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CHARACTERIZATIONS OF VARIOUS DOMAINS OF HOLOMORPHY VIA $\bar{\partial}$ ESTIMATES AND APPLICATIONS TO A PROBLEM OF KOHN

BY

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Abstract

It is shown that the only pseudoconvex sets with smooth boundary in $\mathbf{C}^n$ on which $\bar{\partial}$ satisfies Lipschitz smoothing estimates of order 1/2 are the strongly pseudoconvex ones. Various extensions of this result are made to weakly pseudoconvex domains of finite type and in various norms. It is proved that subelliptic estimates for $\bar{\partial}$ can hold on a pseudoconvex set in $\mathbf{C}^n$ only if the domain is of finite type in the sense of Kohn.

0. Introduction

The purpose of this work is to prove some characterizations of certain types of pseudo-convex domains in terms of estimates for the inhomogeneous Cauchy–Riemann equations on these domains. For instance, we prove that the only pseudo-convex sets in $\mathbf{C}^n$ with C^3 boundary on which the $\bar{\partial}$ operator satisfies the Henkin–Romanov Lipschitz 1/2 estimate [11] are the strongly pseudo-convex domains.

Some of our results, in the context of domains of finite type in $\mathbf{C}^2$ with C^∞ boundary, have been anticipated (in the Sobolev norm) by Greiner [6]. Greiner's results are highly non-elementary, requiring reduction of the problem to the study of an algebra of pseudo-differential operators on the boundary. We, on the other hand, show that the critical feature distinguishing a weakly pseudo-convex point from a strongly pseudo-convex point is the presence of a (possibly low dimensional) complex analytic variety with a high order of contact at the point. Our proofs are fairly elementary, and are inspired by the example of Stein appearing in Kerzman [13].

One consequence of our work is that we are able to give the following partial answer to the question, raised by Kohn,[2] of giving necessary and sufficient conditions for subelliptic estimates for the $\bar{\partial}$ problem:

Let $\mathcal{D} \subseteq \mathbf{C}^n$ be an open set with smooth boundary. Then $\bar{\partial}$ satisfies subelliptic estimates on $\mathcal{D}$ only if $\mathcal{D}$ is of finite type.

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[2] It has recently been announced (see Kohn [16]) that Greiner's methods work in the Sobolev norm on domains in $\mathbf{C}^n$ with C^∞ boundary. Kohn gives some deep sufficient conditions for subellipticity as well.

In what follows, our ambient space $\mathbf{C}^n$ will satisfy $n \geq 2$. For general function theory on $\mathbf{C}^n$ and standard notation, the reader is referred to [12]. We write f_{z_j} for $\partial f/\partial z_j$, $f_{z_j\bar{z}_k}$ for $\partial^2 f/\partial z_j\, \partial \bar{z}_k$, etc. when it is convenient. We recall that if $\mathcal{D} \subset \mathbf{C}^n$ has C^2 boundary then a defining function ρ for $\mathcal{D}$ is a C^2 function on some open $W \supseteq b\mathcal{D}$ with $\operatorname{grad} \rho \neq 0$ on W and

$$\mathcal{D} \cap W = \{z \in W : \rho(z) < 0\}.$$

If $z \in b\mathcal{D}$ and $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$ satisfies $\sum \rho_{z_j} \cdot w_j = 0$ we then say that $w \in T_{1,0}(b\mathcal{D})|_z$. The complex structure on $\mathbf{C}^n$ induces an identification of $T_{1,0}(b\mathcal{D})|_z$ with the maximal complex subspace of $T(b\mathcal{D})|_z$. We let $T_{1,0}(b\mathcal{D})$ be the vector bundle with total space $\bigcup_{z \in b\mathcal{D}} T_{1,0}(b\mathcal{D})|_z$. If T_1, T_2 are real tangent vector fields on $b\mathcal{D}$, we let $[T_1, T_2] = T_1T_2 - T_2T_1$. Of course if S, T are smooth sections of $T_{1,0}$ (holomorphic vector fields) then $[S, T]$ is as well. The same holds for $S, T \in T_{0,1} = \bar{T}_{1,0}$.

To fix notation, we recall that the Levi form at $z \in b\mathcal{D}$ is the map (linear in the first variable, conjugate linear in the second)

$$L_z : T_{1,0}(b\mathcal{D})|_z \times T_{1,0}(b\mathcal{D})|_z \to \mathbf{R}$$

given by $L_z(w, w) = \sum_{j,k=1}^n \rho_{z_j\bar{z}_k}(z) w_j \bar{w}_k$. If the Levi form is positive semi-definite at every $z \in b\mathcal{D}$, we say that $\mathcal{D}$ is pseudo-convex. In case L is positive definite at every $z \in b\mathcal{D}$ we say that $\mathcal{D}$ is strongly pseudo-convex. Pseudo-convexity and strong pseudo-convexity are independent of the choice of defining function. Moreover, these notions make sense in a neighborhood of any point $P \in b\mathcal{D}$ near which $b\mathcal{D}$ is C^2.

Our purpose in the second section is to give an intrinsic indicator, which can be formulated in terms of a weighted exactness criterion for the Dolbeault complex, with which one can detect strongly and weakly pseudo-convex domains. The indicator is invariant under biholomorphic maps which are C^1 to the boundary, yet in principle one need not make direct reference to the boundary in order to apply it.

In Section 3, we develop some refinements and consequences of the main result.

Section 4 contains results in the Sobolev norm and in the case of $\mathbf{C}^2$ we obtain some refinements of Greiner's results. Finally, some remarks about biholomorphic maps are made.

Section 5 addresses itself to analogues of the above results in the context of tangential estimates.

Section 6 contains remarks about L^p smoothing and Orlicz spaces.

It is a pleasure to thank Robert E. Greene for his insightful remarks and suggestions.

1. Function spaces

If $\mathcal{D} \subset \mathbf{C}^n$ is open we let $L^p(\mathcal{D})$ be the Lebesgue space of the pth power integrable functions and $L^\infty(\mathcal{D})$ the essentially bounded functions. For $0<$

$\alpha < 1$ we let

$$\Lambda_\alpha(\mathcal{D}) = \left\{ f \colon \|f\|_{L^\infty(\mathcal{D})} + \sup_{z, z+h \in \mathcal{D}} |f(z+h) - f(z)|/|h|^\alpha \equiv \|f\|_{\Lambda_\alpha(\mathcal{D})} < \infty \right\}.$$

It will be useful to have the auxiliary space

$$\hat{\Lambda}_\alpha(\mathcal{D}) = \left\{ f \colon \sup_{z, z+h \in \mathcal{D}} |f(z+h) - f(z)|/|h|^\alpha \equiv \|f\|_{\hat{\Lambda}_\alpha(\mathcal{D})} < \infty \right\}.$$

Observe that $\| \quad \|_{\Lambda_\alpha}$ is a norm while $\| \quad \|_{\hat{\Lambda}_\alpha}$ is not. We let

$$\Lambda_1(\mathcal{D}) = \left\{ f \colon \|f\|_{L^\infty} + \sup_{z, z+h, z-h \in \mathcal{D}} |f(z+h) + f(z-h) - 2f(z)|/|h| \equiv \|f\|_{\Lambda_1(\mathcal{D})} < \infty \right\}$$

and whenever $\alpha > 1$ we define

$$\Lambda_\alpha(\mathcal{D}) = \left\{ f \colon \|f\|_{L^\infty} + \sum_{j=1}^{n} \|f_{z_j}\|_{\Lambda_{\alpha-1}(\mathcal{D})} + \sum_{j=1}^{n} \|f_{\bar{z}_j}\|_{\Lambda_{\alpha-1}(\mathcal{D})} \equiv \|f\|_{\Lambda_\alpha(\mathcal{D})} < \infty \right\}.$$

For these α, $\hat{\Lambda}_\alpha$ is defined analogously. Finally, let

$$E^k(\mathcal{D}) = C^\infty(\mathcal{D}) \cap \{f \colon \nabla^j f \in L^\infty, 0 \le j \le k\} \quad \text{and} \quad E^\infty(\mathcal{D}) = \textstyle\bigcap_k E^k(\mathcal{D}).$$

It is important to observe that if $\mathcal{D}$ is bounded and has C^m boundary then $E^k(\mathcal{D}) \subseteq C^{k-1}(\bar{\mathcal{D}})$, $k \le m$.

