ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 2, Summer 1980

A DENSITY THEOREM FOR A CLASS OF DIRICHLET SERIES

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I. Introduction and statement of results

Let $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, $a(1) \neq 0$, be a Dirichlet series that converges absolutely for Re $(s) > 1$ and that can be continued to a function analytic on Re $(s) > -1$, except for a finite number of poles in the strip $0 <$ Re $(s) \leq 1$. Let $N(\sigma, T)$ be the number of zeros, ρ, of $f(s)$ with $1 \geq$ Re $(\rho) \geq \sigma$ and $|\text{Im } (\rho)| \leq T$, where $\sigma \geq 1/2$ and $T \geq 1$. The purpose of this paper is to give estimates for $N(\sigma, T)$.

Let $g(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$ be a Dirichlet series that also converges absolutely for Re $(s) > 1$. Let $\Delta(s) = \prod_{j=1}^{N} \Gamma(\alpha_j s + \beta_j)$, where $\alpha_j > 0$ and β_j are complex, $1 \leq j \leq N$. We assume that there exist real numbers C and θ, with $C > 0$, and a complex number δ such that $f(s)$ and $g(s)$ satisfy the functional equation

$$\Delta(s)f(s) = C^{\theta s + \delta}\Delta(1 - s)g(1 - s). \tag{1.1}$$

We shall assume the following estimates on the coefficients of $f(s)$ and $g(s)$:

$$\sum_{n \leq x} |a(n)|^2 \ll x \log^{M_1} x \tag{1.2}$$

and

$$\sum_{n \leq x} |b(n)|^2 \ll x \log^{M_2} x. \tag{1.3}$$

Let $a^{*-1}(n)$ be the Dirichlet convolution inverse of $a(n)$, i.e.,

$$(a^*a^{*-1})(n) = \sum_{d|n} a(d)a^{*-1}(n/d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

This exists since $a(1) \neq 0$. We assume

$$x \log^{M_3} x \ll \sum_{n \leq x} |a^{*-1}(n)|^2 \ll x \log^{M_3} x. \tag{1.4}$$

Note that if $a(n) \geq 0$, then $|a^{*-1}(n)| \leq a(n)$ and so the upper estimate follows from (1.2) with $M_3 = M_1$. Let $W \geq 1$ and

$$c(n) = c(n, W) = \sum_{\substack{d|n, \\ d \leq W}} a(d)a^{*-1}(n/d)$$

Received May 1, 1978.

Then $c(1, W) = 1$ and $c(n, W) = 0$ for $1 < n \le W$. We assume

$$\sum_{n \le x} |c(n)|^2 \ll x \log^{M_4} x. \tag{1.5}$$

Note that (1.5) is independent of W. We cannot prove this in general, but when $a(n) \ge 0$, it is easy to see that the estimate is independent of W, since in that case $|c(n)| \le (a^*a)(n)$. In general the best we can do is an estimate involving the first powers of x and W, which is obtained by using the Cauchy-Schwarz inequality.

We shall prove the following results.

THEOREM 1. *Let $1/2 \le \sigma \le 1$ and let $k \ge 2$ be an integer. If we assume* (1.4), (1.5) *and that there exist constants $\mu(k)$ and $\nu(k)$ such that*

$$\int_{-T}^{T} |f(1/2 + it)|^k \, dt \ll T^{\mu(k)} \log^{\nu(k)} T, \tag{1.6}$$

as $T \to +\infty$, then, as $T \to +\infty$,

$$N(\sigma, T) \ll (T^{2(1-\sigma)} + T^{2(k+2\mu(k))(1-\sigma)/(k+4-4\sigma)}) \log^{M_1(k)} T,$$

where $M_1(k) = \max(M_4 + 10, 3 + (2\nu(k) + (M_3 + 5)k)/(k + 2))$.

