus$ a finite set. We fix any bijection from N to this set and define $p^*(w)$ as that element of $S_{n+1,p}^*$ which corresponds to the positive integer s in this bijection, where s is the last term of w .

Now we look at $S_{q^*(w_{n+1})}$. If it is a singleton, we define $q^*(w) = q^*(w_{n+1})$. Otherwise, we let

$$\begin{aligned} S_{n+1,q}^* &= S_{q^*(w_{n+1})} \left[\left(\bigcup_{i=1}^{n+1} p^*(A_i) \right) \cup \left(\bigcup_{i=1}^n q^*(A_i) \right) \right] \\ &= S_{q^*(w_{n+1})} \setminus \text{a finite set.} \end{aligned}$$

We fix a bijection from N to this set and define $q^*(w)$ as that element of $S_{n+1,q}^*$ which corresponds to the positive integer s in this bijection, where s is the last term of w .

Now we observe that

- (i) $p^*(A_{n+1}) \subset \bigcup_{i=1}^{n+1} S_{p^*(w_i)}$,
- (ii) $q^*(A_{n+1}) \subset \bigcup_{i=1}^{n+1} S_{q^*(w_i)}$, and
- (iii) p^* and q^* are one-to-one on $\bigcup_{i=1}^{n+1} A_i$.

This defines, by induction, p^* and q^* on the whole of FS . When defined in this way, p^* and q^* have disjoint ranges. Suppose $p^*(w_l) = q^*(w_m)$ for some $l \leq m$. Let l_1 and m_1 be the least positive integers such that $p^*(w_{l_1}) = p^*(w_l)$ and $q^*(w_{m_1}) = q^*(w_m)$. If $l_1 \leq m_1$, this contradicts the definition of $q^*(w_{m_1})$. If $l_1 > m_1$, this contradicts the definition of $p^*(w_{l_1})$. Hence these two functions must have disjoint ranges.

Let v_p and v_q be respectively the ranges of p^* and q^* . Suppose $y \in v_p$. Then $y = p^*(w_l)$ for some l . If S_y is a singleton, then certainly $S_y \subset v_p$. If S_y is not a singleton, then a look at the definition of p^* on A_p that is, on the set of elements just extending w_l , gives at once that all but a finite number of points of S_y are in the range of p^* . That is, $S_y \setminus v_p$ is finite. Thus $y \in v_p$ implies that $S_y \setminus v_p$ is finite. This implies that v_p is open, by the fact mentioned just after the definition of S_y in this proof. Similarly v_q is also open. Thus p and q are separated by their disjoint neighborhoods v_p and v_q .

The proof of the theorem is now complete.

THEOREM 1.17. *The following are equivalent for a topological space X :*

- (1) *X is a sequential space with unique sequential limits.*
- (2) *S is a closed continuous image of a Hausdorff scattered sequential space Y .*

Proof. Let X satisfy (1). Then we repeat the proof of Theorem 1.16 to show that X satisfies (2) also. We have only to note that in that proof, Hausdorffness of X was not fully used. We used only the fact that sequential limits are unique, when they exist.

Conversely, let X satisfy (2). Then X is obviously sequential, since sequentialness is invariant under all quotient maps (and therefore under closed continuous maps). If possible let (x_n) be a sequence in X converging to two distinct points x and x' . If (x_n) contains a constant subsequence with value x_0 , then that subsequence must converge to both x and x' ; this implies that $\{x_0\}$ is not closed. But if y is a point of Y which is mapped to x_0 by a closed continuous map ϕ , $\{y\}$ is closed in Y , since Y is Hausdorff; hence $\{x_0\}$ is closed in X , since the map is closed. This contradiction proves that (x_n) cannot contain a constant subsequence (x_{n_i}) whose terms are mutually distinct and distinct from x and x' . For each n_i choose one point y_{n_i} in Y such that $\phi(y_{n_i}) = x_{n_i}$. Let $A = \{y_{n_i} | i = 1, 2, \dots\}$. Then we easily see that $\phi(A)$ is not closed in X and therefore A is not closed in Y . Hence there is a sequence (a_n) in A converging to a point (a) outside A . Since Y is Hausdorff, the compact set

$$B = \{a\} \cup \{y \in A | y = a_n \text{ for some } n\}$$

is closed in Y . Hence $\phi(B)$ is closed in X . But $\phi(B) = \{\phi(a)\} \cup C$ for some infinite subset C of $\{x_{n_i} | i = 1, 2, \dots\}$. Now both x and x' belong to the closure of C . But neither x nor x' belongs to C . Thus on one hand, there are at least two points in $\bar{C} \setminus C$ and on the other hand $C \cup \{\phi(a)\}$ is closed. This contradiction proves that (x_n) cannot converge to two distinct points.