The spaces $\Lambda_\alpha^{(p,q)}$, $E^k_{(p,q)}$, $L^p_{(p,q)}$ are forms on $\mathcal{D}$ with coefficients of the indicated type. Let $C^\infty_{(p,q)}(\mathcal{D})$, $C^\infty_{(p,q)}(\bar{\mathcal{D}})$ be as usual and let

$$A_{(p,q)}(\mathcal{D}) = E^\infty_{(p,q)}(\mathcal{D}) \cap \{\bar{\partial} \text{ closed forms}\}$$

DEFINITION 1.1. Suppose that B^1, B^2 are function spaces which contain $E^\infty(\mathcal{D})$. We say that the $\bar{\partial}$ operator satisfies estimates of type (B^1, B^2) on $\mathcal{D}$ if there is a linear operator $T \colon A_{(0,1)}(\mathcal{D}) \to C^\infty(\mathcal{D})$ so that $\bar{\partial} Tf = f$ for all f and $\|Tf\|_{B^2} \le C\|f\|_{B^1}$. If $P \in b\mathcal{D}$, we say that $\bar{\partial}$ satisfies *weak local estimates* of type (B^1, B^2) at P provided the following holds:

For any $\bar{\partial}$ closed $(0, 1)$ form f with coefficients $B^1(\mathcal{D})$ there is an open $V \ni P$ and a function u on $V \cap \mathcal{D}$ so that $\bar{\partial} u = f$ and $\|u\|_{B^2(V \cap \mathcal{D})} < \infty$. (Here we interpret $\bar{\partial}$ in the distribution sense.)

It follows from the interior ellipticity of $\bar{\partial}$ on functions that $\bar{\partial}$ is hypoelliptic on functions so the above definitions make sense. In what follows, we adhere to the custom of letting C_j, K_i, etc. denote various constants which are different in different contexts. They will be independent of the relevant parameters (made clear in the context) but will be of no intrinsic interest.

2. Principal ideas

In what follows, $\mathcal{D} \subseteq \mathbf{C}^n$ will be an open set, $P \in b\mathcal{D}$, and ρ will be a defining function for $\mathcal{D}$ in a neighborhood of P.

We remark once and for all that our proofs will be seen to be local. Therefore they all hold for domains in a complex manifold, or even at non-singular points of an analytic space. However, all statements in this paper are formulated and proved in $\mathbf{C}^n$, $n \geq 2$.

We will first formulate and prove a basic result, which is not optimal but which contains all the fundamental ideas in its proof. After the proof we give some variants and auxiliary results.

THEOREM 2.1. *Suppose $\mathcal{D} \subset \mathbf{C}^n$ has a C^3 boundary and is pseudo-convex. Then $\mathcal{D}$ is strongly pseudo-convex if and only if $\bar{\partial}$ satisfies $(L^\infty, \Lambda_{1/2})$ estimates on $\mathcal{D}$. If $\mathcal{D}$ is not strongly pseudo-convex at $P \in b\mathcal{D}$ then $\bar{\partial}$ does not satisfy $(L^\infty, \Lambda_{1/3+d})$ estimates on $\mathcal{D}$ for any $d > 0$.*

Proof. In case $\mathcal{D}$ has C^3 boundary and is strongly pseudo-convex, a construction of Fornaess [5] enables one to solve the Cousin II problem with differentiable parameters on $\mathcal{D}$. In particular, we obtain a C^2 function g defined on a neighborhood $\Omega \supseteq \bar{\mathcal{D}} \times \bar{\mathcal{D}}$ so that:

(2.1.1) $g(\zeta, z)$ is holomorphic in z for each fixed ζ.

(2.1.2) For each compact $K \subseteq \Omega$ there is a $\delta_K > 0$ so that

$$\operatorname{Re} g(\zeta, z) \leq \rho(z) - \rho(\zeta) - \delta_K |\zeta - z|^2 \quad \text{for all} \quad (\zeta, z) \in K \times K.$$

(2.1.3) $g(\zeta, \zeta) = 0$ for all $\zeta \in \Omega$.

Using this g, one uses techniques of Henkin and Henkin–Romanov [9], [10], [11] and of Siu [20] to obtain an explicit operator $T: A_{(0,1)}(\mathcal{D}) \to C^\infty(\mathcal{D})$ so that $\bar{\partial} Tf = f$ and $\|Tf\|_{\Lambda_{1/2}} \leq C \|f\|_{L^\infty}$.

Conversely, suppose $\mathcal{D}$ has C^3 boundary, $P \in b\mathcal{D}$, and the Levi form fails to be positive definite at P. Let ρ be the defining function for $\mathcal{D}$ near P. There is a neighborhood U of P and a local holomorphic change of coordinates $\Phi: U \to U_0$ so that if $v = \Phi(z)$ is the new complex variable and $\rho_0(v) \equiv \rho \circ \Phi^{-1}(v)$ then

$$\rho_0(v) = -2 \operatorname{Re} v_1 + \tfrac{1}{2} \sum_{j,k=1}^{n} ((\rho_0)_{v_j \bar{v}_k}(\Phi(P))) \cdot v_j \bar{v}_k + R_0(v). \tag{2.1.4}$$

By boundary smoothness, the remainder satisfies $|R_0(v)| \leq C_1 |v|^3$ for some $C_1 > 0$, v small. In addition, we have $P_0 \equiv \Phi(P) = 0$, $\mathcal{D}_0 \equiv \Phi(\mathcal{D} \cap U)$ satisfies

$$T_{1,0}(b\mathcal{D})|_{P_0} = \{v : v_1 = 0\},$$

and $(1 + 0i, 0, \ldots, 0)$ is the real inward unit normal to $b(\mathcal{D}_0)$ at P_0. We will perform the main construction on D_0, pull the example in a trivial fashion back to $\Phi^{-1}(\mathcal{D}_0)$, then extend it to all of $\mathcal{D}$.

Now define

$$\eta_\varepsilon(v) = -3v_1 - 2C_2(|v|^2 + \varepsilon)^{3/2} + \tfrac{1}{2}(\rho_0)_{v_1 \bar{v}_1}(P_0)\,|v_1|^2 + \sum_{k=2}^{n} (p_0)_{v_1 \bar{v}_k}(P_0) v_1 \bar{v}_k$$

where $C_2>2$ will be chosen momentarily. We suppose, without loss of generality, that

$$L_{P_0}((0, v_2, 0, \ldots, 0), (0, v_2, 0, \ldots, 0))=0.$$

Thus if $v \in \mathscr{D}_0$

$$\rho_0(v) = \mathrm{Re}\, \eta_\varepsilon(v)+2C_2(|v|^2+\varepsilon)^{3/2}+R_0(v)+\mathrm{Re}\, v_1+\left[\tfrac{1}{2} \sum_{j,k=2}^{n} ((\rho_0)_{v_j\bar{v}_k}(P_0))v_j\bar{v}_k\right].$$

Now the expansion in brackets is non-negative since the Levi form is positive semi-definite on span $\{v_2, \ldots, v_n\}$; if we choose C_2 sufficiently large then by the estimate on R_0 we may be sure that $0>\mathrm{Re}\, \eta_\varepsilon(v)+C_2(|v|^2+\varepsilon)^{3/2}+\mathrm{Re}\, v_1$ or

$$\text{(2.1.10)} \qquad \mathrm{Re}\, \eta_\varepsilon(v) \leq -C_2(|v|^2+\varepsilon)^{3/2}-\mathrm{Re}\, v_1 \quad \text{for} \quad v \in \mathscr{D}_0.$$

Now let $b=C_1+1+\|\rho\|_{C^2(\bar{\mathscr{D}}_0)}<\infty$, after shrinking U if necessary. Let C_2 be large enough so that (2.1.10) is true and so that $C_2 \geq 8b$. Then if $v \in \mathscr{D}_0$ and

$$|\mathrm{Re}\, v_1| \leq 2b(|v|^2+\varepsilon)^{3/2}$$

we have

$$\text{(2.1.11)} \qquad \begin{aligned} \mathrm{Re}\, \eta_\varepsilon(v) &\leq -\mathrm{Re}\, v_1-C_2(|v|^2+\varepsilon)^{3/2} \\ &\leq -6b(|v|^2+\varepsilon)^{3/2} \\ &\leq -|\mathrm{Re}\, v_1|-(|v|^2+\varepsilon)^{3/2}. \end{aligned}$$