THEOREM 2. *If we assume* (1.4), (1.5) *and* (1.6), *then for*

$$\sigma \ge (8\mu(k) + 3k - 4)/(8\mu(k) + 4k - 4)$$

we have

$$N(\sigma, T) \ll T^{(4\mu(k)+k)(1-\sigma)/(4-k+(2k-4)\sigma)} \log^{M_2(k)} T,$$

as $T \to +\infty$, where $M_2(k) = \max(M_4 + 6, \nu(k) + 3, M_3 + 6)$.

COROLLARY. *Uniformly on $1/2 \le \sigma \le 1$ we have, as $T \to +\infty$,*

$$N(\sigma, T) \ll T^{(k+2\mu(k))(8\mu(k)+4k-4)(1-\sigma)/(4k\mu(k)+2k^2)} \log^{M_3(k)} T,$$

where $M_3(k) = \max(M_1(k), M_2(k))$.

In most applications we take either $k = 2$ or $k = 4$, which is the reason for Theorem 3 below.

THEOREM 3. *Let $A = \sum_{j=1}^{N} \alpha_j$. If we assume* (1.2) *and* (1.3), *then we may take*

$$\mu(2) = \max(1, 2A - 1) \quad \textit{and} \quad \nu(2) = \max(M_1 + 1, M_2).$$

If we further assume that

$$\sum_{n \le x} |(a^*a)(n)|^2 \ll x \log^{M_5} x \quad \text{and} \quad \sum_{n \le x} |(b^*b)(n)|^2 \ll x \log^{M_6} x,$$

then we may take $\mu(4) = \max(1, 4A - 1)$ and $\nu(4) = \max(M_5 + 1, M_6)$.

In the proofs of Theorems 1 and 2 we adapt the method of Montgomery [13] and in the proof of Theorem 3 we adapt the method of Ramachandra [15].

In [16] Sokolovskii used Ingham's method of convexity theorems to give estimates for $N(\sigma, T)$ for the same class of Dirichlet series as we are concerned with here. He assumes (1.2), (1.4) and (1.5) and the essential tool for him is an estimate for $f(1/2 + it)$. We have replaced this by the estimate (1.6). He uses the functional equation (1.1) to derive his estimate for $f(1/2 + it)$, while we use the functional equation in the proof of Theorem 3 and in the proof of Theorem 1 to guarantee certain behavior of the function $f(s)$.

One could also improve Theorem 2 and its corollary by using further improvements in large value theorems for Dirichlet polynomials. See, for example, Huxley and Jutila [8] or Jutila [10]. We hope to return to this in a latter paper.

In the sequel the c_j, $j = 1, 2, \ldots$, will denote positive absolute constants. We use $\int_{(a)}$ to denote the integral $\int_{a-i\infty}^{a+i\infty}$ and $\int_{(a,T)}$ to denote the integral $\int_{a-iT}^{a+iT}$.

2. Proof of Theorem 1

We state a lemma that we need for the proof of Theorem 1. This is a version of Theorem 2 of [12].

LEMMA 1. *Let M be given and $\{a_n\}$, $1 \le n \le M$, be complex numbers. For $1 \le r \le R$, let $s_r = \sigma_r + it_r$ be arbitrary complex numbers. Let*

$$\tau = \min\{t_a - t_b : 1 \le a < b \le R\},$$
$$S = 1 + \max\{t_r : 1 \le r \le R\} - \min\{t_r : 1 \le r \le R\}$$
$$\omega = \min\{\sigma_r : 1 \le r \le R\}.$$

Then

$$\sum_{r=1}^{R} \left| \sum_{n=1}^{M} a_n n^{-s_r} \right|^2 \ll (S + M)(1 + \tau^{-1}\log^2 M)\log^4 M \sum_{n=1}^{M} \frac{|a_n|^2}{n^2}.$$

To begin the proof of Theorem 1 let

$$M(s) = M(s, W) = \sum_{n \le W} a^{*-1}(n) n^{-s}. \tag{2.1}$$

Then $f(s)M(s) = \sum_{n=1}^{\infty} c(n, W)n^{-s}$. If $r_1 > 1$, then by a standard integration formula we have, if $U > 1$,

$$\begin{aligned} e^{-1/U} + \sum_{n > W} c(n)n^{-s}e^{-n/U} &= \sum_{n=1}^{\infty} c(n)n^{-s}e^{-n/U} \\ &= \frac{1}{2\pi i}\int_{(r_1)} f(s+z)M(s+z)U^z\Gamma(z)\,dz. \end{aligned} \tag{2.2}$$

We assume $W \le U \le T^{c_1}$.