Remark 1.18. (a) While proving Theorem 1.16, we defined a curious topology on Y . Apparently it is unnatural and unmotivated. Here we explain how we are led to consider that topology. Suppose we want to describe it in terms of its convergent sequences. Since we want the topology to be scattered, we safely prescribe the condition that if a sequence (y_n) converges to y in Y , then even-

tually y_n should be $\leq y$. (This would imply that each left ray is open and hence the space is scattered.) At the same time, since we want ϕ to be continuous, we should not allow (y_n) to converge to y unless $\phi(y_n)$ converges to $\phi(y)$. Also, since the chances for ϕ to be closed are higher when the class of convergent sequences in Y is bigger, we shall try to have it as big as possible. Since we want the topology to be Hausdorff, we should not allow a sequence to converge to two distinct points. Thus we have a host of conditions and our job now is to prove their compactibility by exhibiting a topology with such a convergence scheme. Instead of working with sequences, we found it more convenient to work with their underlying sets. Thus in the proof, the members of F are the underlying sets of a "generating" class of convergent sequences. The maximality of F insures that the convergence scheme is going to be big. The method by which we associated some element of Y to each member of F is now well-explained by the above remarks. Clearly our topology Y is the strongest one in which these convergent sequences have these prescribed limits.

(b) It can be shown that our topology on Y is crucial; there is no finer sequential topology on Y that would allow ϕ to be closed; however, there are coarser sequential Hausdorff topologies on Y allowing ϕ to be both continuous and closed. We may also observe that F is not uniquely specified by Y , ϕ , and X . For various choices of F , we get various such topologies on Y , and all these are mutually uncomparable.

(c) Consider the class S of all spaces obtainable as closed continuous images of scattered T_2 spaces. Then one can prove without much difficulty that S is stable under the formation of finer topologies and subspaces. In particular if X admits a one-to-one continuous map into some T_2 sequential space, then $X \in S$. Thus S is quite large. We do not know whether every Hausdorff space belongs to S .

2. Shaffer's problem

We assume continuum hypothesis in this section. We use N to denote the set of all integers > 0 . Let Ω denote the first uncountable ordinal. In general we use greek letters like α, β , etc. to denote a countable ordinal. English letters like m, n , etc. are used to denote members of N . If A, B are subsets of $[0, 1]$ then $\text{dm}(A, B)$ denotes the diameter of $A \cup B$.

DEFINITION 2.1. Let A_1, A_2, A_3, \dots be a sequence of subsets of $[0, 1]$. We say that this sequence converges to an element y in $[0, 1]$ if $\text{dm}(A_n, \{y\}) \rightarrow 0$ as $n \rightarrow \infty$. In this case we also write $A_n \rightarrow y$ or that (A_n) converges to y .

THEOREM 2.2. Let $A_1, A_2, \dots, A_n, \dots$ be a sequence of subsets of $[0, 1]$. Then the following are equivalent:

- (a) The sequence (A_n) converges to some y in $[0, 1]$.
- (b) There is some $y \in [0, 1]$ so that $x_n \rightarrow y$ for every sequence $x_1, x_2, x_3, \dots, x_n, \dots$ so that $x_n \in A_n$ for all $n \in N$.

(c) $\text{dm}(A_n) \rightarrow 0$ as $n \rightarrow \infty$ and there is a sequence $a_1, a_2, \dots, a_m, \dots$ so that $a_n \rightarrow y$ and $a_n \in A_n$, $n \in N$.

Proof. Obvious.

Hereafter we take a fixed function $\phi: N \rightarrow Q$ which is 1-1 and onto the set Q of all rational numbers in $[0, 1]$. Let βN be the Stone-Čech compactification of N with discrete topology. Let $\tilde{\phi}: \beta N \rightarrow [0, 1]$ be the unique continuous extension of ϕ to βN . Of course, $\tilde{\phi}$ is onto.

