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LEFT THICK SUBSETS OF A TOPOLOGICAL SEMIGROUP[1]

BY

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In this paper we will characterize the left X-thick subsets of an X-amenable topological semigroup S. Our result extends a result of T. Mitchell in [12] and Wilde and Witz [15].

1. Preliminaries

Let S be a topological semigroup with separately continuous multiplication and X a subalgebra (under pointwise multiplication) of $CB(S)$, the space of all continuous bounded functions on S, containing constant functions. A mean m (see [5] for the definition) on X is multiplicative if $m(fg) = m(f)m(g)$ for all f and g in X. We denote by M, the set of all means on X and by S_1, the subspace of M consisting of all multiplicative means on X.

Let X be a left translation invariant and left introverted subspace of $CB(S)$ containing the constant functions. Let m and n be in M. We define the Arens product of m and n, denoted by $m \odot n$ to be the functional defined by $m \odot n(f) = m(n_l(f))$ for f in X, where n_l is the left introversion of n. Arens product makes M into a semigroup with the following properties. For s in S let $Q(s)$ denote the evaluation functional at X; then for fixed n in M and s in S the linear mappings $m \to m \odot n$ and $m \to Q(s) \odot m$ are continuous on M, and the map Q is a continuous homomorphism of the semigroup S into M, where M has the weak* topology. If in addition X is a subalgebra of $CB(S)$, it is easy to see that S_1 is a subsemigroup of M. For more details see Day [5] and Rao [14]. Following Wilde and Witz [15], the symbols $k(T)$, $K(T)$ will indicate the convex hull and the weak* closed convex hull of any subset T of M. In particular when $A \subseteq S$, we write kA for $k(QA)$ and KA for $K(QA)$. Let L be the set of all left invariant means on X. If L is nonempty we call S X-amenable. In this case every element in L is a right zero of M (see [5] and [14]). Hence $L = \text{Ker } M$, where Ker M denotes the smallest two sided ideal of M (see [4, Problem 6, p. 6]).

2. The support of a left invariant mean

In this section we shall prove that the support of a left invariant mean on a left translation invariant left introverted closed subalgebra X of $CB(S)$ is a left ideal of S_1. We first define the support of a mean, for this purpose we need the following lemmas.

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LEMMA 2.1. *Let X be a closed subalgebra of $CB(S)$ containing the constant functions. Let S_1 be the set of all multiplicative means on X with weak* topology (S_1 is called the maximal ideal space of X).*

(i) *The map $Q: S \to S_1$ is continuous and $Q(S)$ is dense in S_1.*
(ii) *The induced map $\tilde{Q}: CB(S_1) \to X$ defined by $\tilde{Q}(f_1) = f_1 \circ Q$ is an isometric isomorphism.*

Proof. (i) This follows from [7, Theorem 19, p. 276].
(ii) It is shown in [7, Theorem 18, pp. 274–275] that the map $U: X \to CB(S_1)$ defined by $Uf = f_1$ where $f_1(x^*) = x^*(f)$ for all x^* in S_1, is a linear isometry from X onto $CB(S_1)$. We observe that our map $\tilde{Q}$ is exactly U^{-1}. Hence (ii) follows from [7, Theorem 18, pp. 274–275].

LEMMA 2.2. *Let X be a closed subalgebra of $CB(S)$ containing the constant functions. Let m be a mean on X. Then the functional $m_1 = \tilde{Q}^*m$ is a mean on $CB(S_1)$. If in addition X is left translation invariant, then m is left invariant iff $m_1(f_1 \circ L_s) = m_1(f_1)$ for all f_1 in $CB(S_1)$ and s in S, where L_s is the restriction of operator l_s^* to S_1.*

Proof. It is easy to see that m_1 is a mean on $CB(S_1)$. To see the second assertion we observe that L_s is a continuous map from S_1 into S_1. Hence $f_1 \circ L_s$ is in $CB(S_1)$. Since

$$l_s(f_1 \circ Q)(t) = f_1(Q(st)) = f_1(L_s(Q(t))) = f_1 \circ L_s \circ Q(t)$$

for all t in S, we have $l_s(f_1 \circ Q) = \tilde{Q}(f_1 \circ L_s)$ for all s in S and f_1 in $CB(S_1)$. Hence

$$m_1(f_1 \circ L_s) = \tilde{Q}^*m(f_1 \circ L_s) = m(\tilde{Q}(f_1 \circ L_s)) = m(l_s(f_1 \circ Q))$$

for all s in S and f_1 in $CB(S_1)$.