Let $s = \sigma + it$, where $1/2 < \sigma \leq 1$, and move the contour to $\mathrm{Re}\,(z) = 1/2 - \sigma$. Then we pick up the poles of the integrand, by the residue theorem, which are the poles of $f(s+z)$ in $0 < \sigma \leq 1$ and the pole at $z = 0$ of $\Gamma(z)$. Since $f(s)$ satisfies the functional equation and both $f(s)$ and $g(s)$ are absolutely convergent for $\sigma > 1$, it follows by a standard Phragmen-Lindelöf argument that, if Q is sufficiently small,

$$\begin{aligned}\sum_{n=1}^{\infty} c(n)n^{-s}e^{-n/U} &= \frac{1}{2\pi i}\int_{(1/2-\sigma)} f(s+z)M(s+z)U^z\Gamma(z)\,dz \\ &\quad + \sum_{\lambda} \frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)M(s+z)U^z\Gamma(z)\,dz \\ &= \frac{1}{2\pi i}\int_{(1/2-\sigma)} f(s+z)M(s+z)U^z\Gamma(z)\,dz + f(s)M(s) \\ &\quad + \sum_{\lambda\neq 0} \frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)M(s+z)U^z\Gamma(z)\,dz\end{aligned}$$

where the sum over λ denotes a sum over the poles of the integrand.

If $\lambda - s$ is a pole of $f(s+z)$, let $n(\lambda)$ be its order and let $a_{-j}(\lambda)$, $1 \leq j \leq n(\lambda)$, be the coefficients of the principal part of the Laurent expansion of $f(s+z)$ about $z = \lambda - s$. Then we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)M(s+z)U^z\Gamma(z)\,dz \\ &\qquad = U^{\lambda-s}\Gamma(\lambda-s)\sum_{j=1}^{n(\lambda)} a_{-j}(\lambda) \\ &\qquad\quad \times \sum_{e+f+g=j-1} \frac{(\Gamma^{(e)}/\Gamma)(\lambda-s)M^{(f)}(\lambda-s)\log^g U}{e!\,f!\,g!}.\end{aligned}$$

Suppose $1/2 + 1/\log T \leq \sigma \leq 1$ and $\rho = \beta + i\gamma$, $\beta \geq \sigma$, is a zero of $f(s)$. If $\lambda = u + iv$, let $u^* = \max\{u : \lambda\}$ and $n^* = \max\{n(\lambda) : \lambda\}$. If $|\gamma| \geq \log^2 T$, then the sum of the residue terms is, by Stirling's formula,

$$\begin{aligned}&\ll \sum_{\lambda\neq 0} U^{u-\beta}|v-\gamma|^{u-\beta-1/2}e^{-\pi|v-\gamma|/2}\log^{n(\lambda)-1} U \\ &\ll U^{u^*-\beta}|\gamma|^{u^*-\beta-1/2}e^{-\pi|\gamma|/2}\log^{n^*-1} U \\ &\ll U^{u^*-\beta}|\gamma|^{u^*-\beta-1/2}e^{-(\pi/2)\log^2 T}\log^{n^*-1} U \\ &= o(1)\end{aligned} \tag{2.4}$$

as $T \to +\infty$, since $U \leq T^{c_1}$.

