ILLINOIS JOURNAL OF MATHEMATICS
Volume 22, Number 1, March 1978

ON FORMAL INTEGRATION OF DOUBLE TRIGONOMETRIC SERIES

BY

M. J. KOHN

1. We will be working in two dimensional Euclidean space. We denote points of E_2 by $x = (x_1, x_2) = te^{i\theta}$ and integral lattice points by $n = (n_1, n_2)$. We set $|x| = (x_1^2 + x_2^2)^{1/2}$ and $n \cdot x = n_1 x_1 + n_2 x_2$. By a sum $\sum'$ we mean $\sum_{|n| \neq 0}$. Let

$$(1.1) \qquad T = \sum_{n \in \mathbf{Z}_2} c_n e^{in \cdot x}$$

be a double trigonometric series which is circularly summable at x_0 to finite sum s. Let T^* be the series obtained by formally integrating T once with respect to x_1 and once with respect to x_2:

$$(1.2) \qquad T^* = c_0 x_1 x_2 - \sum_{n_1 n_2 \neq 0} \frac{c_n}{n_1 n_2} e^{in \cdot x} + x_1 \sum_{n_1 = 0}' \frac{c_n}{in_2} e^{in \cdot x} + x_2 \sum_{n_2 = 0}' \frac{c_n}{in_1} e^{in \cdot x}.$$

We are interested in proving a theorem of "Riemann type" for T^*. That is, we want to give conditions on the coefficients of T and on the order of summability of T which will insure that T^* converges at x_0 to a function $F(x)$ which has, in some sense, at x_0 a "second symmetric derivative" with value s.

We define, to this end, the idea of a symmetric derivative of a function $F(x)$ defined in a neighborhood of $x_0 \in E_2$ by expanding a weighted circular mean of $F(x)$, taken about the circle $|x - x_0| = t$, in a Taylor's series of even powers of t. This definition may be thought of as a two dimensional analogue of the formula (1.2) from [8, vol. 2, p. 59]. When the proper weighted circular mean is chosen, we are able to apply it to T^* to prove a two dimensional analogue of results from [8, vol. 1, p. 320].

2. We make the following definition. Let $\Omega(\theta)$ be defined for $\theta \in [0, 2\pi]$ such that $\Omega(\theta + \pi) = \Omega(\theta)$. Let $F(x)$ be defined in a neighborhood of $x_0 \in E_2$ and integrable over each circle $|x - x_0| = t$, for t small. Let $2r$ be an even, positive integer.

DEFINITION. F has, at x_0, a $2r$th *Ω-derivative* with value a_{2r} if

$$(2.1) \qquad \frac{1}{2\pi} \int_0^{2\pi} F(x_0 + te^{i\theta}) \Omega(\theta) \, d\theta = a_0 + \frac{a_2}{2^2 2!} t^2 + \cdots + \frac{a_{2r}}{2^{2r}(r+1)!\,(r-1)!} t^{2r} + o(t^{2r})$$

as $t \to 0$.

Received July 27, 1976.

Copyright © Board of Trustees, University of Illinois

If $\Omega(\theta) \equiv 1$, the expansion of the left side of (2.1) into a series with different coefficients is called the *generalized Laplacian* and is studied in [7]. If $\Omega(\theta) = \cos\theta + \sin\theta$ (which satisfies $\Omega(\theta + \pi) = -\Omega(\theta)$) the expansion of

$$\frac{1}{2\pi}\int_0^{2\pi} F(x_0 + te^{i\theta})\Omega(\theta)\, d\theta$$

in a Taylor's series of *odd* powers of t is considered in [5].

For this paper, we will study (2.1) with $\Omega(\theta) = \cos\theta \sin\theta$. It turns out that the resulting Ω-derivative is well suited for application to the series (1.2).

