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GROTHENDIECK SPACES

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In this paper X will always denote a compact Hausdorff space, E a Banach space over K, the field of real or complex numbers (we shall call K the scalars), and $C(X)$ ($C(X, E)$) all K-valued (E-valued) continuous functions on X with sup norm. A Banach space F is a Grothendieck space if every $\sigma(F', F)$ convergent sequence in F' is also $\sigma(F', F'')$ convergent, where F' and F'' are the topological dual and bidual of F. It is proved in [8] that if X is an F-space, $C(X)$ is a Grothendieck space. In this paper we investigate conditions under which $C(X, E)$ is a Grothendieck space.

Every $\mu \in (C(X, E))' = M(X, E')$ (notations of [7], [9]) can be considered as a regular Borel measure $\mu: \mathscr{B} \to (E', \|\cdot\|)$, with finite variation, where $\mathscr{B}$ is the class of Borel subsets of X; $|\mu|$ will denote the variation of μ. For any $f: X \to E$, f $\mathscr{B}$-simple or $f \in C(X, E)$, we have $|\mu(f)| \leq |\mu|(\|f\|)$ [1], [7], [9].

LEMMA 1. *Let 2^N denote all subsets of N, with product topology. If $\lambda_n: 2^N \to K$ is a sequence of countably additive measures (this implies they are continuous) and $\lim \lambda_n(M) = \lambda(M)$ exists for all $M \subset N$, then $\lambda_n \to \lambda$ uniformly on 2^N. In particular, $\lambda_n(\{n\}) \to 0$.*

Proof. This lemma is a particular case of [4, Lemma 1]; also it follows from the classical Phillips' lemma.

THEOREM 2. *$C(X, E)$ is a Grothendieck space if and only if at least one of the following conditions is satisfied:*

(i) *X is finite and E is a Grothendieck space.*
(ii) *E is finite dimensional and $C(X)$ is a Grothendieck space.*

Proof. If (i) is satisfied then $C(X, E) = E^n$ where n equals the number of elements in X. Since E is a Grothendieck space, E^n is also a Grothendieck space. Now suppose that (ii) is satisfied and let $E = K^p$. This means $C(X, E) = \prod C(X)$, multiplied p times. Since $C(X)$ is a Grothendieck space, it follows that $C(X, E)$ is a Grothendieck space.

Now suppose that neither (i) nor (ii) is satisfied but $C(X, E)$ is a Grothendieck space. Since $C(X, E)''$ contains $(C(X))'' \otimes E$ and $C(X) \otimes E''$, then $C(X)$ and E are Grothendieck spaces (proof by contradiction). Thus X is infinite and E is infinite dimensional. Let $\{x(n)\}$ be a sequence of distinct points of X. If every $\sigma(E', E)$ convergent sequence in E' is norm convergent, then E is finite dimensional [10]. Take a sequence $\{f_n\} \subset E'$ such that $f_n \to 0$ in $\sigma(E', E)$, but

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$\| f_n \| = 1$ for all n. Also take a sequence $\{e(n)\}$ in the closed unit ball of E such that $f_n(e(n)) \geq 1/2$ for all n. Let $\mu_n = e_{x(n)} f_n$, i.e.,

$$\mu_n(A) = \begin{cases} f_n & \text{if } x(n) \in A, \\ 0 & \text{if } x(n) \notin A, \end{cases}$$

for any Borel subset A of X. It is a straightforward verification that $\{\mu_n\} \subset (C(X, E))'$ and $\| \mu_n \| \leq 1$ for all n. We prove that $\mu_n \to 0$ in $\sigma(F', F)$, where $F = C(X, E)$. For a $g \in C(X)$ and $e \in E$, $\mu_n(g \otimes e) = g(x(n)) f_n(e) \to 0$. Since $C(X) \otimes E$ is norm dense in $C(X, E)$ and $\| \mu_n \| \leq 1$ for all n it follows that $\mu_n(h) \to 0$ for all $h \in C(X, E)$ and so $\mu_n \to 0$ in $\sigma(F', F)$. For any $M \subset N$, define $L_M\colon C(X, E)' \to K$, $L_M(\mu) = \sum_{n \in M} \mu(\chi_{x(n)} \otimes e(n))$ (note

$$\begin{aligned} \sum_{n \in M} |\mu(\chi_{x(n)} \otimes e(n))| &\leq \sum_{n \in M} |\mu| (\| \chi_{x(n)} \otimes e(n) \|) \\ &\leq \sum_{n \in M} |\mu| (\chi_{x(n)}) \\ &= |\mu| \left(\sum_{n \in M} \chi_{x(n)} \right) < \infty). \end{aligned}$$

It is easy to verify that $L_M \in F''$. Since F is a Grothendieck space, $\mu_n \to 0$ in $\sigma(F', F'')$. Thus $\langle L_M, \mu_n \rangle \to 0$ for all $M \subset N$. Define $\lambda_n\colon 2^N \to K$, $\lambda_n(M) = \langle L_M, \mu_n \rangle$. By Lemma 1, $\lambda_n(\{n\}) \to 0$ and so $f_n(e(n)) \to 0$. Since $f_n(e(n)) \geq 1/2$ for all n, this is a contradiction. This proves the theorem.

Remark 3. Our result shows that ([5], Theorem 2.2) is not correct.

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