In [1, Theorem 10] it is shown that $f(s)$ has $\ll T\log T$ zeros in the rectangle $0 \leq \mathrm{Re}\,(s) \leq 1$, $|\mathrm{Im}\,(s)| \leq T$. Thus there are $\ll \log^3 T$ zeros with $|\gamma| \leq \log^2 T$. Thus, if we add to the final estimate an $O(\log^3 T)$ term to account for the neglected zeros, we can assume $|\gamma| \geq \log^2 T$. Thus, by (2.3) and (2.4), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c(n)n^{-s}e^{-n/U} \\
(2.5)\qquad &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\tfrac{1}{2}+i(t+u))M(\tfrac{1}{2}+i(t+u))U^{1/2-\sigma+iu}\Gamma(\tfrac{1}{2}-\sigma+iu)\,du \\
&\quad + f(s)M(s) + o(1),
\end{aligned}$$

as $T \to +\infty$.

Since $f(s)$ satisfies the functional equation (1.1) and is absolutely convergent for Re $(s) > 1$ we know that $|f(\sigma+it)|$ is bounded by a power of $|t|$ for σ in any finite fixed vertical strip. Also, by (1.5), we know that $|c(n)| \leq c_2 n^{c_3}$. Thus we have

$$\begin{aligned}
&\int_{\pm(\log^2 T)/2}^{\pm\infty} f(\tfrac{1}{2}+i(t+u))M(\tfrac{1}{2}+i(t+u))U^{1/2-\sigma+iu}\Gamma(\tfrac{1}{2}-\sigma+iu)\,du \\
&\qquad \ll T^{c_4}\int_{(\log^2 T)/2}^{+\infty} u^{c_5}e^{-c_6 u}\,du \\
(2.6)\qquad &\qquad \ll T^{c_4}e^{-(c_6/2)\log^2 T}\int_{1}^{+\infty} u^{c_5}e^{-c_6 u/2}\,du \\
&\qquad = o(1),
\end{aligned}$$

as $T \to +\infty$, and, if $s = \sigma + it$, $1/2 < \sigma \leq 1$, then

$$\begin{aligned}
\sum_{n>u^2} c(n)n^{-s}e^{-n/U} &\ll \sum_{n>U^2} n^{c_7}e^{-n/U} \\
&\ll \int_{U^2}^{+\infty} t^{c_7}e^{-t/U}\,dt \\
(2.7)\qquad &\ll e^{-U^2/2U}\int_{1}^{+\infty} t^{c_7}e^{-t/2U}\,dt \\
&\ll e^{-U/2} \\
&= o(1)
\end{aligned}$$

as $T \to +\infty$, if U tends to $+\infty$ with T.

Thus, by (2.5)–(2.7), we have, since $c(n, W) = 0$ for $1 < n \leq W$,

$$\begin{aligned}
&e^{-1/U} + \sum_{W<n\leq U^2} c(n)n^{-s}e^{-n/U} \\
&\qquad = f(s)M(s) + \frac{1}{2\pi i}\int_{(1/2-\sigma,(\log^2 T)/2)} f(s+z)M(s+z)U^z\Gamma(z)\,dz + o(1),
\end{aligned}$$

as $T \to +\infty$.

Let $\rho = \beta + i\gamma$ be a zero of $f(s)$. Then we have either

$$\left| \sum_{W < n \le U^2} c(n) n^{-\rho} e^{-n/U} \right| \gg 1, \tag{2.8}$$

$$\left| \int_{(1/2-\sigma, (\log^2 T)/2)} f(\rho + z) M(\rho + z) U^z \Gamma(z)\, dz \right| \gg 1 \tag{2.9}$$

or both. Of the zeros ρ with $\beta \ge \sigma$, $|\gamma| \le T$ we take a subset R of them so that if ρ_1, ρ_2 are two zeros, then

$$|\gamma_1 - \gamma_2| \ge 2 \log^2 T. \tag{2.10}$$

By Theorem 3 of [1], we have $N(T + 1) - N(T) \ll \log T$, where $N(T)$ is the total number of zeros of $f(s)$ in the rectangle $0 \le \mathrm{Re}\,(s) \le 1$, $|\mathrm{Im}\,(s)| \le T$, and so

$$N(1/2, t + 1) - N(1/2, t) \ll \log T,$$

for $|t| \le T$. Thus we may choose the subset of R zeros so that

$$N(\sigma, T) \ll (R + 1) \log^3 T.$$

Finally, let R_1 and R_2 be the number of the R zeros such that (2.8) and (2.9), respectively, hold. Then $R \le R_1 + R_2$.