Let X be a topological space with a partition π . Then X/π denotes the quotient space. A subset $A \subset X$ is said to be saturated under π , if A is a union of members of π .

Whenever a set $Y \subset \beta N$ we assume that it is given the subspace topology.

DEFINITION 2.3. A sequence C_1, C_2, C_3, \dots of subsets of βN is called a convergent subsequence if $\tilde{\phi}(C_1), \tilde{\phi}(C_2), \dots, \tilde{\phi}(C_n), \dots$ is a convergent sequence. In this case we say that (C_n) is a convergent sequence.

DEFINITION 2.4. Let Y be an open subset of βN and π a partition of Y by compact subsets of βN . The pair (Y, π) is said to satisfy the condition V_c if the following hold:

- (i) $\tilde{\phi}$ is constant on each member in π .
- (ii) $N \subset Y$ and $\{n\} \in \pi$ for all $n \in N$.
- (iii) Y/π is a countable, locally compact T_2 space.
- (iv) Given a member $A \in \pi$ there is compact open set V of βN so that $A \subset V \subset Y$ and V is saturated under π .

DEFINITION 2.5. Let $A_1, A_2, \dots, A_n, \dots$ be a sequence of compact subsets of βN . The set $F = \bigcup_{n=1}^{\infty} A_n - \bigcup_{n=1}^{\infty} A_n$ is called the growth of the sequence (A_n) .

LEMMA 2.6. Let $A_1, A_2, \dots, A_n, \dots$ be a sequence of compact subsets of βN and let (A_n) converge. Let A be the growth of (A_n) . Then $\tilde{\phi}$ is constant on A .

Proof. If possible, let u, v be distinct numbers in $\tilde{\phi}(A)$. Let $\tilde{\phi}(A_n) \rightarrow y$ where $y \in [0, 1]$. Then either $y \neq u$ or $y \neq v$. Let $y \neq u$. Let $\varepsilon = |y - u|/2$. Let

$$M = \tilde{\phi}^{-1}((y - \varepsilon, y + \varepsilon)) \quad \text{and} \quad H = \tilde{\phi}^{-1}([y - \varepsilon, y + \varepsilon]).$$

Then $\bar{M} \subset H$. Now $(y - \varepsilon, y + \varepsilon)$ contains $\tilde{\phi}(A_n)$ from a certain stage. So $M \supset A_n$ from a certain stage. So $\bar{M} \supset A$. Since $H \supset \bar{M}$ we have that $\tilde{\phi}(A) \subset [y - \varepsilon, y + \varepsilon]$. But $u \in \tilde{\phi}(A)$ and $u \notin [y - \varepsilon, y + \varepsilon]$. This contradiction shows that $\tilde{\phi}$ is constant on A .

LEMMA 2.7. Let Y be an open set in βN and π a partition of Y by compact subsets of βN so that (Y, π) satisfies the condition V_c . Let $F_1, F_2, \dots, F_m, \dots$ be a sequence of distinct members of π so (F_n) converges and the growth F of (F_n) is

disjoint with Y and nonempty. Then there is an open set Y_0 of βN and a partition π_0 of Y_0 by compact sets so that the following hold:

- (a) (Y_0, π_0) satisfies V_c .
- (b) $Y_0 \supset Y$ and $\pi_0 \supset \pi$.
- (c) $Y_0 \cap F \neq \emptyset$.

(In the proof below we will prove that Y_0 in fact contains F though (c) is enough for us).

