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LINEAR MAPPINGS CONTINUOUS IN MEASURE

BY

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In [3], Marcinkiewicz and Zygmund proved the following theorem.

THEOREM 1. *Suppose that* $0 < p \le q \le 2$ *and that* T *is a continuous linear operator on* L^p $(= L^p([0, 1]))$ *with norm* $\| T \|$, *so that*

$$\int_0^1 |Tf(x)|^p \, dx \le \| T \|^p \int_0^1 |f(x)|^p \, dx$$

for every $f \in L^p$. *Then for any* n *and any* $f_1, \ldots, f_n \in L^p$ *we have*

$$\int_0^1 \left[\sum_{i=1}^n |Tf_i(x)|^q \right]^{p/q} dx \le \| T \|^p \int_0^1 \left[\sum_{i=1}^n |f_i(x)|^q \right]^{p/q} dx.$$

Stated differently, Theorem 1 says that a continuous linear operator on L^p extends in a natural way to the space $L^p(l^q)$ of l^q-valued functions on [0, 1]. Now let L be the space of all measurable functions on [0, 1], equipped with the topology of convergence in measure. Our first result is an analog of Theorem 1. It implies that a continuous linear operator on L extends in the same natural way to the space of all l^q-valued measurable functions on [0, 1].

THEOREM 2. *Let* (X, μ) *be a measure space and assume that* μ *is a probability measure on* X. *Let* T *be a linear operator defined on the space of measurable functions on* X, *and assume that* T *is continuous with respect to the topology of convergence in measure on* X. *Fix* q *with* $0 < q \le 2$. *Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* n *and for any measurable functions* $f_1, \ldots, f_n$ *on* X *which satisfy*

$$\mu \left\{ x \in X : \left[\sum_{i=1}^n |f_i(x)|^q \right]^{1/q} \ge \delta \right\} \le \delta,$$

we have

$$\mu \left\{ x \in X : \left[\sum_{i=1}^n |Tf_i(x)|^q \right]^{1/q} \ge \varepsilon \right\} \le \varepsilon.$$

Using Theorem 2 as a lemma we can prove the following theorem, which was proved in [4] with the extra hypothesis that G be abelian.

THEOREM 3. *Let* G *be a compact group and let* $L(G)$ *be the space of all (Haar-) measurable functions on* G, *equipped with the topology of convergence in*

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measure. For $x \in G$ define the left and right translation operators L_x and R_x on $L(G)$ by $L_x f(y) = f(xy)$, $R_x f(y) = f(yx)$ ($f \in L(G)$, $y \in G$). The only continuous linear operators on $L(G)$ which commute with each R_x are the finite linear combinations of the L_x's.

The proof of Theorem 2 rests on the following lemma.

LEMMA. *For $0 < q \leq 2$, the space l^q is topologically isomorphic to a subspace of L.*

Proof. For $1 < q \leq 2$ this is a consequence of Theorem 5.2 of [5], which states that (for these values of q) L^q is topologically isomorphic to a subspace of L. But given the result of [2], the proof of Theorem 5.2 in [5] works for any q with $0 < q \leq 2$.

Proof of Theorem 2. Let X, μ, T, q, and ε be as in the statement of Theorem 2, and let m stand for Lebesgue measure on $[0, 1]$. By the lemma there exist $g_1, g_2, \ldots \in L$ (corresponding to the usual l^q basis) so that the following hold.

(1) There exists $\varepsilon_4 > 0$ such that if $m\{y \in [0, 1]\colon |\sum c_i g_i(y)| \geq \varepsilon_4\} \leq \varepsilon_4$, then $(\sum |c_i|^q)^{1/q} < \varepsilon$;

(2) Given any $\varepsilon_1 > 0$ there exists some $\delta > 0$ such that if $(\sum |c_i|^q)^{1/q} \leq \delta$, then

$$m\{y\colon |\sum c_i g_i(y)| \geq \varepsilon_1\} \leq \varepsilon_1.$$

Let ε_4 be as in (1) and choose $\varepsilon_3 > 0$ with $\varepsilon_3 \leq \varepsilon_4$, $\varepsilon_3/\varepsilon_4 \leq \varepsilon/3$. Since T is continuous, there exists $\varepsilon_2 > 0$ such that if the measurable function h on X satisfies $\mu\{x \in X\colon |h(x)| \geq \varepsilon_2\} \leq \varepsilon_2$, then $\mu\{x\colon |Th(x)| \geq \varepsilon_3\} < \varepsilon_3$. Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 \leq \varepsilon_2$, $\varepsilon_1/(\varepsilon_2\varepsilon_4) \leq \varepsilon/3$. Choose δ as in (2) (corresponding to the present ε_1) and such that $\delta/(\varepsilon_2\varepsilon_4) \leq \varepsilon/3$. Then

$$((\delta + \varepsilon_1)/\varepsilon_2 + \varepsilon_3)/\varepsilon_4 = \delta/\varepsilon_2\varepsilon_4 + \varepsilon_1/\varepsilon_2\varepsilon_4 + \varepsilon_3/\varepsilon_4 \leq \varepsilon. \tag{3}$$