If (2.8) holds, then there is a Y such that $W \le Y \le U^2$ and

$$\left| \sum_{n=Y}^{2Y} c(n) n^{-\rho} e^{-n/U} \right| \gg \log^{-1} U$$

for $\gg R_1 \log^{-1} U$ zeros for which (2.8) holds. If ρ_j, $1 \le j \le R_1$, are the zeros under consideration, then, by Lemma 1,

$$\begin{aligned} R_1 \log^{-3} U &\ll \sum_{j=1}^{R_1} \left| \sum_{n=Y}^{2Y} c(n) n^{-\rho_j} e^{-n/U} \right|^2 \\ &\ll (T + 2Y)(1 + \tau^{-1} \log^2 2Y) \log^4 2Y \sum_{n=Y}^{2Y} |c(n)|^2 n^{-2\sigma} e^{-Y/U}, \end{aligned}$$

where $\tau = \min |\gamma_i - \gamma_j| \ge 2 \log^2 T$, by (2.10). Thus, by (1.5), we have

$$R_1 \ll (T + Y) e^{-Y/U} Y^{1-2\sigma} \log^{M_3+7} T. \tag{2.11}$$

Let $F(Y) = Y^p e^{-Y/U}$ for $Y > 0$. Then

$$F'(Y) = Y^{p-1}(p - Y/U) e^{-Y/U}$$

and

$$F''(Y) = Y^{p-2}(p(p - 1) - 2pY/U + Y^2/U^2) e^{-Y/U}.$$

Now $F'(Y) = 0$ implies $Y = pU$ and $F''(pU) = -p(pU)^{p-2} e^{-Y/U} < 0$. Thus pU yields the maximum for F. Thus, from (2.11), we have

$$R_1 \ll (TW^{1-2\sigma} + U^{2-2\sigma}) \log^{M_3+7} T. \tag{2.12}$$

Suppose (2.9) holds and let ρ_j, $1 \le j \le R_2$, be the zeros under consideration. For these values let t_j be such that $|t_j - \gamma_j| \le (\log^2 T)/2$ and $|f(1/2 + it_j)M(1/2 + it_j)|$ is maximal. Assume that $\beta \ge \sigma \ge 1/2 + 1/\log T$. Then

$$\int_{-\infty}^{+\infty} |\Gamma(1/2 - \beta + iu)|\, du \ll \log T.$$

Thus

$$|f(1/2 + it_j)M(1/2 + it_j)| \gg U^{\sigma - 1/2} \log^{-1} T. \tag{2.13}$$

If ρ_a and ρ_b are zeros with $1 \le a < b \le R_2$ and t_a and t_b are the corresponding values of t, then, by the triangle inequality, the definition of t_j and (2.10), we have $|t_a - t_b| \ge \log^2 T$.

For any integer $k \ge 2$ we have

$$\sum_{j=1}^{R_2} |f(1/2 + it_j)|^k \ll \int_{-T}^{T} |f(1/2 + it)|^k\, dt.$$

Then, by Lemma 1, (1.4) and (2.13), we have

$$\begin{aligned}
&R_2 U^{2k\sigma/(k+2) - k/(k+2)} \log^{-2k/(k+2)} T \\
&\quad \ll \sum_{j=1}^{R_2} |f(1/2 + it_j)M(1/2 + it_j)|^{2k/(k+2)} \\
&\quad \ll \left(\sum_{j=1}^{R_2} |f(1/2 + it_j)|^k\right)^{2/(k+2)} \left(\sum_{j=1}^{R_2} |M(1/2 + it_j)|^2\right)^{k/(k+2)} \\
&\quad \ll (T^{2\mu(k)/(k+2)} \log^{2\nu(k)/(k+2)} T)((T + W) \log^4 W \log^{M_3+1} W)^{k/(k+2)}.
\end{aligned}$$