Proof. The condition (iii) of Definition 2.4 gives that Y is σ -compact. Condition (iv) of that definition gives that Y can be expressed as a countable union $K_1, K_2, \dots, K_n, \dots$ of compact open subsets K_1, \dots, K_n, \dots of βN so that K_n is saturated under π for all $n \in N$. Put $M_1 = K_1$ and $M_n = K_n - \bigcup_{i=1}^{n-1} K_i$ for all $n \in N$ and $n > 1$. Then $Y = \bigcup_{n=1}^{\infty} M_n$ and $\{M_1, M_2, \dots, M_n, \dots\}$ is a pairwise disjoint collection of compact open sets of βN each of which is saturated under π . Now $\tilde{\phi}(Y)$ is countable and $\tilde{\phi}(\beta N) = [0, 1]$ and Y is dense in βN . So Y is not compact. So $M_n \neq \emptyset$ for an infinity of n belonging to N . By dropping the empty sets in the collection (M_n) we can assume without loss of generality that $M_n \neq \emptyset$ for all $n \in N$. Let $T = \bigcup_{n=1}^{\infty} F_n$. Then $T \cap M_k = \bar{T} \cap M_k$ for all $k \in N$. So $T \cap M_k$ is compact for all $k \in N$. Now there is a number $y \in [0, 1]$ so that $\text{dm}(\tilde{\phi}(F_n), \{y\}) \rightarrow 0$ as $n \rightarrow \infty$. So $\text{dm}(\tilde{\phi}(T \cap M_k), \{y\}) \rightarrow 0$ as $k \rightarrow \infty$. So there is a strictly ascending sequence $k_1 < k_2 < \dots < k_r < \dots$ of integers $k_r \in N$ so that $\text{dm}(\tilde{\phi}(T \cap M_{k_r}), \{y\}) < 1/2^r$ if $k \in N$ and $k \geq k_r$ for all $r \in N$. Then given $n \in N$ and $k > k_n$ we can find a compact open set V_k of βN so that $T \cap M_k \subset V_k \subset M_k$ and V_k is saturated under π and $\text{dm} \tilde{\phi}(V_k) < 1/2^n$. Put $V_i = M_i$ if $1 \leq i \leq k_i$. Let $W = \bigcup_{i=1}^{\infty} V_i$. Let $G = \bar{W}$. Then, using the fact that βN is disconnected we get that \bar{W} is open and closed in βN . Clearly (V_i) is a convergent sequence. So $\tilde{\phi}$ is constant on the growth $G - W$ of (V_i) . Clearly $G - W \supset F$. Put $Y_0 = Y \cup (G - W)$ and $\pi_0 = \pi \cup \{G - W\}$. Then it is easily seen that (Y_0, π_0) satisfy conditions (a), (b), (c) of the conclusion of the theorem.

LEMMA 2.8. Let Y_n be an open subset of βN and π_n a partition of Y_n by compact subsets of βN for all $n \in N$. Let the following hold:

- (a) (Y_n, π_n) satisfies condition V_c for all $n \in N$.
- (b) $Y_{n+1} \supset Y_n$ and $\pi_{n+1} \supset \pi_n$ for all $n \in N$.

Let $Y = \bigcup_{n=1}^{\infty} Y_n$ and $\pi = \bigcup_{n=1}^{\infty} \pi_n$. Then (Y, π) satisfies V_c .

Proof. Obvious.

COROLLARY 2.9. Let Y be an open subset of βN and π a partition of Y by compact subsets so that (Y, π) satisfies condition V_c . Let \mathcal{F} be a countable family of subsets of βN so that if $A \in \mathcal{F}$ then there is a convergent sequence (A_n) of distinct members of π whose growth is A and $A \neq \emptyset$.

Then there is an open set Y_0 of βN and a partition π_0 of Y_0 by compact sets so

that the following hold:

- (a) (Y_0, π_0) satisfies the condition V_c .
- (b) $Y_0 \cap A \neq \emptyset$ for all $A \in \mathcal{F}$.
- (c) $Y_0 \supset Y$ and $\pi_0 \supset \pi$.

Proof. Follow the proof of Lemma 1.6 of [20].

Discussion 2.10. We give below a method of constructing a dense open set Y of βN and a partition π of Y by compact sets so that the quotient space $X = Y/\pi$ is a scattered locally compact T_2 locally countable, and there is a map $\Psi: X \rightarrow [0, 1]$ so that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\phi}} & [0, 1] \\ \downarrow \nu & \searrow \Psi & \\ X & & \end{array}$$

where $\nu: Y \rightarrow X$ is the quotient map. Further Y will be so constructed that if (A_n) is a sequence of distinct members of π with a nonempty growth A then $A \cap Y$ will be nonempty. Then X and the map $\Psi: X \rightarrow [0, 1]$ will give a solution to the problem of J. J. Schaffer.

The following method of constructing (Y, π) as above is called the V -process. That method has been used effectively to construct examples and then to solve a problem of C. Scarborough and A. H. Stone [18] and a problem of Z. Semadeni on 0-dimensionality of scattered spaces and scattered compactifications [17] and also a problem of Telgarsky in [19].

DEFINITION 2.11 (V -process). We use transfinite induction. Put $Y_1 = N$ and $\pi_1 = [\{n\} \mid n \in N]$. Let \mathcal{G}_1 be the collection of all closed nonempty sets $A \subset \beta N$ so that $A \in \mathcal{G}_1$ if and only if there is a convergent sequence (A_n) of distinct elements in π_1 whose growth is A . Using (CH) write \mathcal{G}_1 as a well-ordered sequence $A_{11}, A_{12}, \dots, A_{1\alpha}, \dots$ where $\alpha \in [1, \Omega]$. Now suppose that α is a countable limit ordinal and that we have defined (Y_γ, π_γ) and $A_{\gamma\delta}$ for all ordinals $\gamma < \alpha$ and all countable ordinals δ . We put $Y_\alpha = \bigcup_{\gamma < \alpha} Y_\gamma$ and $\pi_\alpha = \bigcup_{\gamma < \alpha} \pi_\gamma$ and \mathcal{G}_α to be the collection of all closed sets $A \neq \emptyset$ where A is the growth of a distinct convergent sequence (A_n) of members of π_α . Well order the set \mathcal{G}_α as $A_{\alpha 1}, A_{\alpha 2}, \dots, A_{\alpha\delta}, \dots$ where $\delta \in [1, \Omega]$. Now suppose that α is a countable successor ordinal and $\alpha = \delta_0 + 1$ where $\delta_0 \in [1, \Omega]$. Let $(Y_{\delta_0}, \pi_{\delta_0})$ and $A_{\gamma\delta}$ be defined for all ordinals $\gamma \leq \delta_0$ and $\delta \in [1, \Omega]$. Let

$$\mathcal{F} = \{A_{\gamma\delta} \mid 1 \leq \gamma \leq \delta_0 \text{ and } 1 \leq \delta \leq \omega^{\delta_0}\}.$$

Use Corollary 2.9 and get an open set Y_α of βN and a partition π_α of Y_α by compact sets so that $Y_\alpha \supset Y_{\delta_0}$, $\pi_\alpha \supset \pi_{\delta_0}$ and (Y_α, π_α) satisfies the condition V_c and $Y_\alpha \cap A_{\gamma\delta} \neq \emptyset$ for $1 \leq \gamma \leq \delta_0$ and $1 \leq \delta \leq \omega^{\delta_0}$. Finally let $Y_{\Omega-} = \bigcup_{\alpha < \Omega} Y_\alpha$ and

$\pi_{\Omega-} = \bigcup_{\alpha < \Omega} \pi_{\alpha}$. This $(Y_{\Omega-}, \pi_{\Omega-})$ is the required pair (Y, π) to be constructed and the V -process ends here.

THEOREM 2.12. *Let $Y_{\alpha}, \pi_{\alpha}, Y_{\Omega-}, \pi_{\Omega-}$ be as in the definition of V -process above. Let $A_1, A_2, \dots, A_n, \dots$ be a sequence of distinct members of $\pi_{\Omega-}$ with growth A . Let $A \neq \emptyset$ be closed. Then $A \cap Y_{\Omega-} \neq \emptyset$.*

Proof. The proof follows essentially the arguments of Theorem 1.9 of [20] and hence is omitted.

THEOREM 2.13. *Let $Y_{\Omega-}, \pi_{\Omega-}, Y_{\alpha}, \pi_{\alpha}$ be as in the definition of V -process 2.11, for all $\alpha \in [1, \Omega)$. Let $X = Y_{\Omega-} / \pi_{\Omega-}$ and $v: Y_{\Omega-} \rightarrow X$ the natural quotient map. Then there exists a unique continuous map $\Psi: X \rightarrow [0, 1]$ from X onto $[0, 1]$ so that $\tilde{\phi} = \Psi \circ v$ on $Y_{\Omega-}$. Moreover X is a T_2 locally compact scattered locally countable first-countable sequentially compact separable space. Thus there exists a countably compact scattered completely regular T_2 space which can be mapped continuously onto $[0, 1]$ and also a scattered T_2 space whose closed continuous image need not be scattered.*

Proof. The proof follows as in the proofs of Theorems 1.8 and 1.9 of [20] except for the properties of the function Ψ . However it is clear that Ψ is well defined on X and that $\tilde{\phi} = \Psi \circ q$. Since X is countably compact and is Ψ continuous and $\Psi(X)$ is dense in $[0, 1]$ then Ψ maps X onto $[0, 1]$.

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