Now if $|\mathrm{Re}\, v_1| \geq |\mathrm{Im}\, v_1|$ we have

$$\text{(2.1.12)} \qquad \mathrm{Re}\, \eta_\varepsilon(v) \leq -\tfrac{1}{2}\, |\mathrm{Re}\, v_1|-\tfrac{1}{2}|\mathrm{Im}\, v_1|-(|v|^2+\varepsilon)^{3/2}.$$

But if $|\mathrm{Re}\, v_1| \leq |\mathrm{Im}\, v_1|$ then by the definition of η_ε, for v small, we have

$$\text{(2.1.13)} \qquad |\mathrm{Im}\, \eta_\varepsilon(v)| \geq |\mathrm{Im}\, v_1| \geq \tfrac{1}{2}|v_1|.$$

In any case, (2.1.11), (2.1.12), (2.1.13) give us

$$\text{(2.1.14)} \quad |\eta_\varepsilon(v)| \geq C(|v_1|+(|v|^2+\varepsilon)^{3/2}) \text{ provided } \quad |\mathrm{Re}\, v_1| \leq 2b(|v|^2+\varepsilon)^{3/2}.$$

On the other hand, if $|\mathrm{Re}\, v_1|>2b(|v|^2+\varepsilon)^{3/2}$ and $|\mathrm{Re}\, v_1| \geq |\mathrm{Im}\, v_1|$ and if it were the case that $\mathrm{Re}\, v_1<0$ then

$$\text{(2.1.15)} \quad \begin{aligned} \rho_0(v) &= -2\, \mathrm{Re}\, v_1+\tfrac{1}{2} \sum_{j,k=1}^{n} (\rho_0)_{v_j\bar{v}_k}(P_0)v_j\bar{v}_k+O(|v|^3) \\ &\geq -2\, \mathrm{Re}\, v_1+\tfrac{1}{2}(\rho_0)_{v_1\bar{v}_1}(P_0)\, |v_1|^2+\mathrm{Re} \sum_{k=2}^{n} (\rho_0)_{v_1\bar{v}_k}(P_0)_{v_1v_k}+O(|v|^3) \end{aligned}$$

by the positive semi-definiteness of the Levi form. For v sufficiently small we thus have $\rho_0(v) \geq -\mathrm{Re}\, v_1>0$ so $v \notin \mathscr{D}_0$. So when $|\mathrm{Re}\, v_1|>2b(|v|^2+\varepsilon)^{3/2}$ and $|\mathrm{Re}\, v_1| \geq |\mathrm{Im}\, v_1|$ we must have $\mathrm{Re}\, v_1>0$ whence it follows, with (2.1.10), that

$$\text{(2.1.16)} \qquad \operatorname{Re} \eta_\varepsilon(v) \le -c(|v_1| + (|v|^2+\varepsilon)^{3/2}).$$

Finally, if $|\operatorname{Re} v_1| > 2b(|v|^2+\varepsilon)^{3/2}$ and $|\operatorname{Im} v_1| > |\operatorname{Re} v_1|$ we have, for v small,

$$\text{(2.1.17)} \qquad |\operatorname{Im} \eta_\varepsilon(v)| \ge |\operatorname{Im} v_1| \ge \tfrac{1}{2}|v_1| \ge c(|v_1| + (|v|^2+\varepsilon)^{3/2}).$$

In summary, (2.1.14)–(2.1.17) tell us that when v is small, $v \in \mathscr{D}_0$, we have

$$\text{(2.1.18)} \qquad |\eta_\varepsilon(v)| \ge C(|v_1| + (|v|^2+\varepsilon)^{3/2}) \quad \text{for some} \quad C>0.$$

After shrinking U, we may suppose that this inequality holds for all $v \in \mathscr{D}_0$.

Now we observe that lines (2.1.11)–(2.1.18) show that $\eta_\varepsilon(\mathscr{D}_0)$ omits the positive real axis. Therefore we may define $\beta_\varepsilon(v) = \bar{z}_2/\log \eta_\varepsilon(v)$, where the "principal branch" for logarithm is selected. Further, we let

$$\gamma_\varepsilon(v) \equiv \bar\partial \beta_\varepsilon(v) = \frac{d\bar{z}_2}{\log \eta_\varepsilon(v)} - \frac{\bar{z}_2\, \bar\partial \eta_\varepsilon(v)}{\eta_\varepsilon(v) \log^2 \eta_\varepsilon(v)}.$$

The first term on the right hand side is clearly in $E^\infty_{(0,1)}$ and is bounded, uniformly, in ε. By the definition of η_ε, and by (2.1.18), the same holds for the second term. Thus $\gamma_\varepsilon \in A_{(0,1)}(\mathscr{D}_0)$ and γ_ε is bounded, uniformly, in ε.

Suppose, seeking a contradiction, that there is a $d>0$ and $u_\varepsilon \in C^\infty(\mathscr{D}_0)$ with $\|u_\varepsilon\|_{\Lambda_{d+1/3}} \le C < \infty$ and $\bar\partial u_\varepsilon = \gamma_\varepsilon$. Then, of necessity, $u_\varepsilon = \beta_\varepsilon + h_\varepsilon$ for some holomorphic functions h_ε on $\mathscr{D}_0$.

We observe that for $0<\delta<C_3$, $\zeta \in \mathbf{C}$, $|\zeta| \le C_3\delta^{1/3}$, C_3 to be selected, one has

$$\rho(\delta, \zeta, 0, \ldots, 0) = -2\delta + \tfrac{1}{2}\rho_{z_1\bar{z}_1}(P_0)\,|\delta|^2 + \operatorname{Re} \sum_{j=2}^{n} \rho_{z_1\bar{z}_j}(P_0)\delta\bar\zeta + O(C_3^3\delta + \delta^3).$$

Clearly if $C_3>0$ is sufficiently small, we have

$$\text{(2.1.19)} \qquad \rho(\delta, \zeta, 0, \ldots, 0) < 0, \quad \text{i.e.} \quad (\delta, \zeta, 0, \ldots, 0) \in \mathscr{D}_0.$$

Now we define, for $0<\delta<C_3/C_2$, $\varepsilon>0$ small,

$$I_\varepsilon(\delta) = \left| \int_{|\zeta| = C_3\delta^{1/3}} u_\varepsilon(\delta, \zeta, 0, \ldots, 0) - u_\varepsilon(C_2\delta, \zeta, 0, \ldots, 0)\, d\zeta \right|.$$

By the above remarks, the integral makes sense. By our hypothesis about the uniform Lipschitz smoothness of u_ε, we have

$$\text{(2.1.20)} \qquad I_\varepsilon(\delta) \le C\delta^{d+2/3}$$

with C independent of δ, ε for all δ sufficiently small and positive. On the other hand, by the Cauchy integral theorem,

(2.1.21)

$$\begin{aligned}I_\varepsilon(\delta) &= \left| \int_{|\zeta|=C_3\delta^{1/3}} \beta_\varepsilon(\delta, \zeta, 0, \ldots, 0) - \beta_\varepsilon(C_2\delta, \zeta, 0, \ldots, 0)\, d\zeta \right| \\ &= \left| \int_{|\zeta|=C_3\delta^{1/3}} \bar{\zeta}\{1/\log[-3\delta - 2C_2(\delta^2 + C_3^2\delta^{2/3} + \varepsilon)^{3/2} + C'\delta^2 + C''\delta\bar{\zeta}] \right. \\ &\qquad \left. - 1/\log[-3C_2\delta - 2C_2(C_2^2\delta^2 + C_3^2\delta^{2/3} + \varepsilon)^{3/2} + C'C_2^2\delta^2 + C''C_2\delta\bar{\zeta})]\}\, d\zeta \right|\end{aligned}$$

where C', C'' are constants. If we let $\varepsilon = \delta^2$ then we see that for δ sufficiently small we have:

(2.1.22) The argument of the first logarithmic term has negative real part which is essentially -3δ.

(2.1.23) The argument of the second logarithmic term has negative real part which is essentially $-3C_2\delta$.

(2.1.24) The arguments of each of the logarithm terms have moduli essentially

$$(3+2C_2)\delta, \quad (3+2C_3^2)C_2\delta$$

respectively.

Using the elementary formula

$$\frac{1}{\log A} - \frac{1}{\log B} = \frac{\log(B/A)}{\log A \log B}$$

and observing that in our case, B/A is both bounded and bounded in modulus away from 1, and using (2.1.22)–(2.1.24), it is a simple matter to verify that line (2.1.21) is not less than $C_4\delta^{2/3}$. Combining this with (2.1.20) we have

$$C_4\delta^{2/3} \le I_{\delta^2}(\delta) \le C\delta^{d+2/3}.$$

Letting $\delta \to 0$, we obtain a contradiction.

What we have done is to show that $\bar{\partial}$ does not satisfy $(L^\infty, \Lambda_{d+1/3})$ estimates on $\mathscr{D}_0$. However the entire proof takes place locally. The same proof, using $b_\varepsilon \equiv \beta_\varepsilon \circ \Phi$ and $g_\varepsilon \equiv \bar{\partial} b_\varepsilon$ shows that $\bar{\partial}$ does not satisfy $(L^\infty, \Lambda_{d+1/3})$ estimates on $\mathscr{D} \cap U$. Using a Whitney extension operator (see [21]), we may extend the b_ε smoothly to all of $\mathscr{D}$ so that $\bar{\partial} b_\varepsilon$ are uniformly bounded in ε, and the same proof shows that $\bar{\partial}$ does not satisfy $(L^\infty, \Lambda_{d+1/3})$ estimates on $\mathscr{D}$, as desired. In subsequent proofs, all of the analysis will be done on $\mathscr{D}_0$ and we make no further mention of the local-global dialectic. ■

3. Further results for Lipschitz spaces

The smoothness hypotheses of $b\mathcal{D}$ can be considerably relaxed. In fact it suffices to know that the error term of the second order Taylor expansion for ρ is $O(|z|^\beta)$, some $\beta > 2$. Moreover, only smoothness near P is important. Further, using modifications introduced in [17], we may deal with higher order Lipschitz classes as well. Thus using 2.1 and 3.1 we have:

THEOREM 3.1. *Let $\mathcal{D} \subseteq \mathbf{C}^n$ be pseudoconvex with $P \in b\mathcal{D}$ and suppose that there is a neighborhood $U \ni P$ so that $b\mathcal{D} \cap U$ is Λ_β, some $\beta > 2$. If the Levi form is not positive definite at P then $\bar\partial$ does not satisfy $(L^\infty, \hat{\Lambda}_{1/2})$ estimates at P.*

More generally, if $b\mathcal{D} \cap U$ is Λ_β, $2 < \beta < 3$, then $\bar\partial$ does not satisfy $(L^\infty, \hat{\Lambda}_{1/\beta + d})$ estimates on $\mathcal{D}$ for any $d > 0$. In addition, for any $\alpha > 0$, $\bar\partial$ does not satisfy $(\Lambda_\alpha, \hat{\Lambda}_{\alpha+1/\beta+d})$ estimates for any $d > 0$. In case $b\mathcal{D} \cap U$ is C^3, $\bar\partial$ does not satisfy $(L^\infty, \Lambda_{1/3+d})$ estimates nor $(\Lambda_\alpha, \Lambda_{\alpha+1/3+d})$ estimates for any $d > 0$.

REMARK 3.2. Theorem 3.1 persists with weaker conditions on the modulus of continuity of $b\mathcal{D}$ than Λ_β continuity, $\beta > 2$. The most natural condition would be that $b\mathcal{D}$ is C^2, but we are unable to use the techniques of this paper to weaken the hypotheses that far.

Remark 3.3. If $\mathcal{D} \Subset \mathbf{C}^n$ is strongly pseudoconvex with Λ_β boundary, $\beta > 2$, it is well known that near a point $P \in b\mathcal{D}$ there is a holomorphic change of coordinates to make $b\mathcal{D}$ strongly convex near P. One may then mimic the example appearing in [13] to prove that for any $d > 0$, one cannot have estimates of type $(L^\infty, \Lambda_{1/2+d})$ on $\mathcal{D}$. In short, there are no domains on which $\bar\partial$ can satisfy better than $(L^\infty, \Lambda_{1/2})$ estimates.

The proof of 2.1 would have been simpler had we only been proving the nonexistence of weak local estimates. In this case, we would let $\eta(z) = \eta_0(z)$, so that ε doesn't appear. Then with no changes, the proof of 2.1 gives:

THEOREM 3.4. *Let $\mathcal{D} \subseteq \mathbf{C}^n$ be pseudoconvex, $P \in b\mathcal{D}$, $U \ni P$ a neighborhood so that $b\mathcal{D} \cap U$ is Λ_β, some $\beta > 2$, and suppose that the Levi form is not positive definite at P. Then $\bar\partial$ does not satisfy weak local estimates of type $(L^\infty, \Lambda_{1/2})$ on $\mathcal{D}$ at P, nor of type $(\Lambda_\alpha, \Lambda_{\alpha+1/2})$ at P for $\alpha > 0$. More precisely, if $b\mathcal{D}$ is Λ_β, $2 < \beta < 3$, then $\bar\partial$ does not satisfy weak local estimates of type $(L^\infty, \Lambda_{1/\beta+d})$ nor of type $(\Lambda_\alpha, \Lambda_{\alpha+1/\beta+d})$, any α, $d > 0$.*

A corollary of Theorem 3.4 is the surprising fact that if $b\mathcal{D}$ is C^3 near P and $\mathcal{D}$ is not strongly pseudoconvex at P, then $\bar\partial$ cannot satisfy better estimates than $(L^\infty, \Lambda_{1/3})$ near P. Put another way, we have:

THEOREM 3.5. *Suppose $\mathcal{D} \subseteq \mathbf{C}^n$ has C^3 boundary, is pseudoconvex, and for each $P \in b\mathcal{D}$ there are open neighborhoods $U_P \supseteq V_p \ni P$ and C_P, $d_P > 0$ so that* *for every* $f \in A_{(0,1)}(\mathcal{D} \cap U_P)$ *there is a* $u \in C^\infty(V_p \cap \mathcal{D})$ *with* $\bar\partial u = f$ *and*

$$\|u\|_{\Lambda_{1/3+d_P}}(V_P) \le C_P \|f\|_{L^\infty(U_P)}.$$

Then $\mathcal{D}$ *is strongly pseudoconvex at every boundary point of* $\mathcal{D}$.

The reader will observe that the key ingredient of the proof of 2.1 is the fact that there is a one-dimensional complex analytic variety with high order of contact to $b\mathcal{D}$ at P in case the Levi form is not positive definite at P. In fact the proof makes clear that the order of contact is a measure of (and an obstruction to) available Lipschitz estimates for $\bar\partial$ near P. This is so in part because the order of contact determines the diameter of the contour over which one integrates. Therefore we are able to prove:

THEOREM 3.6. *Suppose* $\mathcal{D} \subseteq \mathbf{C}^n$ *is open. Let* $P \in b\mathcal{D}$ *and* U *be a neighborhood of* P. *Suppose that* $b\mathcal{D} \cap U$ *is* Λ_β, *that* ρ *is a defining function for* $\mathcal{D}$ *on* U, *and that there is a non-singular complex analytic variety* V *of dimension* $n-1$ *in* U *with* $\rho(z) = O(|z-P|^\beta)$, *some* $\beta > 2$, *all* $z \in V$. *Then* $\bar\partial$ *does not satisfy estimates of type* $(L^\infty, \Lambda_{d+1/\beta})$ *on* $\mathcal{D}$, *any* $d > 0$, *nor of type* $(\Lambda_\alpha, \Lambda_{\alpha+1/\beta+d})$, *any* $d, \alpha > 0$.

Proof. After a change of coordinates we may suppose that

$$V = \{z \in U \colon z_1 = 0\}$$

and that $P = 0$. It follows that there is a defining function ρ near P of the form

$$\rho(z) = -2 \operatorname{Re} z_1 + \Psi(z) + R(z)$$

where $|R(z)| \le C_1 |z|^\beta$, some $C_1 > 0$, z small and $\Psi(0, z_2, \ldots, z_n) = 0$. Let us write

$$\Psi(z) = \sum_{\substack{s,t \\ 2 \le |s|+|t| < \beta}} a_{st} z^s \bar z^t$$

where we have used standard multi-index notation. Now Ψ is real and if we let

(3.6.1) $A_1(z)$ be all terms in Ψ with $s_1 \ne 0$, $t_1 \ne 0$,
(3.6.2) $A_2(z)$ be all terms in Ψ with $s_1 \ne 0$, $t_1 = 0$,

it follows that $\operatorname{Re}(A_1(z) + 2A_2(z)) = \Psi(z)$. Imitating the proof of 2.1, we let

$$\eta_\varepsilon(z) = -3z_1 + A_1(z) + 2A_2(z) - 2C_2(|z|^2 + \varepsilon)^{\beta/2},$$

C_2 large. By an argument similar to that in 2.1 we find that, after shrinking U if necessary:

(3.6.3) $\eta_\varepsilon(\mathcal{D}_0)$ omits the positive real axis.
(3.6.4) $|\eta_\varepsilon(z)| \ge C(|z_1| + |z|^\beta)$ on $\mathcal{D}_0$.