Now, assume that m is left invariant. Then

$$m_1(f_1 \circ L_s) = m(l_s(f_1 \circ Q)) = m(f_1 \circ Q) = \tilde{Q}^*m(f_1) = m(f_1)$$

for all s in S and f_1 in $CB(S_1)$.

Conversely, if $m_1(f_1 \circ L_s) = m_1(f_1)$ for each s in S and f_1 in $CB(S_1)$ then

$$m(l_s(f_1 \circ Q)) = m_1(f_1 \circ L_s) = m_1(f_1) = \tilde{Q}^*m(f_1) = m(f_1 \circ Q)$$

for all s in S and f_1 in $CB(S_1)$, and this completes the proof.

DEFINITION 2.3. Let X be a closed subalgebra of $CB(S)$ containing the constant functions. Let m be a mean on X and m_1 the mean defined in Lemma 2.2. Then by Riesz's representation theorem, m_1 induces a unique regular probability measure μ_1 on the Borel sets of S_1. Let $s(m)$ be the support of μ_1. We call $s(m)$ the support of m. For discrete semigroups our definition agrees with one given in [15]. If E is a collection of means on X, we denote by $s(E)$ the support of E, which is by definition the weak* closure of $\bigcup_{m \in E} s(m)$ in S_1 (see [16]).

LEMMA 2.4. *Let X be a left translation invariant closed subalgebra of $CB(S)$ containing the constant functions. Let m be a left invariant mean on X. Then $L_s(s(m)) = s(m)$ for each s in S.*

Proof. For simplicity write $F_1 = s(m)$.

Let s be in S and f_1 the element in $CB(S_1)$ satisfying $f_1(L_sF_1) = 1, 0 \leq f_1 \leq 1$. Let μ_1 be the probability measure induced by m_1, where m_1 is as in Lemma 2.2. Then, by Lemma 2.2,

$$\int_{F_1} f_1 \, d\mu_1 = m_1(f_1) = m_1(f_1 \circ L_s) = \int_{F_1} f_1 \circ L_s \, d\mu_1 = \int_{F_1} 1 \, d\mu_1.$$

Hence $\int_{F_1} (1 - f_1) \, d\mu_1 = 0$. Since $1 - f_1$ is continuous and nonnegative we have $1 - f_1 \equiv 0$ on F_1. Therefore by Urysohn's lemma, $F_1 \subseteq L_sF_1$. A similar argument implies that $L_sF_1 \subseteq F_1$, for each s in S.

THEOREM 2.5. *Let X be a left introverted and left translation invariant closed subalgebra of $CB(S)$ containing the constant functions. Let m be a left invariant mean on X. Then $Q(s) \odot s(m) = s(m)$ for all s in S, and $s(m)$ is a left ideal of S_1.*

Proof. First we observe that, for s in S and μ in S_1,

$$L_s\mu(f) = \mu(l_s f) = Q(s) \odot \mu(f)$$

for any f in X. Hence $L_s \mu = Q(s) \odot \mu$ for each s in S and μ in S_1. Now we apply Lemma 2.4 to get $Q(s) \odot s(m) = s(m)$ for all s in S. Let ν be any element in S_1. Then there is a net $Q(s_\alpha)$ converging to ν in S_1. Let γ be any element in $s(m)$. Then $Q(s_\alpha) \odot \gamma \rightarrow \nu \odot \gamma$. Since each element of the net $Q(s_\alpha) \odot \gamma$ is in $s(m)$ and $s(m)$ is closed, we deduce that $\nu \odot \gamma$ is in $s(m)$. But this implies that $S_1 \odot s(m) \subseteq s(m)$, that is $s(m)$ is a left ideal of S_1.

Remarks. (a) Applying Theorem 2.5, we can show the following formally stronger result: For any subset L_0 of L, $s(L_0)$ is a left ideal of S_1. We omit this simple proof.

(b) By the preceding remark, for discrete cancellative semigroups S and $X = B(S)$, Theorem 2.5 reduces to a result of Wilde and Witz [15, Theorem 4.3, p. 583].

(c) An analogue of Lemma 2.4 is shown by Argabright in [2, pp. 197–200] for completely regular semigroups S and $X = C(S)$, the space of all continuous (not necessarily bounded) functions on S.

(d) It is possible to give a different proof of Lemma 2.4, using the techniques employed by Argabright in [1].

3. The maximal ideal space of almost periodic functions

In this section we shall improve a result due to Loomis [11] and Burckel [3], by applying Theorem 2.5. See [3] for the definition of (weakly) almost periodic functions $(W(S))A(S)$. First we need the following lemma.

LEMMA 3.1. (i) *Let S_1 be the set of all multiplicative means on $X = W(S)$. Then for each f in $W(S)$,*

$$\textit{weak closure of } \{r_s f\colon s \in S\} = \{n_l(f)\colon n \textit{ in } S_1\}.$$

(ii) *Let S_1 be the set of all multiplicative means on $X = A(S)$. Then for each f in $A(S)$,*

$$\textit{norm closure of } \{r_s f\colon s \in S\} = \{n_l(f)\colon n \textit{ in } S_1\}.$$

Proof. (i) Let f be in $W(S)$. Suppose s_α is a net in S, such that $r_{s_\alpha} f$ converges weakly to g. Then certainly $r_{s_\alpha} f$ converges pointwise to g. Since S_1 is compact, passing to a subnet if necessary, we may assume that $Q(s_\alpha)$ converges to an element n in S_1, that is, $Q(s_\alpha) \to n$ weak*. This implies that

$$r_{s_\alpha} f = (Q(s_\alpha))_l f \to n_l(f) \quad \text{pointwise.}$$

Therefore $g = n_l(f)$ and consequently g is in $\{n_l(f)\colon n \text{ in } S_1\}$. Hence weak closure of $\{r_s f\colon s \text{ in } S\} \subseteq \{n_l(f)\colon n \text{ in } S_1\}$. Let n be any element in S_1 and $Q(s_\alpha)$ a net in $Q(S)$ converging weak* to n. Then

$$(Q(s_\alpha))_l f = r_{s_\alpha} f \to n_l(f) \quad \text{pointwise.}$$

Since f is in $W(S)$, $\{r_s f\colon s \in S\}$ is relatively weak compact. Hence passing to a subnet if necessary we may assume that $r_{s_\alpha} f$ converges weakly. But this implies that $r_{s_\alpha} f$ converges to $n_l(f)$ weakly. That is,

$$\{n_l(f)\colon n \text{ in } S_1\} \subseteq \text{weak closure of } \{r_s f\colon s \in S\}.$$

(ii) A similar argument as above and the fact that, for f in $A(S)$, $\{r_s f\colon s \text{ in } S\}^-$ is norm compact implies this part.

The following is a well known result. (See, for example, [6], [13], or [16]). For completeness we give a different proof here.

LEMMA 3.2. (i) *S_1, the maximal ideal space of weakly almost periodic functions under Arens multiplication, is a compact semigroup.*

(ii) *S_1, the maximal ideal space of almost periodic functions under Arens multiplication, is a compact semigroup with jointly continuous multiplication.*

Proof. (i) All we need is to show that the map $n \in S_1 \to m \odot n \in S_1$ is continuous for each m in S_1. Let n_α be a net in S_1 converging to n in S_1. We should prove $m \odot n_\alpha \to m \odot n$ weak* for any m in S_1, that is, for each f in $W(S)$ and m in S_1, $m \odot n_\alpha(f) \to m \odot n(f)$. Let f be any element in $W(S)$. By Lemma 3.1 (i) and definition, $\{n_l(f)\colon n \text{ in } S_1\}$ is weakly compact, since this set is pointwise compact we deduce that the pointwise and weak topology coincide on $\{n_l(f)\colon n \text{ in } S_1\}$. Now since $n_\alpha \to n$ weak*, we conclude that $(n_\alpha)_l f \to n_l(f)$ pointwise and hence weakly. That is,

$$m((n_\alpha)_l f) = m \odot n_\alpha(f) \to m(n_l(f)) = m \odot n(f)$$

for each m in S.

(ii) We should show that the map $(m, n) \in S_1 \times S_1 \to m \odot n \in S_1$ is continuous. Let m_α and n_α be two nets in S_1 converging to m and n in S_1 respectively and let f be in $A(S)$. Then $(n_\alpha)_l f \to n_l(f)$ pointwise. But the pointwise and norm topology coincide on $\{n_l(f)\colon n \text{ in } S_1\}$ (this follows from an argument similar to one given in part (i)). Hence $(n_\alpha)_l f \to n_l(f)$ strongly. Therefore

$$
\begin{aligned}
&|m_\alpha \odot n_\alpha(f) - m \odot n(f)| \\
&\qquad \le |m_\alpha \odot n_\alpha(f) - m_\alpha \odot n(f)| + |m_\alpha \odot n(f) - m \odot n(f)| \\
&\qquad \le \| m_\alpha \| \, \| (n_\alpha)_l f - n_l(f) \| + |m_\alpha(n_l(f)) - m(n_l(f))| \to 0.
\end{aligned}
$$

(Notice that $n_l(f)$ is in $A(S)$ for each n in S_1.) Since f is arbitrary this completes the proof.

Now we are ready for the main result.

THEOREM 3.3. *Let S be a topological semigroup. Then the following statements are equivalent:*

(i) *There is a two sided invariant mean m on $A(S)$ such that $m(f) \neq 0$ for each nonnegative nonzero f in $A(S)$.*

(ii) *S_1, the maximal ideal space of $A(S)$, is a compact group.*

Proof. (i) → (ii). Suppose (i) holds. Let $m_1 = \tilde{Q}^* m$ and let μ_1 be the measure induced by m_1 on S_1. We claim that $s(m)$ equals the support of $\mu_1 = S_1$. Suppose $s(m) \subset_{\neq} S_1$. Let ν be in $S_1 - s(m)$. Then there is a function f_1 in $CB(S_1)$ such that

$$0 \le f_1 \le 1, \quad f_1(\nu) = 1 \quad \text{and} \quad f_1(s(m)) = 0.$$

Now, $m(\tilde{Q}(f_1)) = \tilde{Q}^* m(f_1) = m_1(f_1) = \int_{s(m)} f_1 \, d\mu_1 = 0$. Since $f_1 \ge 0$ we have $\tilde{Q}(f_1) \ge 0$; therefore, by assumption, $\tilde{Q}(f_1) = 0$. But this implies that $f_1 = 0$ on S_1, which is a contradiction. Hence $s(m) = S_1$. Since $A(S)$ is left introverted we can apply Theorem 2.5 to get $Q(s) \odot S_1 = S_1$, for each s in S. We claim that $S_1 \subseteq \mu \odot S_1$ for each μ in S_1. To see this let $Q(s_\alpha)$ be a net in S_1 converging to μ weak* and ν in S_1. Since for each α, $Q(s_\alpha) \odot S_1 = S_1$. Corresponding to each α, there is an element μ_α in S_1 such that $Q(s_\alpha) \odot \mu_\alpha = \nu$.

Since S_1 is compact passing to a subnet if necessary, we may assume that $\mu_\alpha \to \gamma$ in S_1. Now joint continuity of Arens multiplication (Lemma 3.2) implies that $Q(s_\alpha) \odot \mu_\alpha \to \mu \odot \gamma = \nu$. Hence $S_1 \subseteq \mu \odot S_1$ for μ in S_1. Therefore $S_1 = \mu \odot S_1$ for each μ in S_1. Similarly we can show that $S_1 \odot \mu = S_1$ for each μ in S_1 (note that due to separate continuity of Arens multiplication a right handed version of Theorem 2.5 holds for $X = W(S)$ and $X = A(S)$). Therefore S_1 is a group (see Clifford and Preston [4, p. 6]). Since multiplication is jointly continuous on S_1 we deduce that S_1 is a compact group.

(ii) → (i). Suppose (ii) holds. Let I be the normalized Haar integral on S_1. Then $I(f_1 \circ L_s) = I(f_1)$ for each s in S and f_1 in $CB(S_1)$. Let $m = (\tilde{Q}^*)^{-1} I$ (note

L_s and $\tilde{Q}$ are as in Lemma 2.2). Then by Lemma 2.2, m is a left invariant mean on $A(S)$. Similarly we can show that m is right invariant. Let f be a nonnegative nonzero element of $A(S)$. Then $f_1 = \tilde{Q}^{-1}f$ is a nonnegative and nonzero element of $CB(S_1)$. Hence $I(f_1) > 0$ and consequently $m(f) = m(\tilde{Q}(\tilde{Q}^{-1}f)) = I(f_1) > 0$, and this completes the proof.

Remarks. (a) By Krein–Smulian Theorem (see [8]), for each f in $W(S)$ (resp. $A(S)$), the weak (resp. norm) closure of $Co\{r_s f: s \in S\}$ is weak (resp. norm) compact. Using this and an argument similar to the proof of Lemma 3.1, we can show that for any f in $W(S)$ (resp. $A(S)$), the weak closure of $Co\{r_s f: s \in S\}$, equals $\{m_l(f): m \text{ a mean on } W(S)\}$ (resp. the norm closure of $Co\{r_s f: s \in S\}$ equals $\{m_l(f): m \text{ a mean on } A(S)\}$). This together with the fact that $W(S)$ (resp. $A(S)$) is a weak (resp. norm) closed subspace of $CB(S)$ implies the known result (see [6]) that $W(S)$ (resp. $A(s)$) is a left introverted subspace of $CB(S)$.

(b) If the topological semigroup S contains an identity, then the map $n \to n_l$, where n is in S_1 and S_1 is the maximal ideal space of $A(S)$ (resp. $W(S)$), designates a homeomorphism between S_1 and S^α (resp. S^ω), the deLeeuw-Glicksberg's (resp. weakly) almost periodic compactification of S (see [6] or [3] for details of this compactification). The same map is also an algebraic isomorphism, where S_1 has the Arens multiplication and S^α (resp. S^ω) the composition of operators as multiplication.

(c) Let S be a topological group. Then $A(S)$ admits a two-sided invariant mean satisfying (i) of Theorem 3.3. (See [10, Theorem 18.8, p. 250].) Hence implication (i) $\to$ (ii) of Theorem 3.3 reduces to a result of Loomis [11, Theorem 41C, pp. 167–168].

(d) If S is a group and a topological semigroup, then $A(S)$ admits a two-sided invariant mean satisfying (i) of Theorem 3.3 (see [3, Corollary 1.26, p. 15]). Therefore Theorem 3.3 reduces to a result of Burckel [3, Corollary 2.27, p. 32].

(e) Let S be a nontrivial regular Hausdorff space such that $CB(S)$ consists of constant functions. (For existence of such spaces see [9].) Define a multiplication on S by $st = t$ for s and t in S. Then S is a topological semigroup which is not a group and has no identity. Trivially $A(S)$ admits a two-sided invariant mean satisfying (i) of Theorem 3.3. Hence we deduce from Theorem 3.3 that S_1 is a one point compact group. Neither Loomis's Theorem (due to lack of group structure on S) nor Burckel's result (due to lack of identity in S) can be applied to such a topological semigroup.

4. Left thick subsets of a topological semigroup

In this section we will characterize the left thick subsets of an X-amenable topological semigroup. Our result implies a result due to Mitchell [12].

DEFINITION 4.1. Let X be a left translation invariant, left introverted closed subalgebra of $CB(S)$ containing the constant functions. Let S_1 be the maximal ideal space of X. A subset A of S is called left X-thick in S if $Q(A)^-$ (weak* closure of $Q(A)$ in S_1) contains a left ideal of S_1.

THEOREM 4.2. *Let X be as in Definition 4.1. Suppose X admits a left invariant mean. Then the following are equivalent:*

(i) *There is a left invariant mean m on X such that $m(f)=1$ for each f in X satisfying $\chi_A \leq f \leq 1$.*
(ii) *A is left X-thick in S.*

Proof. (i)$\rightarrow$(ii). Suppose (i) holds. We claim that $s(m) \subseteq Q(A)^-$. Let $m_1 = \tilde{Q}^*m$ and μ_1 be the measure induced by m_1 on S_1. If we show $\mu_1(Q(A)^-)=1$, we will have $s(m) \subseteq Q(A)^-$. For this purpose let f_1^n be a sequence of functions in $CB(S_1)$ such that $\chi_{Q(A)^-} \leq f_1^n \leq 1$ and $f_1^n \geq f_1^{n+1}$ for all n, and f_1^n converges to $\chi_{Q(A)^-}$ μ_1-a.e. (for construction of such sequence see Hewitt and Ross [19, p. 129]). Then $f^n = f_1^n \circ Q \in X$ and $\chi_A \leq f^n \leq 1$ for each n and hence

$$\begin{aligned}\mu_1(Q(A)^-) &= \int \chi_{Q(A)^-}\, d\mu_1 = \int \lim_n f_1^n \, d\mu_1 = \lim_n \int f_1^n \, d\mu_1 \\ &= \lim_n m_1(f_1^n) = \lim_n (\tilde{Q}^*m)(f_1^n) = \lim_n m(f^n) = 1\end{aligned}$$

by the monotone convergence theorem and assumption. Hence $s(m) \subseteq Q(A)^-$. Since, by Theorem 2.5, $s(m)$ is a left ideal of S_1, we deduce that $Q(A)^-$ contains a left ideal of S_1; that is, A is left X-thick in S.

(ii)$\rightarrow$(i). Suppose (ii) holds. Let $I \subseteq Q(A)^-$ be a left ideal of S_1, then KI is a left ideal of M. By compactness of M, every left ideal contains a minimal left ideal (see [3, Theorem 2.1, p. 19]). Since $L = \operatorname{Ker} M = \emptyset$ (see Section 1), $L \cap KI$ is nonempty. Hence $L \cap KA \neq \emptyset$. Let m be any element in $KA \cap L$. Define $m_1 = \tilde{Q}^*m$ and let μ_1 be the measure induced by m_1 on S_1. We show that $\mu_1(Q(A)^-)=1$. To see this let ϕ_α be a net in KA converging to m weak*. Let f_1^n be the sequence introduced in the first part of the proof. Then $\tilde{Q}^*\phi_\alpha(f_1^n)=1$ for each α and n. Hence

$$\begin{aligned}\mu_1(Q(A)^-) &= \int \chi_{Q(A)^-}\, d\mu_1 = \int \lim_n f_1^n \, d\mu_1 = \lim_n \int f_1^n \, d\mu_1 \\ &= \lim_n m_1(f_1^n) = \lim_n \tilde{Q}^*m(f_1^n) = \lim_n m(f_1^n \circ Q) \\ &= \lim_n \left(\lim_\alpha \phi_\alpha(f_1^n \circ Q)\right) = \lim_n \left(\lim_\alpha \tilde{Q}^*\phi_\alpha(f_1^n)\right) = 1.\end{aligned}$$

Now let f be any element in X satisfying $\chi_A \leq f \leq 1$. Let $f_1 = \tilde{Q}^{-1}f$. Then by continuity of f_1, $\chi_{Q(A)^-} \leq f_1 \leq 1$ and therefore

$$m(f) = m(\tilde{Q}(\tilde{Q}^{-1}f)) = \tilde{Q}^*m(f_1) = m_1(f_1) = \int_{Q(A)^-} f_1 \, d\mu_1 = 1,$$

which completes the proof.

Remarks. (a) Let S be a discrete semigroup. In [12, p. 256], T. Mitchell defines the left thick subsets of S as follows: a subset, A, of S is called left thick in S if for each finite subset F of S there exists s in S such that $Fs \subseteq A$. According to Wilde and Witz [15, Lemma 5.1, p. 589] this definition agrees with our definition and hence for discrete semigroups S and $X = B(S)$, Theorem 4.2 reduces to a result of Mitchell [12, Theorem 7, p. 257].

(b) Due to antisymmetry of Arens product the right handed version of Theorem 4.2 may not be true in general.

(c) Let S be an X-amenable topological semigroup, I any left ideal of S. Then $Q(I)^-$ is a left ideal of S_1 and therefore I is left X-thick in S.

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