Thus

$$\begin{aligned}
R_2 \ll\ & T^{2\mu(k)/(k+2)}(T + W)^{k/(k+2)} U^{(k - 2k\sigma)/(k+2)} \\
& \times \log^{(2\nu(k) + (M_3+5)k)/(k+2)} T.
\end{aligned} \tag{2.14}$$

Thus, by (2.12) and (2.16), we have

$$\begin{aligned}
N(\sigma, T) &\ll (R + 1) \log^3 T \\
&\ll (TW^{1-2\sigma} + U^{2-2\sigma}) \log^{M_4+10} T \\
&\quad + (T + W)^{k/(k+2)} T^{2\mu(k)/(k+2)} U^{(1-2\sigma)k/(k+2)} \\
&\quad \times \log^{3 + (2\nu(k) + (M_3+5)k)/(k+2)} T.
\end{aligned}$$

If we choose $W = T$ and $U = T^{(k + 2\mu(k))/(k+4-4\sigma)}$, we have

$$N(\sigma, T) \ll (T^{2(1-\sigma)} + T^{2(k+2\mu(k))(1-\sigma)/(k+4-4\sigma)}) \log^{M_1(k)} T,$$

which completes the proof of Theorem 1.

3. Proof of Theorem 2 and its corollary

We first state a lemma that we shall need.

LEMMA 2. *Under the hypotheses of Lemma* 1, *if* V *satisfies*

$$V^2 \gg S^{1/2}(\log^{3/2} S)(\log\log M) \sum_{n=1}^{M} |a_n|^{2n-2\omega},$$

then the number of r, $1 \leq r \leq R$, *such that* $|\sum_{n=1}^{M} a_n n^{-s_r}| \geq V$ *is*

$$\ll MV^{-2}(1+\tau^{-1}\log M) \sum_{n=1}^{M} |a_n|^2 n^{-2\omega}.$$

This is Theorem 3 of [12].

Throughout the proof of Theorem 2 we assume that

$$\sigma \geq (8\mu(k)+3k-4)/(8\mu(k)+4k-4).$$

We take $V = \log^{-1} U$ in Lemma 2. The hypotheses of the lemma will be satisfied if

$$W^{2\sigma-1} \gg V^{-2}T^{1/2}\log^{M_4+2} T \gg T^{1/2}\log^{M_4+4} T. \tag{3.1}$$

Thus, subject to (3.1), we have, by Lemma 2 and (1.5),

$$\begin{aligned} R_1 \log^{-1} U &\ll YV^{-2}(1+\tau^{-1}\log Y) \sum_{n=Y}^{2Y} |c(n)|^2 n^{-2\sigma} \\ &\ll Y(\log^2 U)Y^{1-2\sigma}(\log^{M_4} Y)e^{-Y/U}. \end{aligned}$$

Thus

$$R_1 \ll U^{2-2\sigma}\log^{M_4+3} T. \tag{3.2}$$

Let V_1 be a positive quantity. Then the number of r for which $|f(1/2+it_r)| \geq V_1$ is

$$\ll V_1^{-k}T^{\mu(k)}\log^{\nu(k)} T. \tag{3.3}$$

By (2.13), we have for the remaining r,

$$|M(1/2+it_r)| \geq U^{\sigma-1/2}V_1^{-1}\log^{-1} T.$$

We now take $V = U^{\sigma-1/2}V_1^{-1}\log^{-1} T$ in Lemma 2. The hypotheses of Lemma 2 will be satisfied if

$$\begin{aligned} U^{2\sigma-1} &\gg U_1^2 T^{1/2}\log^4 T \sum_{n\leq W} |a^{*-1}(n)|^2 n^{-1} \\ &\gg V_1^2 T^{1/2}\log^{M_3+5} T, \end{aligned} \tag{3.4}$$

by (1.4), since $W \leq T^{c_1}$. