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ORIENTATION REVERSING SQUARE ROOTS OF INVOLUTIONS

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1. Introduction

Let X be a smooth, orientable, compact surface of genus n and let $g: X \to X$ be a smooth orientation reversing self homeomorphism with the property that $f = g^2$ has prime order p. In this and a subsequent paper [6] we give a geometric description of such maps g as well as a classification of their conjugacy classes in the group of diffeomorphisms of X. An analogous classification of orientation preserving maps has been given by Nielsen [3] and Gilman [2]. In the present paper we consider the case $p = 2$, which appears to be somewhat different from the case in which p is odd [6].

Our main theorem is the following.

THEOREM 1.1. *Let X be a smooth, orientable, compact surface of genus n and let $g_i: X \to X$, $i = 1, 2$ be two orientation reversing maps with the property that g_1^2 and g_2^2 both have order two. Then g_1 and g_2 are conjugate in the group of diffeomorphisms of X if and only if g_1^2 and g_2^2 have the same number of fixed points.*

Although our results are purely topological we sometimes use theorems from, or give results in, the theory of Riemann surfaces. It is well known that it is possible to put a complex structure on X so that g and f are respectively, anti-conformal and conformal. An (anti-)conformal self homoeomorphism of a compact Riemann surface is called an (anti-)automorphism. Such maps must always have finite order if $n \geq 2$, so that the study of (anti-)automorphisms of compact Riemann surfaces is in a sense equivalent to the study of periodic maps of compact smooth 2-manifolds of genus $n \geq 2$.

We say that a Riemann surface is embeddable if it is conformally equivalent to a smooth surface which is embedded in $\mathbf{R}^3$. If f is an orientation preserving self-homeomorphism of a smooth surface X, then we say f is metrically embeddable if there exists a smooth injection $d: X \to \mathbf{R}^3$ so that dfd^{-1} is the restriction of a rotation. If f is orientation reversing, then we say that f is metrically embeddable if dfd^{-1} is the restriction of a reflection in some plane. Finally, if X is a Riemann surface, f is conformal or anti-conformal and d is conformal, then we simply say that f is embeddable.

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R. Rüedy [4] has shown that any map of order two is embeddable and that every embeddable map has an even number of fixed points.

We now fix some notation used throughout this paper. The surface X is orientable, smooth and compact of genus n and g: $X \to X$ is an orientation reversing self homeomorphism. The map $f = g^2$ has $2a$ fixed points, where a is an integer. The map g induces a map g' on $X' = X/\langle f \rangle$ which is orientation reversing and of order two. Let $\pi: X \to X'$ denote the (possibly branched) covering. The surface X' has genus m, where by the Riemann-Hurwitz formula $n-1 = 2(m-1)+a$. All surfaces will be understood to be (at least) smooth, and the word "map" will always mean smooth homeomorphism. If h is an embeddable map then let $\alpha(h)$ denote the angle of rotation. We will normalize by assuming that $0 < \alpha(h) < 2\pi$. Thus $\alpha(f) = \pi$.

The following theorem describes a large class of orientation reversing square roots of involutions geometrically.

THEOREM 1.2. *If X, f, g and a are as above, and if $m \geq a-1$, then $g = H \circ K$, where H is a metrically embeddable map such that $H^2 = f$ and K is orientation reversing of order two. Also H and K commute. In addition, if X is given a complex structure so that H is conformal, then $\alpha(H) = \pi/2$. If a is odd then $X/\langle K \rangle$ is orientable of genus $3(a-1)/2$ with $2(m-a)+3$ boundary components. If a is even and positive then $X/\langle K \rangle$ is orientable of genus $3(a-2)/2$ with $2(m-a)+6$ boundary components. If $a = 0$, then $X/\langle K \rangle$ is orientable of genus zero with $m+1$ boundary components.*

Before beginning we need one more definition. By a canonical homology basis on X (henceforth known as a CHB) we mean a collection of $2n$ loops $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ such that $A_i \times B_j = \delta_{ij}$, $A_i \times A_j = B_i \times B_j = 0$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. Here x denotes the intersection number.

2. Preliminaries

We prove here some results which are used in proving 1.1 and 1.2.

LEMMA 2.1. *If Y is a smooth orientable surface and ϕ: $Y \to Y$ is an orientation reversing map of finite order which has fixed points, then ϕ has order two.*

Proof. It is possible to put a Riemann surface structure on Y so that ϕ is anti-conformal. We let x be a point fixed by ϕ. It is easy to find a disc D containing x which ϕ maps onto itself. If Δ denotes the unit disc, and h: $D \to \Delta$ is a conformal map with the property that $h(x) = 0$, then $h\phi h^{-1}$: $\Delta \to \Delta$ is an anti-moebius transformation. Thus

$$h\phi h^{-1}(z) = (a\bar{z}+b)(\overline{bz}+\bar{a})^{-1} \quad \text{where} \quad |a|^2-|b|^2 = 1.$$

Since $h\phi h^{-1}(0) = 0$, $b = 0$. Thus $|a| = 1$, so $a = \exp i\theta$ and

$$h\phi h^{-1}(z) = (\exp 2i\theta)\bar{z}.$$

It is easy to check that $h\phi^2h^{-1}(z)=z$. Thus ϕ^2 is the identity.

LEMMA 2.2. *The map g' has no fixed points.*

Proof. Let $x'=\pi(x)$. If $g'(x')=x'$, then either $g(x)=x$ or $g(x)=f(x)$. Since $g^2=f$ the second equation reduces to the first. By 2.1 this implies that g has order two, a contradiction. Thus g, and hence g', have no fixed points.

LEMMA 2.3. $m-a\equiv 1 \bmod 2$.

Proof. We prove this by induction on a. Thus assume first that $a=0$. We assume the contrary, i.e., m is even. In this case it is shown in [5, pp. 225–226] that there is a dividing cycle A on X' with the property that $g'(A)=A$. Since A is a dividing cycle, it is easy to show that it lifts to two loops on X which are interchanged by f. We label these loops A_1 and A_2. If $g(A_i)=A_i$, $i=1,2$, then $f(A_i)=A_i$, a contradiction. If g interchanges A_1 and A_2 then $f(A_i)=A_i$, again a contradiction. Thus m must be odd.

Now let $a>0$. We assume that X has been given a complex structure so that g is anti-conformal. Suppose f has fixed points q and $g(q)$ which are contained in closed discs D and $D'=g(D)$, respectively. Assume that $f(D)=D$ so that $f(D')=D'$. Let $\phi: D\to\Delta$, $\psi: D'\to\Delta'$ be two holomorphic homeomorphisms, where Δ and Δ' are respectively the closed unit disc and the closure of the exterior of the unit disc. Assume $\phi(q)=0$ and $\psi(g(q))=\infty$. The maps $\psi g\phi^{-1}=g_1$ and $\phi g\psi^{-1}=g_2$ are both anti-conformal and $g_1: \Delta\to\Delta'$, $g_2: \Delta'\to\Delta$. Thus g_1 and g_2 are anti-moebius transformations, so we may write

$$g_1(z)=(a+b\bar{z})(b+\overline{az})^{-1}, \quad |a|^2-|b|^2=1,$$

and

$$g_2(z)=(c+\overline{dz})(d+\overline{cz})^{-1}, \quad |c|^2-|d|^2=1.$$

Since $g_1(0)=\infty$ and $g_2(\infty)=0$ we must have $b=d=0$. Thus we may write

$$g_1(z)=(\exp i\alpha)/\bar{z} \quad \text{and} \quad g_2(z)=(\exp i\beta)/\bar{z},$$

where $-\pi\le\alpha\le\pi$ and $-\pi\le\beta\le\pi$. This implies that $g_1(\exp i\theta)=\exp i(\theta+\alpha)$ and $g_2(\exp i\theta)=\exp i(\theta+\beta)$.

We now define a map $l: \partial D\to\partial D'$ with the property that $lgl=g$. Let

$$l(\phi^{-1}(\exp i\theta))=\psi^{-1}(\exp i(\theta+(\alpha-\beta)/2)).$$

We now calculate

$$\begin{aligned} lgl(\phi^{-1}(\exp i\theta)) &= lg\psi^{-1}(\exp i(\theta+(\alpha-\beta)/2)) \\ &= l\phi^{-1}(\exp i(\theta+(\alpha+\beta)/2)) \\ &= \psi^{-1}(\exp i(\theta+(\alpha+\beta)/2+(\alpha-\beta)/2) \\ &= \psi^{-1}(\exp i(\theta+\alpha)) \\ &= \psi^{-1}g_1(\exp i\theta) \\ &= g\phi^{-1}(\exp i\theta). \end{aligned}$$

Thus $lgl=g$.

We now construct a surface of genus $n+1$ on which g induces a map. First remove the interiors of D and D' from X and identify $x \in \partial D$ with $l(x) \in \partial D'$ to obtain a surface Y of genus $n+1$. The condition $lgl = g$ implies that g induces an orientation reversing map G on Y. Clearly G^2 has order two with $2(a-1)$ fixed points. The surface $Y/\langle G^2 \rangle$ has genus $m+1$. Thus by the induction hypothesis $m+1-(a-1) = 1 \bmod 2$, so that $m-a = 1 \bmod 2$. This completes the proof.

LEMMA 2.4. *Suppose Y and Z are compact surfaces of genus m, each with 2a distinguished points $p_i \in Y$ and $q_i \in Z$, $i = 1, 2, \ldots, 2a$. Suppose further that there are orientation reversing involutions $g_1: Y \to Y$ and $g_2: Z \to Z$ such that*

$$g_1(q_i) = q_{i+a} \ (\mathrm{mod}\ 2a), \quad g_2(p_i) = p_{i+a} \ (\mathrm{mod}\ 2a).$$

Also assume that $Y/\langle g_1 \rangle \cong Z/\langle g_2 \rangle$ and that these are both non-orientable surfaces without boundary curves. Then there is a map $h: Y \to Z$ such that $g_2 = hg_1h^{-1}$ and

$$h\{p_i, p_{i+a}\} = \{q_i, q_{i+a}\}, \quad i = 1, 2, \ldots, a.$$

Proof. There are covering maps $\pi_1: Y \to Y/\langle g_1 \rangle$ and $\pi_2: Z \to Z/\langle g_2 \rangle$. Let $r_i = \pi_1(p_i)$ and $s_i = \pi_2(q_i)$, $i = 1, 2, \ldots, a$. Let $h': Y/\langle g_1 \rangle \to Z/\langle g_2 \rangle$ be a smooth homeomorphism. We may adjust h', if necessary, so that $h'(r_i) = s_i$, $i = 1, 2, \ldots, a$. Now it may be shown (see [1, pp. 57–88]) that h' lifts to a map $h: Y \to Z$. Thus $g_2 = hg_1h^{-1}$. Furthermore, $h\{p_i, p_{i+a}\} = \{q_i, q_{i+a}\}$ $i = 1, 2, \ldots, a$.

LEMMA 2.5. *If $m \geq a-1$, then there is an embedding $d: X' \to \mathbf{R}^3$ so that $dg'd^{-1} = h \circ k$, where h is a rotation about the z-axis with fixed points $q_i = d(p_i)$ and k is a reflection in the x-y plane. Also $d(X')/\langle k \rangle$ has genus $(a-2)/2$ with $m-a+3$ boundary components if a is even, and has genus $(a-1)/2$ with $m-a+2$ boundary components if a is odd.*

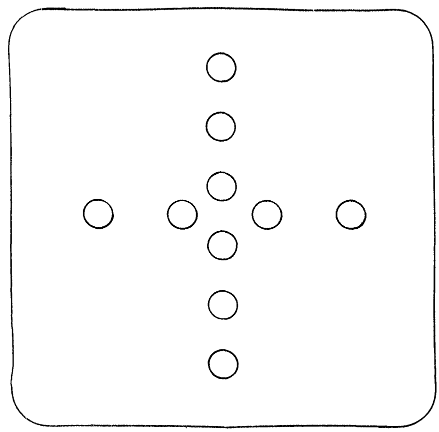
Proof. By 2.3, $m-a \equiv 1 \bmod 2$. It suffices to give an example of a surface Y embedded in $\mathbf{R}^3$ with an involution of the form $G = h \circ k$ such that $Y/\langle g \rangle$ is homeomorphic to $X'/\langle g' \rangle$, since by 2.4 there is a map $d: X \to Y$ with the desired property. To construct such a surface first let S denote the closed square in the y-z plane

$$S = \{(y, z): -1 \leq y \leq 1, -1 \leq z \leq 1\}$$

and let $S' = S - \bigcup D_t - \bigcup E_j$, $t = 1, 2, \ldots, a-1$, $j = 1, 2, \ldots, m-a+1$. The sets D_t and E_j are open discs constructed as follows. Let N be a positive integer such that

$$N \geq 2 \max\{2a-1, 2m-2a+3\}.$$

Then D_t is a disc with center $y = 0$, $z = 1-(3+4t)/(2a-1)$ and radius $1/N$ and E_j is a disc with center $z = 0$, $y = 1-(3+4j)/(2m-2a+3)$ and radius



$1/N$. It is easy to check that these discs are disjoint. If we construct a regular neighborhood of S' with smooth boundary in $\mathbf{R}^3$ which is invariant under a reflection in the x-y plane and a rotation about the z-axis, then this surface may serve as our example Y. See the figure.

3. Proofs of main theorems

We prove Theorem 1.2 by lifting loops from a specially chosen CHB on X'. Theorem 1.1 then follows from Theorem 1.2.

Proof of Theorem 1.2. We first consider the case in which $a=0$. We remark that by [5, pp. 225–226] there exists a CHB Σ on X' with the following properties. First

$$\Sigma=\{A_0, B_0, \dots, A_{2s}, B_{2s}\}$$

and $g'(A_0)=A_0$, $g'(B_0)\approx -B_0$, ($\approx$ means homologous), $g'(A_t)=A_{t+s}$, $g'(B_t)=-B_{t+s}$ (mod $2s$). Here, of course, $A_t\times B_j=\delta_{tj}$, $A_t\times A_j=B_t\times B_j=0$, $t, j=0, 1, 2, \dots, 2s$, $m=1+2s$.

We first assert that the loop A_0 lifts to one loop on X. If A_0 lifts to two loops, then these loops are interchanged by f. If they are interchanged by g, then both are fixed by f, a contradiction. If both are fixed by g, then they are both fixed by f, again a contradiction. Thus A_0 must lift to one loop.

We now claim that we may assume that A_t, $t>0$, and B_t, $t\geq 0$, may be chosen so that each of these loops lifts to two loops on X. Consider first the case $1\leq t\leq s$. Then there exist integers $m_t\geq 0$ and $n_t\geq 0$ such that loops A_t' and B_t', homologous to $A_t+m_tA_0$ and $B_t+n_tA_0$, respectively, both lift to two loops on X. the loops

$$A'_{t+s}=g(A_t')\approx A_{t+s}+m_tA_0 \quad \text{and} \quad B'_{t+s}=-g'(B_t')\approx B_{t+s}-n_tA_0$$

also lift to two loops. Finally there is an integer $n \geq 0$ such that a loop

$$B_0' \approx B_0 + nA_0 - \Sigma(m_t(A_t + A_{t+s}) + n_t(B_t - B_{t+s})), \quad t = 1, 2, \ldots, s,$$

lifts to two loops on X. If we replace A_t by A_t' and B_t by B_t' then we obtain a CHB with the desired property. It is still true that $g'(A_t) = A_{t+s}$ and $g'(B_t) = -B_{t+s}$ (mod $2s$), although it may happen that $g'(B_0) \neq B_0$.

We now construct a planar surface on which g induces a mapping. The set of lifts of A_t and B_t, $t > 0$, the lift of A_0, and any lift of B_0 forms a CHB of X, as can be seen by calculating the intersection numbers and counting the number of loops. Thus the lifts of A_t, $t \geq 0$, do not disconnect X. If we cut along these lifts we obtain a planar surface Z bounded by $4(m-1)+2$ boundary components. The map g induces a self homeomorphism of this surface, which by abuse of notation we also call g, such that the two boundary components which come from the lifts of A_0 are interchanged by this map.

Now there exists a quasi-conformal map $h: Z \to R$, where $R \subset \mathbf{C}$ is bounded by the circles $|z| = 2/3$, $|z| = 3/2$ and by $4(m-1)$ circles of radius $1/8$ and centers

$$(5/4) \exp(j\pi i/(m-1)), \quad j = 1, \ldots, 2(m-1)$$

and the reflections of these circles in $|z| = 1$. The map h may be chosen so that the circles $|z| = 2/3$ and $|z| = 3/2$ correspond to the boundary components obtained from A_0 and the circle $|z - (5/4)\exp(j\pi i/(m-1))| = 1/8$ and its reflection in $|z| = 1$ correspond to a boundary component obtained from a lift of one of the A_k, $k > 0$. Furthermore h may be chosen so that the identification on the boundary components of Z becomes inversion in $|z| = 1$.

The map hgh^{-1} extends to a map of $\hat{\mathbf{C}}$, so it is an anti-moebius transformation. Since this extension interchanges 0 and ∞, we have $hgh^{-1}(z) = b/c\bar{z}$, where $bc = -1$. Since $|hgh^{-1}(3/2)| = 2/3$, $|b/c| = 1$ and thus hgh^{-1} fixes the unit circle. The map $(hgh^{-1})^2$ is a moebius transformation of order two which fixes 0 and ∞. Thus $(hgh^{-1})^2 = -z$. We must therefore have either (1) $hgh^{-1}(z) = i/\bar{z}$ or (2) $hgh^{-1}(z) = -i/\bar{z}$. If (2) holds then conjugate by $z \to \bar{z}$ to obtain (1). Identify boundary components of R under the map $z \to 1/\bar{z}$. We can thus obtain a representation of g as a product of a rotation through an angle of $\pi/2$ about the z-axis followed by a reflection in the x-y planes. This completes the proof if $a = 0$.

Now assume that $a > 0$. We first consider the case in which a is odd. We assume that $X' \subset \mathbf{R}^3$ and $g' = h \circ k$, where h and k are as in 2.5. Since a is odd there are loops $A_0, A_1, \ldots, A_{2s}$, $s = (m-a+1)/2$, which divide X' into two components, and such that $g'(A_0) = A_0$, $g'(A_t) = A_{t+s}$ (mod $2s$), $t > 0$.

Our first assertion is that A_0 lifts to one loop on X. The proof of this is similar to the proof that A_0 lifts to one loop in the case $a = 0$. Also, if A_t, $t = 1, 2, \ldots, s$, lifts to one loop then it may be replaced by $A_t' \approx A_t + A_0$, which lifts to two loops. We then replace A_{t+s} by $A'_{t+s}=g'(A'_t)\approx A_{t+s}+A_0$, which also lifts to two loops. The loops $A_0, A_1, \ldots, A'_t, \ldots, A_{t+s}, \ldots, A_{2s}$ divide X'. Thus we may assume that each A_t lifts to two loops.

Now let A' be an annular region about A_0 which does not contain any branch points and with the property that $g'(A')=A'$. This lifts to an annular region A containing the lift of A_0 and with the property that $g(A)=A$. We claim that by a proof similar to that used in the case $a=0$, A may be embedded onto an annular region $\{z: r\le|z|\le 1/r\}$ by a map $l: A\to\mathbf{C}$ so that $lgl^{-1}(z)=i/\bar{z}$.

We now embed annular regions about lifts of the loops A_t into $\mathbf{R}^3$. First denote a point in $\mathbf{R}^3$ by cylindrical coordinates (d, r, θ). Embed the annular region A onto the region $\alpha=\{d, 1, \theta): -1\le d\le 1, 0\le\theta\le 2\pi\}$ by a map e_0, such that

$$e_0 g e_0^{-1}(d, 1, \theta)=(-d, 1, \theta+\pi/2).$$

Let A_{tj}, $t=1, 2, \ldots, 2s$, $j=1, 2$, denote the lifts of A_t and let β_{tj} denote a closed annular region with smooth boundary which contains A_{tj} in its interior, and which contains no fixed points of f. Assume that these regions are chosen and numbered so that

$$g(\beta_{tj})=\beta_{t+s\,j} \quad\text{and}\quad g(\beta_{t+s\,j})=\beta_{t\,j+1}, \quad t=1, 2, \ldots, s.$$

Now let $\alpha_{kj}=\{(d, r, \theta): -1\le d\le 1, |r\exp(i\theta)-(j+1)\exp(\pi ik/2)|=1/4\}$, $k=1, 2, \ldots, 2s$, $j=1, 2,$. Clearly α_{kj} is a circular cylinder. It is easy to construct a map

$$e_1: \bigcup\beta_{tj}\to\bigcup\alpha_{tj}$$

such that $e_1(\beta_{tj})=\alpha_{tj}$ and $e_1 g e_1^{-1}(d, r, \theta)=(-d, r, \theta+\pi/2)$.

We now embed $X-\bigcup\beta_{tj}-A$ in $\mathbf{R}^3$. To do this we first construct a surface Z_1 in $\mathbf{R}^3$ of genus $3(a-1)/2$ with $rs+1=2(m-a+1)+1$ boundary components, which is invariant under a rotation ϕ through an angle of $\pi/2$ about the x-axis. Assume that the boundary components of Z_1 coincide with the boundary components of $(\bigcup\alpha_{tj})\cup\alpha$ which lie below the x-y plane.

Let $Y_1=\pi^{-1}(X_1)-\bigcup\beta_{tj}-A$. We will show that $Y_1\cong Z_1$. Here $\cong$ means homeomorphic. Clearly $Y_1\cong\pi^{-1}(X_1)$ and $\pi^{-1}(X_1)$ is a branched covering, with a branch points, of X_1. Now X_1 has genus $(a-1)/2$ and has $2s+1=m-a+2$ boundary components. As we have shown, $2s$ of the boundary components of X_1 each lift to two boundary components and the remaining one lifts to one boundary component. Thus from the Riemann-Hurwitz formula $\phi^{-1}(X_1)$ has genus $3(a-1)/2$. Hence X_1, and therefore also Y_1, is homeomorphic to Z_1. By [3, p. 53], or [2] there exists a map $e_2: Y_1\to Z_1$ so that $e_2 f e_2^{-1}=\phi^2$.

Now let $Y_2=\pi^{-1}(X_2)-\bigcup\beta_{tj}-A$ and define the map $e_3: Y_2\to\mathbf{R}^3$ by $e_3(x)=(d, r, \theta)$, where $e_2(g(x))=(-d, r, \theta+\pi/2)$. This map is well-defined since g induces a homeomorphism from Y_2 onto Y_1. Also the maps e_0, e_1,

e_2 and e_3 agree where their domains intersect. Thus we may define a map $e: X \to \mathbf{R}^3$ by $e \mid Y_t = e_{t+1}$, $t = 1, 2$, $e \mid \bigcup \beta_{tj} = e_1$ and $e \mid A = e_0$. It is clear that

$$ege^{-1}(d, r, \theta) = (-d, r, \theta + \pi/2),$$

so that $ege^{-1} = \phi \circ \psi$, where ψ is reflection in the x-y plane. Clearly ϕ and ψ commute, and we may set $H = e^{-1}\phi e$ and $K = e^{-1}\psi e$. It is easy to verify that $X/\langle K\rangle \cong \pi^{-1}(X_1)$, that $H^2 = f$, and that H and K commute. This finishes the proof if a is odd.

We now consider the case in which $a > 0$ is even. Let q and $g(q)$ be fixed points of f which are contained in discs D and $g(D) = D'$, respectively. Assume that $f(D) = D$ so that, of course, $f(D') = D'$. By the argument used in 2.3 there is a map $l: \partial D \to \partial D'$ such that, if we remove the interiors of D and D' and identify the boundaries via l, then g induces a map on the resulting surface. We call the resulting surface Y, and the map which g induces G. Now G^2 has $2(a-1)$ fixed points so that by what we have shown $G = H \circ K$, where K and H satisfy the conclusion of 1.2.

Let δ be the curve obtained by identifying ∂D and $\partial D'$. We claim that H and K may be chosen so that δ is fixed pointwise by K. First let δ_0 be the projection of δ onto $Y/\langle G^2\rangle$. If G' is the map induced by G on $Y/\langle G^2\rangle$, then $G'(\delta_0) = \delta_0$. By a slight modification of the argument used in 2.4 and 2.5, it may be shown that there is a map d: $Y/\langle G^2\rangle$ onto a surface embedded in $\mathbf{R}^3$ (as in the figure), so that (1) $dG'd^{-1} = h \circ k$, where h is a rotation about the z-axis through an angle of π, and k is reflection in the x-y plane, and (2) $d(\delta_0) = A_0$. Here A_0 is the curve used in the case a is odd. It is fixed pointwise by k. Now by repeating the construction of H and K in the case in which a is odd, it is clear that K fixes δ pointwise.

To finish the proof we cut Y along δ and glue discs to each of the resulting components to recover X. Clearly H and K may be extended to X to produce new maps, which we also call H and K, so that $g = H \circ K$. It is easy to check that H and K satisfy the conclusion of the theorem.

Before beginning the proof of 1.1 we make several remarks. Necessity in 1.1 is trivial. Also it follows from the work in [3, p. 53], or [2] that the conjugacy class of a map of order two is determined by the number of fixed points. Thus the condition that g_1^2 and g_2^2 have the same number of fixed points is equivalent to the condition that g_1^2 is conjugate to g_2^2. To prove sufficiency it is not hard to show that this condition may be replaced by $g_1^2 = g_2^2$. We prove 1.1 by considering separately the cases $m \geq a-1$ and $m < a-1$. The first case follows directly from 1.2 while the second case requires a more complicated argument.

Proof of Theorem 1.1. We first consider the case $m \geq a-1$. We assume $g_1^2 = g_2^2$. By 1.2, $g_i = H_i \circ K_i$, $i = 1, 2$, where H_i and K_i commute, $H_i^2 = f$, $\alpha(H_i) = \pi/2$ and K_i is orientation reversing of order two with the properties that $X/\langle K_i\rangle$ is orientable and $X/\langle K_1\rangle \cong X/\langle K_2\rangle$. The maps H_i

induce self-homeomorphisms H_i' on $X/\langle K_i\rangle$. In fact $X/\langle K_i\rangle$ may be embedded in $\mathbf{R}^3$ so that H_i' becomes a rotation about the z-axis through an angle of $\pi/2$. The map H_i' fixes either two, zero, or one boundary components depending on whether $a=0$, a is even, or a is odd, respectively, and permutes the remaining boundary components. Thus by [3, p. 53], or [2] there is a map

$$h: X/\langle K_1\rangle \to X/\langle K_2\rangle$$

so that $hH_1' = H_2'h$. Since X may be obtained from $X/\langle K_i\rangle$ by doubling across the boundary components, this map h may be lifted to a map k: $X\to X$ so that $kK_1 = K_2k$ and $kH_1 = H_2k$. Thus $kH_1K_1k^{-1} = H_2K_2$, or $kg_1k^{-1} = g_2$.

We consider now the case $m < a-1$. We construct a CHB Σ on X' such that (1) $g'(\Sigma)=\Sigma$ (up to homology), (2) the loops of Σ do not pass through any branch points, and (3) each loop in Σ lifts to two loops on X. We first construct a CHB which satisfies (1) and (2) by drawing an appropriate set of loops on X', as it was represented in 2.5. See the figure for example. Let A be a loop in the CHB and let σ be a small loop about a branch point. If A lifts to one loop then replace A by a loop A' homologous to $A\sigma$. Then A$'$ lifts to two loops. We replace A by A' and still have a CHB. If $g'(A)\neq A$ then $g'(A)$ is another loop in this CHB which lifts to one loop. Also $g'(\sigma)$ is a small loop about another branch point. The loop $g'(A')$ is homologous to $g'(A)g(\sigma)$ and this loop lifts to two loops. We continue in this way, each time using loops about different branch points, until all loops are replaced by loops which lift to two loops. Since $2m < 2a-2$ we may do this. We thus obtain a CHB satisfying (1), (2) and (3).

If q is a fixed point of f denote by $\alpha(f, q)$ the angle f makes at q with respect to the orientation of X. Now label the branch points $p_1, p_2, \ldots, p_{2a}$ so that if q_i is a lift of p_i, we have that $\alpha(f, q_i)+\alpha(f, q_{i+1})=0$. By [4] we may do this. It is not hard to find non-intersecting Jordan curves δ_i, $i = 1, 2, \ldots, a$, such that δ_i joins p_{2i-1} and p_{2i} and does not intersect the loops in the CHB constructed in the previous paragraph. Assume that these curves are chosen so that $g'(\delta_i)=\delta_t$, $t=a-i+1$, $i=1, 2, \ldots, s$, and $2s=a-1-m$. By a slight modification of the argument used in Lemma 2 [4], it follows that the lifts of δ_i, $i=1, 2, \ldots, a$, divide X into two components which are interchanged by f.

We now construct a surface Y of genus $n+m+1-a$ from X on which g induces a map G with the property that G^2 has $2(m+1)$ fixed points. First lift δ_i to X, $i=1, 2, \ldots, s$, and $i=a-s+1, \ldots, a$. If we cut along the lifts we obtain a surface of genus $n-2s=n+m+1-a$ with $2\cdot 2s=2(a-1-m)$ boundary components. See Lemma 2 and Figure 2 in [4]. Each boundary component consists of two arcs, and the map f induces a map f', defined on these two arcs, which interchanges them and which has order two. Now identify these two arcs via f' and call the resulting surface Y. We call the arc obtained by identifying the two arcs of a boundary component δ_{ij}, $i = 1, 2, \ldots, s, a-s+1, \ldots, a$ and $j = 1, 2$. Clearly g induces a map G on Y and $F = G^2$ has order two with $2a - 4s = 2(m+1)$ fixed points. Also F interchanges δ_{i1} and δ_{i2}. From the Riemann-Hurwitz formula the surface $Y' = Y/\langle F \rangle$ has genus m. If we assume that these arcs are numbered so that $G(\delta_{ij}) = \delta_{tj}$, then we must have $G(\delta_{tj}) = \delta_{i\,j+1}$, where the second subscript is taken mod 2.

Since Y' has genus m and F has $2(m+1)$ fixed points we may apply Theorem 1.2. We showed that $G = H \circ K$, where H and K have certain properties. We now claim that H and K may be chosen so that K fixes each δ_{ij} pointwise. To see why this is so we examine the proof of 1.2. If $m+1$ is even we have a loop A_0 on Y' which is fixed by G'. This loop lifts to a loop on Y which is fixed pointwise by K. It is easy to replace A_0 by a freely homotopic loop which contains the arcs δ_i, $i = 1, 2, \ldots, s, a-s+1, \ldots, a$, and which is fixed by G'. The lift of this freely homotopic loop, and hence also δ_{ij}, will thus be fixed pointwise by K.

If $m+1$ is odd then an analogous argument can be used. We now have two loops A_1 and A_2, each of which lifts to two loops on Y which are interchanged by G'. We replace A_1 by a freely homotopic loop A_1' which contains each δ_i, $i = 1, 2, \ldots, s$, and we replace A_2 by $G'(A_1')$. The loop $G(A_1')$ contains each of the arcs δ_i, $i = a-s+1, \ldots, a$. If A_1' and $G'(A_1')$ are used in place of A_1 and A_2 in the proof of 1.2 then the lifts of these loops will be fixed pointwise by K. Hence each δ_{ij} will be fixed pointwise by K.

Now suppose that $g_i: X \to X$, $i = 1, 2$, are two orientation reversing maps such that $g_1^2 = g_2^2 = f$. We may construct surfaces Y_i on which the maps g_i induce mappings G_i, $i = 1, 2$, as was just done. The surfaces Y_1 and Y_2 are homeomorphic. Also Y_i contains a set of curves δ_{ijk}, $i = 1, 2$, $j = 1, 2$, $k = 1, 2, \ldots, s, a-s+1, \ldots, a$, $s = (a-1-m)/2$, and by cutting along these curves and reglueing one can recover X. Now, as was previously shown, $G_i = H_i \circ K_i$ where H_i and K_i satisfy the conditions of H and K in Theorem 1.2. Furthermore K_i fixes the curves δ_{ijk}. Now $Y_1/\langle K_1 \rangle \cong Y_2/\langle K_2 \rangle$ and H_i induces a mapping H_i' on $Y_i/\langle K_i \rangle$ which has $m+1$ fixed points and either one or no fixed boundary components, depending on whether $m+1$ is odd or even.

We now construct a map $h: X \to X$ such that $hg_1h^{-1} = g_2$. First observe that by 1.2 we may embed Y_i in $\mathbf{R}^3$ so that H_i becomes a rotation about the z-axis and K_i becomes reflection in the x-y plane. Thus we may identify $Y_i/\langle K_i \rangle$ with that part of Y_i which lies beneath and in the x-y plane. The maps H_i' are induced by rotations about the z-axis through an angle of $\pi/2$. Thus by [3, p. 53] or [2] there is a map $e: Y_1/\langle K_1 \rangle \to Y_2/\langle K_2 \rangle$ so that $eH_1' = H_2'e$.

Let λ_{ijk} be the image of δ_{ijk} in $Y_i/\langle K_i \rangle$. We now show that we may find a map

$$e': Y_1/\langle K_1 \rangle \to Y_2/\langle K_2 \rangle$$

such that $e'H_1' = H_2'e'$ and such that $e'(\lambda_{1jk}) = \lambda_{2rt}$ for some r and t, where $r = 1$ or 2 and $1 \leq t \leq s$ or $a - s + 1 \leq t \leq a$. To construct this map we first observe that e induces a map $l: Z_1 \to Z_2$, where $Z_i = (Y_i/\langle K_i \rangle)/\langle H_i' \rangle$. Let δ_{ik} denote the projection onto Z_i of λ_{ijk}. Then all of the curves δ_{ik} lie on one boundary component of Z_i, so that we may continuously deform l to a map $l': Z_1 \to Z_2$ with the property that $l'(\delta_{1k}) = \delta_{2t}$ for some t. Then the map l' lifts to a map $e': Y_1/\langle K_1 \rangle \to Y_2/\langle K_2 \rangle$ and $e'(\lambda_{1jk}) = \lambda_{2rt}$ and $e'H_1' = H_2'e'$.

We now use e' to construct a map h such that $hg_1h^{-1} = g_2$. First, Y_i may be recovered from $Y_i/\langle K_i \rangle$ by doubling across the boundary components. We may thus lift e' to a map $\phi: Y_1 \to Y_2$ such that $\phi H_1 = H_2 \phi$ and $\phi K_1 = K_2 \phi$. Therefore $\phi H_1 K_1 \phi^{-1} = H_2 K_2$ or $\phi G_1 \phi^{-1} = G_2$. Since $\phi(\delta_{1jk}) = \delta_{2rt}$, we may cut along the curves δ_{1jk} and δ_{2jk} and ϕ induces a map of the resulting surfaces. If we reglue to recover X, then it is easy to check that we obtain a map $h: X \to X$. Also $hg_1h^{-1} = g_2$. The proof of 1.1 is now complete.

References

1. N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein Surfaces*, Lecture Notes in Math., vol. 219, Springer-Verlag, New York, 1971.
2. J. Gilman, *On conjugacy classes in the Teichmüller modular group*, Michigan Math. J., vol. 23 (1976), pp. 53–64.
3. J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*, Danske Vid. Selsk. Mat.-Fys. Medd., no. 1 (1937), pp. 1–77.
4. R. Rüedy, *Symmetric embeddings of Riemann surfaces, discontinuous groups and Riemann surfaces*, Ann. of Math. Studies 79, Princeton University Press, 1974.
5. R. Zarrow, *A canonical form for symmetric and skew-symmetric extended symplectic modular matrices with applications to Riemann surface theory*, Trans. Amer. Math. Soc., vol. 204 (1975), pp. 207–227.
6. ———, *Orientation reversing maps of surfaces*, Illinois J. Math., vol. 23 (1979), pp. 82–92 (this issue).

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