3. The value of our Ω-derivative is given by the following theorem.

THEOREM 1. *Let $\Omega(\theta) = \cos\theta \sin\theta$. Let $r \geq 1$. Suppose $F(x)$ and all partial derivatives of F of order $\leq 2r + 1$ exist and are continuous in a neighborhood of $x_0 \in E_2$. Then F has at x_0 a $2r$-th Ω-derivative with value*

$$a_{2r} = \frac{\partial^2}{\partial x_1\, \partial x_2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^{r-1} F(x_0).$$

Proof. We may assume $x_0 = 0$. We abbreviate

$$\left.\frac{\partial^{m+n}F}{\partial x_1^m\, \partial x_2^n}\right|_{x=0}$$

by $F(m, n)$. By Taylor's formula,

$$F(te^{i\theta}) = \sum_{j=0}^{2r} \frac{1}{j!}\left(t\cos\theta\, \frac{\partial}{\partial x_1} + t\sin\theta\, \frac{\partial}{\partial x_2}\right)^j F(0)$$
$$+ \frac{1}{(2r+1)!}\left(t\cos\theta\, \frac{\partial}{\partial x_1} + t\sin\theta\, \frac{\partial}{\partial x_2}\right)^{2r+1} F(\mu e^{i\theta})$$

for some $\mu \in (0, t)$. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} F(te^{i\theta}) \cos\theta \sin\theta\, d\theta \tag{3.1}$$
$$= \sum_{j=0}^{2r} \frac{t^j}{j!}\frac{1}{2\pi}\int_0^{2\pi}\left(\cos\theta\, \frac{\partial}{\partial x_1} + \sin\theta\, \frac{\partial}{\partial x_2}\right)^j F(0) \cos\theta \sin\theta\, d\theta$$
$$+ \frac{t^{2r+1}}{(2r+1)!}\frac{1}{2\pi}\int_0^{2\pi}\left(\cos\theta\, \frac{\partial}{\partial x_1} + \sin\theta\, \frac{\partial}{\partial x_2}\right)^{2r+1} F(\mu e^{i\theta}) \cdot \cos\theta \sin\theta\, d\theta$$
$$= \sum_{j=0}^{2r} a_j t^j + R_{2r+1}.$$

Here,

$$(3.2)\quad a_j = \frac{1}{j!}\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}\cos^k\theta\sin^{j-k}\theta F(k, j-k)\cdot\cos\theta\sin\theta\, d\theta$$
$$= \sum_{k=0}^{j}\frac{1}{k!\,(j-k)!}\cdot\frac{1}{2\pi}\int_0^{2\pi}\cos^{k+1}\theta\sin^{j-k+1}\theta\, d\theta\cdot F(k, j-1)$$
$$= \sum_{k=0}^{j}\frac{1}{k!\,(j-k)!}c_{kj}F(k, j-k),$$

where

$$c_{kj} = \frac{1}{2\pi}\int_0^{2\pi}\cos^{k+1}\theta\sin^{j-k+1}\theta\, d\theta.$$

Clearly, $c_{kj} = 0$ if j is odd. When j is even, we find using reduction formulae,

$$c_{kj} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dfrac{k!\,(j-k)!}{2^j\left(\dfrac{j+2}{2}\right)!\left(\dfrac{k-1}{2}\right)!\left(\dfrac{j-k-1}{2}\right)!} & \text{if } k \text{ is odd.} \end{cases}$$

We set $m = \frac{1}{2}j$, $s = \frac{1}{2}(k-1)$. Returning to (3.2), if j is odd then $a_j = 0$, and if j is even then

$$(3.3)\quad a_j = \sum_{k=0}^{j}\frac{1}{k!\,(j-k)!}c_{kj}F(k, j-k)$$
$$= \sum_{\substack{k=0,\\ k \text{ odd}}}^{j}\frac{1}{k!\,(j-k)!}\,\frac{k!\,(j-k)!}{2^j\left(\dfrac{j+2}{2}\right)!\left(\dfrac{k-1}{2}\right)!\left(\dfrac{j-k-1}{2}\right)!}F(k, j-k)$$
$$= \sum_{s=0}^{m-1}\frac{1}{2^{2m}(m+1)!\,s!\,(m-1-s)!}F(2s+1, 2m-2s-1)$$
$$= \frac{1}{2^{2m}(m+1)!\,(m-1)!}\sum_{s=0}^{m-1}\binom{m-1}{s}F(2s+1, 2m-2s-1)$$
$$= \frac{1}{2^{2m}(m+1)!\,(m-1)!}\frac{\partial^2}{\partial x_1\,\partial x_2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)^{m-1}F(0).$$

For the estimate of R_{2r+1} we obtain,

$$(3.4)\qquad R_{2r+1} = t^{2r+1}\int_0^{2\pi}0(1)\cos\theta\sin\theta\, d\theta = o(t^{2r}).$$

Applying (3.3) and (3.4) to (3.1), the proof of Theorem 1 is complete.