We thus define $\beta_\varepsilon(z) \equiv \bar{z}_2/\log \eta_\varepsilon(z)$, $\gamma_\varepsilon(z) \equiv \bar{\partial}\beta_\varepsilon(z)$ and observe that:

(3.6.5) β_ε is well defined by (3.6.3).

(3.6.6) $\gamma_\varepsilon \in A_{(0,1)}(\mathscr{D}_0)$ and the γ_ε are uniformly bounded by (3.6.4).

We observe from the formula for ρ that there is a $C_3 > 0$ so that whenever $0 < \delta < C_3$ and $\zeta \in \mathbf{C}^{n-1}$, $|\zeta| \leq C_3\delta^{1/\beta}$ then $(\delta, \zeta) \in \mathscr{D}_0$.

Following the proof of 2.1 closely, we seek a contradiction by supposing that there is a $d > 0$ and $u_\varepsilon \in C^\infty(\mathscr{D}_0)$ with $\bar{\partial} u_\varepsilon = \gamma_\varepsilon$ and $\|u_\varepsilon\|_{\Lambda_{d+1/\beta}} \leq C$, some $C > 0$.

We let, for $0 < \delta < C_3/C_2$,

$$I_\varepsilon(\delta) = \left| \int_{\substack{\zeta \in \mathbf{C} \\ |\zeta| = C_3\delta^{1/\beta}}} u_\varepsilon(\delta, \zeta, 0, \ldots, 0) - u_\varepsilon(C_2\delta, \zeta, 0, \ldots, 0)\, d\zeta \right|$$

and obtain a contradiction just as in 2.1. ■

COROLLARY 3.7. *Suppose $\mathscr{D}$ is an open set, $\mathbf{C}^n$, $P \in b\mathscr{D}$, and for each $0 < k \in \mathbf{N}$ there is a neighborhood $U_k \supseteq P$ with $b\mathscr{D} \cap U_k$ smooth of order C^k and a variety V_k with $P \in V_k \subseteq U_k$ so that $\rho(z) = O(|z-P|^k)$ for all $z \in V_k$ (some defining function ρ). Then ∂ does not satisfy estimates of type (L^∞, Λ_d) nor of type $(\Lambda_\alpha, \Lambda_{\alpha+d})$, any $\alpha, d > 0$.*

Remark 3.8. It is important to observe here that Range [19, Section 2] has proved a version of 3.6, 3.7 for domains of the form

$$B_m = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^n |z_j|^{2m_j} < 1 \right\}, \quad m \in \mathbf{N}^n.$$

Moreover, he has constructed a Henkin–Ramierz kernel for these domains and computed $(L^\infty, \Lambda_{\alpha(m)})$ estimates for them [19, Theorem 1.2, Proposition 1.5] which show that 3.6, 3.7 are best possible. This means that the indices $d + 1/\beta$, $\alpha + d + 1/\beta$ in the statement of 3.6 cannot be decreased and likewise for 3.7.

Finally, let $\mathscr{D} \Subset \mathbf{C}^n$ have C^∞ boundary and be pseudoconvex. Consider the semi-exact sequence

$$\Lambda^{p,0}(\bar{\mathscr{D}}) \xrightarrow{\bar{\partial}_0} \Lambda^{p,1}(\bar{\mathscr{D}}) \xrightarrow{\bar{\partial}_1} \Lambda^{p,2}(\bar{\mathscr{D}}) \tag{3.7}$$

where $\Lambda^{p,q}(\bar{\mathscr{D}})$ denotes the differential (p, q) forms on $\bar{\mathscr{D}}$ with coefficients in $C^\infty(\bar{\mathscr{D}})$. It follows from the work in [15] that in fact the sequence is exact at $\Lambda^{p,1}(\bar{\mathscr{D}})$. What Theorem 2.1 shows is that if there are $C, d > 0$ so that every $f \in \ker \bar{\partial}_1$ has a preimage u under $\bar{\partial}_0$ satisfying $\|u\|_{\Lambda_{1/3+d}} \leq C\|f\|_{L^\infty}$ then $\mathscr{D}$ must be strongly pseudo-convex. Observe that the non-zero p in (3.7) does not alter the validity of our assertion since holomorphic differentials may be carried without effect through the proof of 2.1.

4. Domains of finite type and Sobolev norms

Let $\mathcal{D} \subseteq \mathbf{C}^n$. Let $P \in b\mathcal{D}$ and $U \ni P$ be open with $b\mathcal{D} \cap U$ of class C^∞. We let $\mathcal{L}_0$ be the module of vector fields, over $C^\infty(b\mathcal{D} \cap U)$, spanned by $T_{1,0}(b\mathcal{D} \cap U)$ and $T_{0,1}(b\mathcal{D} \cap U)$. Moreover, for $1 \leq k \in \mathbf{N}$ we let $\mathcal{L}_k$ be the module spanned by $\mathcal{L}_{k-1}$ and by elements of the form $[F, G]$ with $F \in \mathcal{L}_{k-1}$, $G \in \mathcal{L}_0$. We have:

(4.1) $\mathcal{L}_k$ is closed under conjugation.
(4.2) $\mathcal{L}_{k-1} \subseteq \mathcal{L}_k$.
(4.3) $\mathcal{L} \equiv \bigcup_{k=0}^{\infty} \mathcal{L}_k$ is a Lie algebra.

Let ρ be a defining function for $\mathcal{D}$ on U. Following [14], [2], we define P to be a point of type m, $0 < m \in \mathbf{N}$ provided $\langle F, \partial\rho \rangle|_P = 0$ for all $F \in \mathcal{L}_{m-1}$ while $\langle F, \partial\rho \rangle|_P \neq 0$ for some $F \in \mathcal{L}_m$. We say that $\mathcal{D}$ is of type m if every $P \in b\mathcal{D}$ is of type *at most* m. It is well known (see [4]) that strongly pseudo-convex domains are of type 1 and that a point P on the boundary of the polydisc (but not on the distinguished boundary) is not of finite type m for any m.

Using the terminology of [2], we say that a complex analytic manifold V is tangent to $b\mathcal{D}$ at P to order s provided $\rho|_V$ vanishes to order $s+1$. Equivalently, $\rho(z) = O(|z-P|^{s+1})$ for $z \in V$. We observe that one only needs that $b\mathcal{D} \cap U$ be C^{s+1} in order to define finite type s at P or tangency to order s.

The following theorem was first proved in [14] for domains in $\mathbf{C}^2$ and was generalized to all $\mathbf{C}^n$ in [2].

THEOREM 4.4 (Kohn, Bloom, Graham). *Let $\mathcal{D} \subseteq \mathbf{C}^n$ be an open set, $P \in b\mathcal{D}$, U a neighborhood of P so that $b\mathcal{D} \cap U$ is C^{m+1}. Then P is a point of type m if and only if there is an $n-1$ dimensional complex manifold with order of contact m at P.*

The first result of this section is:

THEOREM 4.5. *Let $\mathcal{D} \subseteq \mathbf{C}^n$ be open, $P \in b\mathcal{D}$, U a neighborhood of P, and suppose $b\mathcal{D} \cap U$ is C^{m+1}. Suppose that P is a point of type m. Then $\bar{\partial}$ does not satisfy estimates of type $(L^\infty, \Lambda_{d+1/(m+1)})$ nor of type $(\Lambda_\alpha, \Lambda_{d+1/(m+1)+\alpha})$ for any $\alpha, d > 0$.*

Further, if $0 < \alpha \leq 1/2$ and $\bar{\partial}$ satisfies weak local estimates of type $(L^\infty, \hat{\Lambda}_\alpha)$ at P, and if U is a neighborhood of P with $b\mathcal{D} \cap U$ smooth of order $C^{2+[1/\alpha]}$ then P is a point of finite type $m_0 \leq [1/\alpha] - 1$.