Now suppose that $\mu\{x\colon (\sum |f_i(x)|^q)^{1/q} \geq \delta\} \leq \delta$; we will show that

$$\mu\{x\colon (\sum |Tf_i(x)|^q)^{1/q} \geq \varepsilon\} \leq \varepsilon.$$

Since $m\{y\colon |\sum f_i(x) g_i(y)| \geq \varepsilon_1\} > \varepsilon_1$ implies $(\sum |f_i(x)|^q)^{1/q} > \delta$ by (2), we have

$$\mu\{x\colon [m\{y\colon |\sum f_i(x) g_i(y)| \geq \varepsilon_1\} > \varepsilon_1]\} \leq \delta.$$

Writing $\phi_1(x, y)$ for $\sum f_i(x) g_i(y)$ and E_1 for the set

$$\{x\colon [m\{y\colon |\phi_1(x, y)| \geq \varepsilon_1\} > \varepsilon_1]\},$$

we get $\mu(E_1) \leq \delta$. If $x \notin E_1$, then $m\{y\colon |\phi_1(x, y)| \geq \varepsilon_1\} \leq \varepsilon_1$. It follows (from Fubini's theorem) that

$$(\mu \times m)\{(x, y) \in X \times [0, 1]\colon |\phi_1(x, y)| \geq \varepsilon_1\} \leq \delta + \varepsilon_1.$$

Since $\varepsilon_2 \geq \varepsilon_1$ we have $(\mu \times m)\{(x, y)\colon |\phi_1(x, y)| \geq \varepsilon_2\} \leq \delta + \varepsilon_1$. Another application of Fubini's theorem then yields

$$(4) \qquad m\{y\colon [\mu\{x\colon |\phi_1(x, y)| \geq \varepsilon_2\} \geq \varepsilon_2]\} \leq (\delta + \varepsilon_1)/\varepsilon_2.$$

Write $\phi_2(x, y)$ for $\sum Tf_i(x)g_i(y) = T(\sum f_i g_i(y))(x)$ and recall that

$$\phi_1(x, y) = \sum f_i(x)g_i(y).$$

By the choice of ε_2, the inequality (for fixed y) $\mu\{x\colon |\phi_2(x, y)| \geq \varepsilon_3\} \geq \varepsilon_3$ implies the inequality $\mu\{x\colon |\phi_1(x, y)| \geq \varepsilon_2\} \geq \varepsilon_2$. Thus if we write E_2 for the set

$$\{y\colon [\mu\{x\colon |\phi_2(x, y)| \geq \varepsilon_3\} \geq \varepsilon_3]\},$$

(4) yields $m(E_2) \leq (\delta + \varepsilon_1)/\varepsilon_2$. If $y \notin E_2$, though, $\mu\{x\colon |\phi_2(x, y)| \geq \varepsilon_3\} < \varepsilon_3$. Therefore

$$(\mu \times m)\{(x, y)\colon |\phi_2(x, y)| \geq \varepsilon_3\} \leq [(\delta + \varepsilon_1)/\varepsilon_2] + \varepsilon_3,$$

and so

$$(\mu \times m)\{(x, y)\colon |\phi_2(x, y)| \geq \varepsilon_4\} \leq [(\delta + \varepsilon_1)/\varepsilon_2] + \varepsilon_3$$

since $\varepsilon_4 \geq \varepsilon_3$. A last application of Fubini's theorem gives

$$\mu\{x\colon [m\{y\colon |\phi_2(x, y)| \geq \varepsilon_4\} \geq \varepsilon_4]\} \leq \left(\frac{\delta + \varepsilon_1}{\varepsilon_2} + \varepsilon_3\right)\Big/ \varepsilon_4.$$

Taking into account (3) and the definition of ϕ_2, we have

$$(5) \qquad \mu\{x\colon [m\{y\colon |\textstyle\sum Tf_i(x)g_i(y)| \geq \varepsilon_4\} \geq \varepsilon_4]\} \leq \varepsilon.$$

Now (1) implies that $m\{y\colon |\sum Tf_i(x)g_i(y)| \geq \varepsilon_4\} \geq \varepsilon_4$ if $(\sum |Tf_i(x)|^q)^{1/q} \geq \varepsilon$, so (5) yields the desired result:

$$\mu\{x\colon (\textstyle\sum |Tf_i(x)|^q)^{1/q} \geq \varepsilon\} \leq \varepsilon.$$

Proof of Theorem 3. Write $L^p(G)$ for the Lebesgue space formed with respect to Haar measure on G and write $\| f \|_p$ for the norm of a function in $L^p(G)$. Let T be as in the statement of Theorem 3. The only part of the proof in [4] which does not go over *mutatis mutandis* to the present situation is the demonstration that T is bounded on $L^2(G)$. In [4], where the compact group G was abelian, this was an easy observation. Here we shall use Theorem 2 to show that T is bounded on $L^2(G)$ without the hypothesis that G be abelian. Our method is based on an adaptation of the central idea in [1]. We shall show the following.