4. We now apply Definition (2.1) to study formally integrated double trigonometric series. Let β be a nonnegative number. We will say the series (1.1) is Bochner-Riesz-β summable at x_0 to s if

$$\lim_{R\to\infty} \sum_{|n|<R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^\beta c_n e^{in\cdot x_0} = s.$$

THEOREM 2. *Suppose series (1.1) is Bochner-Riesz-β summable at x_0 to finite sum s, for some number β with $0 \le \beta < 3/2$. Suppose the coefficients c_n of (1.1) satisfy*

$$\sum_{n_1 n_2 \neq 0} |n_1 n_2|^{-2} |n|^{1+\varepsilon} |c_n|^2 + \sum_{n_1=0}' |n_2|^{-2} |n|^{1+\varepsilon} |c_n|^2 + \sum_{n_2=0}' |n_1|^{-2} |n|^{1+\varepsilon} |c_n|^2 < \infty \tag{4.1}$$

for some $\varepsilon > 0$.

Let

$$F_R(x) = c_0 x_1 x_2 - \sum_{\substack{n_1 n_2 \neq 0,\\ |n|<R}} \frac{c_n}{n_1 n_2} e^{in\cdot x} + x_1 \sum_{\substack{n_1=0,\\ |n|<R}}' \frac{c_n}{in_2} e^{in\cdot x} + x_2 \sum_{\substack{n_2=0,\\ |n|<R}}' \frac{c_n}{in_1} e^{in\cdot x}.$$

Then, as $R \to \infty$, $F_R(x)$ converges a.e. on T_2 to a function $F(x)$ which is integrable on each circle $|x - x_0| = t$. Moreover, F has at x_0 a second Ω-derivative, with $\Omega(\theta) = \cos\theta \sin\theta$, equal to s.

We can think of Bochner-Riesz-β summability as a two dimensional version of Cesaro-β summability. Thus Theorem 2 may be considered as an analogue, of sorts, of part of the result on p. 66, vol. 2, of [8]. Note that the order of summability required in the two dimensional version is somewhat weaker than in the one dimensional case.

5. Before we give the proof of Theorem 2 we need to establish a lemma. In what follows, $J_\nu(z)$ indicates the Bessel's function of order ν.

LEMMA. *Let $n = (n_1, n_2)$, $|n| \neq 0$. Define, for $x \in E_2$,*

$$g_n(x) = \begin{cases} \dfrac{-\exp(in\cdot x)}{n_1 n_2} & \text{if } n_1 n_2 \neq 0, \\ x_1 (in_2)^{-1} \exp(in\cdot x) & \text{if } n_1 = 0, \\ x_2 (in_1)^{-1} \exp(in\cdot x) & \text{if } n_2 = 0. \end{cases} \tag{5.1}$$

Then,

$$\frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta}) \cos\theta \sin\theta \, d\theta = \frac{J_2(|n|t)}{|n|^2}. \tag{5.2}$$

Proof. We first assume $n_1 n_2 \neq 0$. Let $n_1/|n| = \cos\phi$ and $n_2/|n| = \sin\phi$. Then,

$$(5.3)\quad \frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\cos\theta\sin\theta\, d\theta$$

$$= \frac{-1}{n_1 n_2}\frac{1}{2\pi}\int_0^{2\pi} \exp\,(in\cdot te^{i\theta})\cos\theta\sin\theta\, d\theta$$

$$= \frac{-1}{n_1 n_2}\frac{1}{2\pi}\int_0^{2\pi} \exp\,\{i|n|t\,(\cos\phi\cos\theta + \sin\phi\sin\theta)\}\cdot\cos\theta\sin\theta\, d\theta$$

$$= \frac{-1}{n_1 n_2}\frac{1}{2\pi}\int_0^{2\pi} \exp\,\{i|n|t\cos(\theta-\phi)\}\cos\theta\sin\theta\, d\theta.$$