Proof. By Theorem 4.4, if P is of type m there is an $n-1$ dimensional complex manifold V with $\rho(z) = O(|z-P|^{m+1})$ for $z \in V$. It follows from Theorem 3.6 that $\bar{\partial}$ does not satisfy estimates better than $(L^\infty, \Lambda_{1/(m+1)})$ nor better than $(\Lambda_\beta, \Lambda_{\beta+1/(m+1)})$ on $\mathcal{D}$.

On the other hand, if P is not a point of type at most $[1/\alpha]-1$, then in particular $\langle F, \partial\rho \rangle|_P = 0$ for all $F \in \mathcal{L}_{[1/\alpha]-1}$. It follows from the proof of Lemma

2.12 in [2] that there is an $(n-1)$ dimensional complex analytic manifold tangent to $b\mathcal{D}$ at P to order at least $[1/\alpha]$. By Theorem 3.6, $\bar\partial$ does not satisfy estimates of type $(L^\infty, \Lambda_{d+1/(1+[1/\alpha])})$, any $d>0$. Since $1/(1+[1/\alpha])<\alpha$ we have that $\bar\partial$ does not satisfy estimates of type $(L^\infty, \Lambda_\alpha)$ on $\mathcal{D}$. A variant of 3.6 along the lines of 3.4 yields that $\bar\partial$ does not satisfy weak local estimates of type $(L^\infty, \hat\Lambda_\alpha)$ near P. ■

In [14] Kohn proves that if P is a point of finite type m in $b\mathcal{D}$, if $\mathcal{D}$ is pseudoconvex, and if $n=2$ then m is *odd*. The fact persists in $\mathbf{C}^n$ as an examination of proofs in [14] and [2] show (or one can use the reference in the footnote, p. 528 [14]). This yields the following refinement:

COROLLARY 4.6. *Suppose $\mathcal{D}\subseteq\mathbf{C}$ is pseudo-convex with C^5 boundary and suppose that for each $P\in b\mathcal{D}$, $\bar\partial$ satisfies weak local estimates of type $(L^\infty, \Lambda_{1/4+d})$ at P, some $d(P)>0$. Then $\mathcal{D}$ is of type 1, hence is strongly pseudo-convex.*

Similarly, if $\bar\partial$ satisfies estimates of type $(L^\infty, \Lambda_{d+1/4})$, some $d>0$, then $\mathcal{D}$ is of type 1 hence is strongly pseudo-convex.

We can now introduce the Nikol'skii spaces, a variant of Sobolev spaces, which will enable us to obtain results in the Sobolev norm and, in addition, to obtain in $\mathbf{C}^2$ a new proof of Greiner's Theorem [6] on sharp subelliptic estimates for domains of finite type. We will also be able to obtain some refinements and generalizations thereof, and to study a problem of Kohn.

For $0<\alpha<1$, we define the L^p Nikol'skii spaces $N_p^\alpha(\mathbf{C}^n)$ of order α, $1\le p\le\infty$, to be functions $f\colon \mathbf{C}^n\to\mathbf{C}$ for which

$$\|f\|_{L^p(\mathbf{C}^n)}+\sup_{h\neq 0}\,[\|f(z+h)-f(z)\|_{L^p(\mathbf{C}^n)}/|h|^\alpha]\equiv\|f\|_{N_p^\alpha(\mathbf{C}^n)}<\infty.$$

Here the L^p norm of $f(z+h)-f(z)$ is computed in the z variable for each fixed h. Similarly, one defines N_p^α for $\alpha\ge 1$ like the Lipschitz spaces of order α with the modification that the L^∞ norm is replaced by the L^p norm. One defines, for $\mathcal{D}\subset\mathbf{C}^n$ open with C^1 boundary, $N_p^\alpha(\mathcal{D})$ to be those functions on $\mathcal{D}$ which are *restrictions* to $\mathcal{D}$ of functions in $N_p^\alpha(\mathbf{C}^n)$.

THEOREM 4.7. *Suppose $\mathcal{D}\subseteq\mathbf{C}^n$ is open, $P\in b\mathcal{D}$, U is a neighborhood of P, $b\mathcal{D}\cap U$ is C^{m+1}, and P is a point of type m. Then $\bar\partial$ does not satisfy estimates of type $(L^p, N_p^{1/(m+1)+d})$, nor of type $(N_p^\alpha, N_p^{\alpha+1/(m+1)+d})$ on $\mathcal{D}$ for any $\alpha, d>0$, $1\le p\le\infty$.*

Proof. The proof follows already familiar lines that have been laid down in Section 2. For the sake of brevity, we consider only the case $m=3$, $p=2$. Let $P\in b\mathcal{D}$ be a point of type m.

By Theorem 4.4 there is an $(n-1)$ dimensional analytic manifold V which is tangent to order m at P. It follows, just as in the proof of Theorem 3.6, that we may construct a function $\eta(z)$ on $\mathcal{D}_0 \equiv U \cap \mathcal{D}$ so that:

(4.7.1) $\eta(\mathcal{D}_0)$ omits the positive real axis.
(4.7.2) $|\eta(z)| \geq c(|z_1| + |z|^4)$ for $z \in \mathcal{D}_0$.

Here the coordinates have been normalized, as in 2.1, so that $V = \{z_1 = 0\}$, $(1, 0, \ldots, 0)$ is the real inward unit normal to $b\mathcal{D}$ at P, and $P = 0$.

We define, for $\varepsilon > 0$ small,

$$\beta_\varepsilon(z) = \bar{z}_2/[(\eta(z) - \varepsilon)^{(n+3)/4} \log(\eta(z) - \varepsilon)].$$

This is well defined on $\mathcal{D}_0$ by (4.7.1). Further, let

$$\begin{aligned}\gamma_\varepsilon(z) &\equiv \bar{\partial}\beta_\varepsilon(z)\\ &= d\bar{z}_2/[(\eta(z) - \varepsilon)^{(n+3)/4} \log(\eta(z) - \varepsilon)]\\ &\quad -\frac{(n+3)}{4}\bar{z}_2\bar{\partial}\eta(z)/[(\eta(z) - \varepsilon)^{(n+7)/4} \log(\eta(z) - \varepsilon)]\\ &\quad -\bar{z}_2\bar{\partial}\eta(z)/[(\eta(z) - \varepsilon)^{(n+7)/4} \log^2(\eta(z) - \varepsilon)].\end{aligned}$$

Since $\bar{\partial}\eta(z) = O(|z_1| + |z|^3)$ while $\eta(z) \geq c(|z_1| + |z|^4)$, an easy calculation yields

$$|\gamma_\varepsilon(z)| \leq C(|z_1| + |z|^4)^{-(n+3)/4}/|\log(|z_1| + |z|^4)| \equiv A(z).$$

Let us write $z' = (z_2, \ldots, z_n)$. Thus

$$\int_{\{|z| \leq 1/2\}} |\gamma_\varepsilon(z)|^2 \leq \int_{\substack{|z_1| \leq |z'|^4\\ |z| \leq 1/2}} A^2(z) + \int_{\substack{|z'|^4 \leq |z_1|\\ |z| \leq 1/2}} A^2(z) \equiv A_1(z) + A_2(z).$$

Now

$$\begin{aligned}A_1(z) &\leq c\int_{\substack{|z_1|, |z'| \leq 1/2\\ |z_1| \leq |z'|^4}} 1/[|z'|^{2n+6} \log^2 |z'|]\\ &\leq c\int_{|z'| \leq 1/2} 1/[|z'|^{2n-2} \log^2 |z'|]\\ &\leq C.\end{aligned}$$

The term $A_2(z)$ is handled similarly. It follows that $\gamma_\varepsilon \in L^2(\mathcal{D}_0)$, *uniformly in* ε, and of course we have $\gamma_\varepsilon \in A_{(0,1)}(\mathcal{D}_0)$.

Let us also observe, as in 3.6, that for $C_3 > 0$ sufficiently small it follows from the existence of V that

(4.7.3) $(\delta, \zeta, 0, \ldots, 0) \in \mathcal{D}_0$ provided $0 < \delta < C_3$, $\zeta \in \mathbf{C}$, $|\zeta| \leq c_3\delta^{1/4}$.

We suppose, seeking a contradiction, that there are function $u_\varepsilon \in C^\infty(\mathcal{D}_0)$ with $\bar{\partial}u_\varepsilon = \gamma_\varepsilon$ and $\|u_\varepsilon\|_{N_2^{1/4+d}(\mathcal{D}_0)} \leq C$, some $d > 0$. Then, of necessity, $u_\varepsilon = \beta_\varepsilon + h_\varepsilon$, some holomorphic h_ε on $\mathcal{D}_0$. We define, for $C_4 = C_3/16(C_2 + 1)n$,

$\varepsilon > 0$ small, $0 < \delta < C_3$,

$$I_\varepsilon(\delta) = \left| \underset{\substack{\delta \le \operatorname{Re} z_1 \le 2\delta \\ |\operatorname{Im} z_1| \le \delta \\ |z_j| \le C_4\delta^{1/4}, j \ge 2}}{\int\int \cdots \int} \left| \int_{|\zeta| = C_4\delta^{1/4}/2} u_\varepsilon(z_1 + C_2\delta, z_2 + \zeta, z_3, \ldots, z_n) - u_\varepsilon(z_1 + \delta, z_2 + \zeta, z_3, \ldots, z_n)\, d\zeta \right|^2 d\bar{z} \wedge dz \right|^{1/2}$$

Notice that since $(1, 0, \ldots, 0)$ is the real inward unit normal at P, the same argument which gives (4.7.3) also gives

$$(z_1 + C_2\delta, z_2 + \zeta, z_3, \ldots, z_n), (z_1 + \delta, z_2 + \zeta, z_3, \ldots, z_n) \in \mathcal{D}_0$$

when z, ζ range over their respective domains of integration. By our hypothesis about the Nikol'skii class of the u_ε and by Minkowski's integral inequality (see [21]) we have

$$I_\varepsilon(\delta) \le C\delta^{1/2+d}, \quad \text{uniformly in } \varepsilon. \tag{4.7.4}$$

On the other hand, the h_ε integrate out by the Cauchy integral theorem and we have

$$\begin{aligned} I_\varepsilon(\delta) = \Bigg| \int\int \cdots \int \Bigg| \int_{|\zeta| = C_4\delta^{1/4}} & (\bar{\zeta} + \bar{z}_2) \\ & \times \{1/[(\eta(z_1 + C_2\delta, z_2 + \zeta z_3, \ldots, z_n) - \varepsilon)^{(n+3)/4} \\ & \times \log(\eta(z_1 + C_2\delta, z_2 + \zeta, z_3, \ldots, z_n) - \varepsilon)] \\ & - 1/[\eta(z_1 + \delta, z_2 + \zeta, z_3, \ldots, z_n) - \varepsilon)^{(n+3)/4} \\ & \times \log(\eta(z_1 + \delta, z_2 + \zeta, z_3, \ldots, z_n) - \varepsilon)]\}\, d\zeta \Bigg|^2 d\bar{z}\, dz \Bigg|^{1/2}. \end{aligned}$$

A computation similar to that in the proof of 2.1, letting $\varepsilon = \delta/10$, and using (4.7.1), (4.7.2), and the existence of V yields

$$I_{\delta/10}(\delta) \ge C \left| \int\int (\delta^{1/2}/(\delta^{(n+3)/4} |\log^2 \delta|)^2\, d\bar{z}\, dz \right|^{1/2}. \tag{4.7.5}$$

If we observe that the region of integration in the z variable has volume $(\delta^{1/4})^{2n-2}\delta^2$ we obtain

$$I_{\delta/10}(\delta) \ge C\sqrt{\delta}/|\log^2 \delta|. \tag{4.7.6}$$

Now combining (4.7.4) and (4.7.6) and letting $\delta \to 0$ yields a contradiction.
The counterexamples for $(N_2^\alpha, N_2^{\alpha+1/m+1+d})$ are constructed similarly. ■

Let now H_p^α denote the classical L^p Sobolev spaces on $\mathcal{D}_0$ (see [1] or [4]). One has for any $\varepsilon > 0$, $\alpha > 0$, $1 \le p \le \infty$,

$$N_p^{\alpha+\varepsilon}(\mathcal{D}_0) \subseteq H_p^\alpha(\mathcal{D}_0) \subseteq N_p^\alpha(\mathcal{D}_0) \tag{4.8}$$

provided $\bar{\mathscr{D}}_0$ is compact and $b\mathscr{D}_0$ is C^1. If one combines these inclusions with Theorem 4.7 one has:

THEOREM 4.9. *Suppose $\mathscr{D} \subseteq \mathbf{C}^n$ has C^∞ boundary. Then $\bar\partial$ satisfies subelliptic estimates on $\mathscr{D}$ only if $\mathscr{D}$ is of finite type. If $\mathscr{D}$ is of finite type m then $\bar\partial$ cannot satisfy subelliptic estimates of order better than $1/(m+1)$.*

This partially answers a question of Kohn mentioned in [3]. The result is due to Greiner in case $\mathscr{D} \subseteq \mathbf{C}^2$. It is clear from our methods that one can considerably relax the smoothness conditions on the boundary. Using some positive results which are due to Kohn [14], we can state a more complete result in $\mathbf{C}^2$:

THEOREM 4.10. *Suppose $\mathscr{D} \Subset \mathbf{C}^2$ has C^∞ boundary. If $\mathscr{D}$ is pseudo-convex of type m then the subelliptic estimate*

$$\|u\|_{H_2^{\alpha+1/(m+1)-\varepsilon}} \leq C_\varepsilon(\|\bar\partial u\|_{H_2^\alpha} + \|u\|_{L^2}) \tag{4.10.1}$$

holds on $\mathscr{D}$ for all $\varepsilon > 0$.

Conversely, if for some $1 \leq m \in \mathbf{N}$, $\mathscr{D}$ has C^{m+1} boundary and (4.10.1) *holds on $\mathscr{D}$ (with L^2 Sobolev spaces replaced by L^p Sobolev spaces for any $1 \leq p \leq \infty$ if desired) then $\mathscr{D}$ is a domain of finite type at most m.*

REMARK 4.10.1. In fact the second statement of the theorem holds locally. That is, if (4.10.1) holds near a point P then P must be of finite type at most m.

We conclude this section with some remarks about a connection between our results and nonexistence of biholomorphic maps. Let $1 \leq m_1 < m_2$, $m_i \in \mathbf{N}$, and suppose $\mathscr{D}_1, \mathscr{D}_2 \subseteq \mathbf{C}^2$ are domains with smooth boundary of types m_1, m_2 respectively. If $\mathscr{D}_1$ and $\mathscr{D}_2$ were biholomorphically equivalent via a map φ, then one would pull $\bar\partial$ closed $(0, 1)$ forms from $\mathscr{D}_2$ back to $\mathscr{D}_1$ via φ and similarly with functions, and vice-versa. If φ, φ^{-1} were C^1 to the boundary, it would follow that the Jacobians of φ, φ^{-1} were bounded.

Therefore to solve $\bar\partial u = f$ on $\mathscr{D}_2$ with $f \in A_{(0,1)}(\mathscr{D}_2)$, one could pull f back to $\mathscr{D}_1$, solve there where $\bar\partial$ satisfies $(L^2, H_{1/(m_1+1)-\varepsilon})$ estimates, and pull the solution back to $\mathscr{D}_2$. This would contradict Theorem 4.10. Hence φ and φ^{-1} cannot be C^1 to the boundary. A more careful analysis shows that there is a constant $\omega = \omega(m_1, m_2)$ with $0 < \omega < 1$ so that φ, φ^{-1} cannot be Λ_ω to the boundary. This should be interpreted as strong evidence that type is a biholomorphic invariant.

5. Tangential estimates

In [22], certain non-isotropic Lipschitz spaces were introduced which are an appropriate context for regularity for the $\bar\partial$ operator. In several other sources, these spaces have been exploited, and optimal regularity on strongly pseudo-convex domains has been computed (the best reference is [7]).