(6) There exists a $\delta > 0$ such that if f and Tf are continuous and if $\| f \|_2 \leq \delta/2$, then $\| Tf \|_2 \leq (1 + \delta)/2$.

(7) Tf is continuous whenever f is a trigonometric polynomial on G.

From (6) and (7) it follows immediately that T is bounded on $L^2(G)$.

First we establish (6). Write μ for normalized Haar measure on G and let

$\delta > 0$ be such that

$$\mu\{x \in G\colon (\textstyle\sum |Tf_i(x)|^2)^{1/2} \geq 1/2\} \leq 1/2 \tag{8}$$

if

$$\mu\{x \in G\colon (\textstyle\sum |f_i(x)|^2)^{1/2} \geq \delta\} \leq \delta$$

for $f_i \in L(G)$. Such a δ exists by Theorem 2. Let f be a continuous function on G such that Tf is continuous, and suppose that $\| f \|_2 \leq \delta/2$. By the uniform continuity of f and Tf, there exists a Borel partition $\{E_i\}_{i=1}^n$ of G such that

$$|f(xy_1) - f(xy_2)|,\ |Tf(xy_1) - Tf(xy_2)| \leq \delta/2 \tag{9}$$

for any $x \in G$ whenever $y_1, y_2 \in E_i$ $(1 \leq i \leq n)$. For $i = 1, \ldots, n$, fix $x_i \in E_i$ and let $f_i(x) = f(xx_i)\mu(E_i)^{1/2}$. Writing χ_i for the characteristic function of E_i $(1 \leq i \leq n)$ and, for arbitrary fixed $x \in G$, putting $g(y) = \sum f(xx_i)\chi_i(y)$, we have

$$(\textstyle\sum |f_i(x)|^2)^{1/2} = (\sum |f(xx_i)|^2 \mu(E_i))^{1/2} = \| g \|_2.$$

Now $\| g - L_x f \|_\infty \leq \delta/2$ by (9), and so

$$\| g \|_2 \leq \| L_x f \|_2 + \| g - L_x f \|_2 \leq \| f \|_2 + \| g - L_x f \|_\infty \leq \delta.$$

Thus $(\sum |f_i(x)|^2)^{1/2} \leq \delta$. As this holds for any $x \in G$, we have

$$\mu\{x \in G\colon (\textstyle\sum |Tf_i(x)|^2)^{1/2} \geq 1/2\} \leq 1/2 \tag{10}$$

by (8). Since T commutes with each R_{x_i}, $Tf_i(x) = Tf(xx_i)m(E_i)^{1/2}$. If for fixed $x \in G$ we put $h(y) = \sum Tf(xx_i)\chi_i(y)$, then we have, as before, $(\sum |Tf_i(x)|^2)^{1/2} = \| h \|_2$. Since $\| h - L_x Tf \|_\infty \leq \delta/2$ by (9), we get

$$(\textstyle\sum |Tf_i(x)|^2)^{1/2} \geq \| Tf \|_2 - \delta/2,$$

and this holds for each $x \in G$. Now (10) implies that $\| Tf \|_2 \leq 1/2 + \delta/2$, and so (6) is established.

We conclude the proof of Theorem 3 by establishing (7). Each trigonometric polynomial f on G is a finite linear combination of trigonometric polynomials u which satisfy functional equations of the form

$$u(xy) = \sum_{l=1}^{m} u_{jl}(x)u_{lk}(y).$$

Here the u_{jl}'s and the u_{lk}'s are again trigonometric polynomials. For such a u and for each fixed $y \in G$ we have

$$\begin{aligned} Tu(xy) &= (R_y Tu)(x) = (TR_y u)(x) \\ &= \left(T \sum_{l=1}^{m} u_{jl}u_{lk}(y)\right)(x) = \sum_{l=1}^{m} Tu_{jl}(x)u_{lk}(y), \end{aligned} \tag{11}$$

for almost all $x \in G$. Thus there exists some x in G such that (11) holds for almost all $y \in G$, and so Tu is (equal almost everywhere to) a continuous function on G.

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