Let $\mu = \theta - \phi$. Then

$$\begin{aligned}\cos\theta\sin\theta &= \tfrac{1}{2}\sin 2\theta\\ &= \tfrac{1}{2}\sin(2\mu + 2\phi)\\ &= \tfrac{1}{2}\sin 2\mu\cos 2\phi + \tfrac{1}{2}\cos 2\mu\sin 2\phi.\end{aligned}$$

So returning to (5.3),

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\cos\theta\sin\theta\, d\theta$$

$$\begin{aligned}&= \frac{-1}{n_1 n_2}\,\frac{\cos 2\phi}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\exp\,(i|n|t\cos\mu)\sin 2\mu\, d\mu\\ &\quad + \frac{-1}{n_1 n_2}\,\frac{\sin 2\phi}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\exp\,(i|n|t\cos\mu)\cos 2\mu\, d\mu\\ &= 0 + \frac{-1}{n_1 n_2}\cdot\frac{n_1 n_2}{|n|^2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\exp\,(i|n|t\cos\mu)\cos 2\mu\, d\mu\\ &= \frac{J_2(|n|t)}{|n|^2},\end{aligned}$$

by formula 2 from [1, p. 81].

We next consider the case when $n_1 = 0$. Then,

$$(5.4)\quad \frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\cos\theta\sin\theta\, d\theta$$

$$\begin{aligned}&= \frac{1}{2\pi}\int_0^{2\pi}\frac{t\cos\theta}{in_2}\exp\,(in_2 t\sin\theta)\cos\theta\sin\theta\, d\theta\\ &= \frac{t}{in_2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\exp\,(in_2 t\sin\theta)\tfrac{1}{2}\sin 2\theta\, d\theta.\end{aligned}$$

We integrate the last integral by parts. Then (5.4) becomes

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\cos\theta\sin\theta\,d\theta &= \frac{-t}{in_2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(in_2 t\sin\theta)}{in_2 t}\cos 2\theta\,d\theta\\ &= \frac{1}{n_2^2}\frac{1}{2\pi}\int_0^{2\pi}\exp(in_2 t\sin\theta)\cos 2\theta\,d\theta\\ &= \frac{J_2(n_2 t)}{n_2^2}\\ &= \frac{J_2(|n|t)}{|n|^2},\end{aligned}$$

since $|n| = \pm n_2$ and $J_2(-z) = J_2(z)$.

A similar argument applies for the case when $n_2 = 0$. Thus the proof of the lemma is complete.

6. Having established the lemma, the proof of Theorem 2 is now very similar to the proof of the theorem in [4]. We will give the proof in detail for the case $\beta = 1$. If $1 < \beta < 3/2$ the proof becomes much more complicated, so we just sketch the idea and refer the reader to [4] for some details.

Without loss of generality we may assume $c_0 = 0$, $x_0 = 0$, and $s = 0$. Write $S_R = S_R(0) = \sum_{|n|<R} c_n$, and for $\eta > 0$ set

$$S_R^\eta = \frac{1}{\Gamma(\eta)}\int_0^R (R-u)^{\eta-1} S_u\,du.$$

We are assuming that series (1.1) is Bochner-Riesz-1 summable to 0 at $x_0 = 0$. Therefore (see [2]) $\sum_{|n|<R} c_n(R - |n|) = o(R)$ as $R \to \infty$. Hence,

$$S_R^1 = o(R) \quad \text{as } R \to \infty. \tag{6.1}$$

The condition (4.1) insures that $F(x) = \lim_{R\to\infty} F_R(x)$ exists a.e. on each circle $|x| = t$ and that $\sup_{R>0}\int_0^{2\pi} |F_R(te^{i\theta})|\,d\theta < M$, (see [3]). Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} F(te^{i\theta})\Omega(\theta)\,d\theta = \lim_{R\to\infty}\frac{1}{2\pi}\int_0^{2\pi} F_R(te^{i\theta})\Omega(\theta)\,d\theta.$$