We recall the definitions of these spaces. Let $\mathscr{D} \subseteq \mathbf{C}^n$ have C^2 boundary. For each $z \in b\mathscr{D}$, let ν_2 denote the outward unit normal. Let $N(b\mathscr{D})|_z = \{\mathbf{C}\nu_z\}$ be the complex linear subspace of $\mathbf{C}^n$ generated by ν_z, each $z \in b\mathscr{D}$. If $T(b\mathscr{D})|_z$ denotes the $(2n-1)$ dimensional real tangent space to $b\mathscr{D}$ at z, write $T(b\mathscr{D})|_z = \mathscr{T}(b\mathscr{D})|_z \oplus (N(b\mathscr{D})|_z \cap T(b\mathscr{D})|_z)$ where these summands are orthogonal in the Hermitian metric inherited from $\mathbf{C}^n$. Then $\mathbf{C} \otimes_{\mathbf{R}} \mathscr{T}(b\mathscr{D})|_z \cong T_{1,0} \oplus T_{0,1}$ with notation as in Section 0. Let $W \supseteq b\mathscr{D}$ be an open set with the property that the orthogonal projection $\pi\colon W \to b\mathscr{D}$ is a well defined retraction. For any $z \in W$, let $\mathscr{T}(z) = \mathscr{T}(b\mathscr{D})|_{\pi(z)}$ and $N(z) \equiv N(b\mathscr{D})|_{\pi(z)}$. Define

$$\mathscr{C}^k(\mathscr{D}) = \{\gamma\colon (0,1) \to \mathscr{D} \cap W\colon |\gamma^{(1)}| \leq 1, \ldots, |\gamma^{(k)}| \leq 1\},$$

$$\mathscr{C}_1^k(\mathscr{D}) = \{\gamma \in \mathscr{C}^k\colon \gamma(t) \in \mathscr{T}(\gamma(t)) \quad \text{for all} \quad t \in (0,1)\}.$$

We shall say that, for $0 < \alpha \in \mathbf{R}$, $f \in \Upsilon_\alpha(\mathscr{D})$ provided f is continuous and

$$\lim_{\gamma \in \mathscr{C}_1^{[\alpha]+1}} \|f \circ \gamma\|_{\Lambda_\alpha(0,1)} \equiv \|f\|_{\Upsilon_\alpha(\mathscr{D})} < \infty.$$

The definition of Υ_α apparently depends upon W, but in the context of boundary regularity the choice of W is inconsequential and we merely fix W at the outset. We also define $\Gamma_{\alpha,\tau}(\mathscr{D})$, for $0 < \alpha$, $\tau \in \mathbf{R}$, to be those f for which

$$\|f\|_{\Lambda_\alpha(\mathscr{D})} + \|f\|_{\Upsilon_\tau(\mathscr{D})} \equiv \|g\|_{\Gamma_{\alpha,\tau}(\mathscr{D})} < \infty.$$

Now for almost all of the results in Sections 2, 3, and 4, there are analogues in the context of the Υ_α and $\Gamma_{\alpha,\tau}$ spaces. In the proofs of the new facts, one uses the same η_ε, β_ε, γ_ε. The only difference is that one defines

$$I_\varepsilon(\delta) = \left| \int_\zeta u_\varepsilon(\delta^\beta, \zeta + C_2\delta, 0) - u_\varepsilon(\delta^\beta, \zeta + \delta, 0)\, d\zeta \right|$$

for C, δ appropriately small, β the order of vanishing of ρ restricted to the analytic variety $V = \{z_1 = 0\}$ at P (as in Theorem 3.6}, and the integration taking place over an appropriate contour. Of course the modification takes place because one is now studying smoothness in the holomorphic tangential directions. We now state some of the results for these new Lipschitz classes.

THEOREM 5.1. *Suppose $\mathscr{D} \subseteq \mathbf{C}^n$ is pseudoconvex, $P \in b\mathscr{D}$, $U \ni P$ is a neighborhood so that $\mathscr{D} \cap U$ has Λ_β boundary, some $2 < \beta < 3$. If $\mathscr{D}$ is not strongly pseudo-convex at P then $\bar\partial$ does not satisfy $(L^\infty, \Upsilon_{2/\beta+d})$ estimates on $\mathscr{D}$, nor $(\Lambda_\alpha, \Upsilon_{\alpha+2/\beta+d})$ estimates on $\mathscr{D}$, any $d > 0$, $\alpha > 0$. Moreover, $\bar\partial$ does not satisfy weak local estimates of these types either.*

Conversely, if $\mathscr{D} \Subset \mathbf{C}^n$ has sufficiently smooth boundary and is strongly pseudo-convex, then $\bar\partial$ does satisfy estimates of type

$$(L^\infty, \Lambda_{1/2,1}) \quad \textit{and} \quad (\Lambda_\alpha, \Gamma_{\alpha+1/2,\alpha+1})$$

on $\mathscr{D}$ for all $\alpha > 0$.

Proof. The first part follows from our general technique and the remarks preceding the statement of the theorem.

The second part is due variously to Henkin–Romanov, Krantz, Alt, Phong, Greiner, and Stein, and can be found in [7]. ■

THEOREM 5.2. *Let $\mathcal{D}$ be as in Theorem 3.6. Then $\bar\partial$ does not satisfy estimates of type $(L^\infty, \Upsilon_{2/\beta+d})$ nor of type $(\Lambda_\alpha, \Upsilon_{\alpha+2/\beta+d})$ for any $\alpha, d>0$. Moreover $\bar\partial$ does not satisfy weak local estimates of these types.*

COROLLARY 5.2.1. *Let $\mathcal{D}$ be as in Corollary 3.7. Then $\bar\partial$ does not satisfy estimates of type (L^∞, Υ_d) nor of type $(\Lambda_\alpha, \Upsilon_{\alpha+d})$ for any $\alpha, d>0$. Moreover, $\bar\partial$ does not satisfy weak local estimates of these types.*

THEOREM 5.3. *Suppose $\mathcal{D}\subseteq \mathbf{C}^n$ has C^3 boundary, is pseudoconvex, and for each $P\in b\mathcal{D}$ there are open neighborhoods $U_P\supseteq V_P\ni P$ and C_P, $d_P>0$ so that for every $f\in A_{(0.1)}(\mathcal{D}\cap U_P)$ there is a $u\in C^\infty(V_P\cap\mathcal{D})$ with $\bar\partial u=f$ and*

$$\|u\|_{\Upsilon_{1/2+\alpha_P}}\le C_p\,\|f\|_{L^\infty(U_p)}.$$

Then $\mathcal{D}$ is strongly pseudo-convex at every boundary point of $\mathcal{D}$.

One may define Sobolev classes which reflect additional smoothness in the holomorphic tangential directions in a manner analgous to that for the Υ_α, $\Gamma_{\alpha,\beta}$. A great deal is known about regularity for $\bar\partial$ on a strongly pseudo-convex domain with respect to these non-isotropic Sobolev classes. The reader should consult [7] for details.

Although we will not give definitions here, we remark that our constructions work in the context of these Sobolev classes. One studies the analogous Nikol'skii classes and uses appropriate imbedding theorems. However, the imbedding theorems are difficult and we will say no more about this matter here.

6. L^p Smoothing and Orlicz norms

In [18] the following theorem is proved:

THEOREM 6.1. *Let $\mathcal{D}\Subset\mathbf{C}^n$ be strongly pseudo-convex with C^5 boundary. Let $1<p<2n+2$. Then $\bar\partial$ satisfies (L^p, L^q) estimates on $\mathcal{D}$ where*

$$\frac{1}{q}=\frac{1}{p}-\frac{1}{2n+2}.$$

The value of q given here cannot be taken to be any larger in case $\mathcal{D}$ is the unit ball in $\mathbf{C}^n$.

Now let $2n+2<p\le\infty$. Then $\bar\partial$ satisfies $(L^p, \Lambda_{(1/2)-(n+1)/p})$ estimates on $\mathcal{D}$ and the Lipschitz order cannot be made any larger when $\mathcal{D}$ is the unit ball in $\mathbf{C}^n$.

The smoothness hypotheses in Theorem 6.1 can be relaxed, but we will not concern ourselves with that here. Our purpose is rather to observe that any of the estimates cited in Theorem 6.1 characterize strongly pseudoconvex domains. One merely constructs η_ε as usual and let $\beta_\varepsilon(z)=\bar{z}_2/[\eta_\varepsilon(z)]^a\,|\log\eta_\varepsilon(z)|^b$ where a and b are chosen appropriately. Then $\gamma_\varepsilon=\bar{\partial}\beta_\varepsilon$ and an appropriate integral I_ε leads to a contradiction.

One may also prove a variety of negative results on domains of finite type, just as in Section 4, for (L^p, L^q) smoothing and (L^p, Λ_α) smoothing. Since no positive results are known in this category, it is perhaps best to leave the details to the interested reader.

We conclude by noting that a variety of Orlicz classes, such as $L^p(\operatorname{Log} L)^q$, are susceptible to examples such as those in Section 4. In all cases, the basic result is that "smoothing of order 1/2" or "tangential smoothing of order 1" characterizes strongly pseudoconvex domains. Moreover, if smoothing of some order occurs, then the domain must be of finite type.

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