We apply the lemma to the integral on the right.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} F_R(te^{i\theta})\Omega(\theta)\,d\theta &= \sum_{|n|<R} c_n\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta\\ &= \sum_{|n|<R} c_n|n|^{-2}J_2(|n|t)\\ &= t^2\sum_{|n|<R} c_n\gamma(|n|t),\end{aligned}$$

where $\gamma(z) = z^{-2}J_2(z)$. We change the last sum into an integral and integrate twice by parts.

$$(6.2)\qquad \sum_{|n|<R} c_n\gamma(|n|t) = S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)\,du$$
$$= S_R\gamma(Rt) - S_R^1 \frac{d}{dR}\gamma(Rt) + \int_0^R S_u^1 \frac{d^2}{du^2}\gamma(ut)\,du.$$

Note that the hypothesis (4.1) implies $\sum_{n\in \mathbf{Z}_2} |n|^{\varepsilon-1}|c_n|^2 < \infty$ for some $\varepsilon > 0$. Thus, using Holder's inequality,

$$(6.3)\qquad \begin{aligned} S_R &= \sum_{|n|<R} c_n \\ &= \sum_{|n|<R} (|n|^{(\varepsilon-1)/2}|c_n|)(|n|^{(1-\varepsilon)/2}) \\ &\le \left(\sum_{n\in\mathbf{Z}_2} |n|^{\varepsilon-1}|c_n|^2\right)^{1/2}\left(\sum_{|n|<R} |n|^{1-\varepsilon}\right)^{1/2} \\ &= C\cdot R^{(3-\varepsilon)/2} \\ &= o(R^{3/2}). \end{aligned}$$

Using formula (51) from [1, p. 11] and the fact that $J_\nu(z) = O(z^{-1/2})$ as $z\to\infty$, it is clear that

$$(6.4)\qquad \gamma^{(n)}(z) = O(z^{-5/2}) \quad \text{as } z\to\infty \text{ for } n = 0, 1, 2, \ldots.$$

Combining (6.1), (6.3), and (6.4), the integrated terms on the right of (6.2) drop out as $R\to\infty$. Thus,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} F(te^{i\theta})\Omega(\theta)\,d\theta &= t^2 \lim_{R\to\infty} \sum_{|n|<R} c_n\gamma(|n|t) \\ &= t^2\int_0^\infty S_u^1 \frac{d^2}{du^2}\gamma(ut)\,du \\ &= 0 + 0\cdot t^2 + t^2B(t). \end{aligned}$$

We will complete the proof by showing $B(t)\to 0$ as $t\to 0$.

$$\begin{aligned} B(t) &= \int_0^{1/t} S_u^1 \frac{d^2}{du^2}\gamma(ut)\,du + \int_{1/t}^\infty S_u^1 \frac{d^2}{du^2}\gamma(ut)\,du \\ &= B_1(t) + B_2(t). \end{aligned}$$

To estimate $B_1(t)$ we note that $\gamma(z)$ is an entire function, so for $|z| < 1$,

$$\left|\frac{d^2}{dz^2}\gamma(z)\right| < C.$$

Thus, when $0 < u < 1/t$,

$$\left| \frac{d^2}{du^2} \gamma(ut) \right| \leq Ct^2,$$

and

$$B_1(t) = \int_0^{1/t} o(u) \cdot Ct^2 \, du = O(t^2) \int_0^{1/t} o(u) \, du = o(1).$$

To estimate $B_2(t)$ we use (6.4).

$$\begin{aligned} B_2(t) &= \int_{1/t}^{\infty} S_u^1 \frac{d^2}{du^2} \gamma(ut) \, du \\ &= \int_{1/t}^{\infty} o(u) \cdot t^2 O(ut)^{-5/2} \, du \\ &= O(t^{-1/2}) \int_{1/t}^{\infty} o(u^{-3/2}) \, du \\ &= o(1). \end{aligned}$$

This completes the proof of Theorem 1 in the case when $0 \leq \beta \leq 1$.

If $1 < \beta < 3/2$ write $\beta = 1 + \alpha$. We begin as in the proof above, but at equation (6.2) we integrate by parts once again. We obtain, after showing the integrated terms tend to 0,

$$\frac{1}{2\pi} \int_0^{2\pi} F(te^{i\theta})\Omega(\theta) \, d\theta = -t^2 \int_0^{\infty} S_u^2 \frac{d^3}{du^3} \gamma(ut) \, du. \tag{6.5}$$

If $f(u)$ is a function defined for $u > 0$ and η is a positive number we denote by

$$I^{\eta}(f)(u) = \frac{1}{\Gamma(\eta)} \int_0^u (u - z)^{\eta - 1} f(z) \, dz,$$

the fractional integral of order η of f (see [6]). Now if we let $f(u) = S_u$, then

$$S_u^2 = I^2(f)(u) = I^{1-\alpha} I^{1+\alpha}(f)(u) = I^{1-\alpha} S_u^{1+\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^u (u - z)^{-\alpha} S_z^{1+\alpha} \, dz.$$

Returning to (6.5),

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} F(te^{i\theta})\Omega(\theta) \, d\theta &= -t^2 \lim_{R \to \infty} \int_0^R S_u^2 \frac{d^3}{du^3} \gamma(ut) \, du \\ &= -t^2 \lim_{R \to \infty} \int_0^R \frac{1}{\Gamma(1-\alpha)} \int_0^u (u - z)^{-\alpha} S_z^{1+\alpha} \, dz \frac{d^3}{du^3} \gamma(ut) \, du \\ &= -t^2 \lim_{R \to \infty} \int_0^R S_z^{1+\alpha} \frac{1}{\Gamma(1-\alpha)} \int_z^R (u - z)^{-\alpha} \frac{d^3}{du^3} \gamma(ut) \, du \, dz \end{aligned}$$

$$= t^2 \lim_{R\to\infty} \int_0^R S_z^{1+\alpha} H(z, t, R)\, dz$$

$$= t^2 \lim_{R\to\infty} \left\{ \int_0^{1/t} + \int_{1/t}^R \right\}$$

$$= t^2\{P + Q\}.$$

Using estimates similar to those in [4], we find

$$|H(z, t, R)| \le \begin{cases} Ct^2 \left(\frac{1}{t} - z\right)^{-\alpha} & \text{for } 0 < z < 1/t, \\ Ct^{-5/2}(R - z)^{-\alpha} R^{-5/2} & \text{for } z > 1/t. \end{cases}$$

Hence

$$P = \int_0^{1/t} o(z^{1+\alpha}) O(t^2) \left(\frac{1}{t} - z\right)^{-\alpha} dz = o(1)$$

and

$$Q = \lim_{R\to\infty} \int_{1/t}^R o(z^{1+\alpha}) O(t^{-5/2})(R - z)^{-\alpha} R^{-5/2}\, dz = o(1).$$

This completes the proof of Theorem 2.

7. It seems probable that many other weights $\Omega(\theta)$ (for example, surface harmonics of even order) may be used with Definition (2.1) to derive theorems of Riemann type for multiple trigonometric series. The key step in establishing such a result is the verification of the lemma of Section 5. For general surface harmonics and for application to T_k for $k > 2$ the proof of the lemma may be aided by the Funk-Hecke Theorem [1, p. 247], which facilitates the computation of some surface integrals involving surface harmonics. Details will be given elsewhere.

Bibliography

1. A. Erdelyi et. al., *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.
2. G. H. Hardy, *The second theorem of consistency for summable series*, Proc. London Math. Soc. (2), vol. 15 (1916), pp. 72–88.
3. M. Kohn, *Spherical convergence and intergrability of multiple trigonometric series on hypersurfaces*, Studia Math., vol. 44 (1972), pp. 345–354.
4. ———, *Riemann summability of multiple trigonometric series*, Indiana Univ. Math. J., vol. 24 (1975), pp. 813–823.
5. ———, *Lebesgue summability of double trigonometric series*, Trans. Amer. Math. Soc., vol. 225 (1977), pp. 199–209.
6. M. Riesz, *L'intergrale de Riemann-Liouville et le problem de Cauchy*, Acta Math., vol. 81 (1949), pp. 1–233.
7. V. Shapiro, *Circular summability C of double trigonometric series*, Trans. Amer. Math. Soc., vol. 76 (1954), pp. 223–233.
8. A. Zygmund, *Trigonometric series*, vols. I and II, Cambridge University Press, Cambridge, 1968.

Brooklyn College of the City University of New York
Brooklyn, New York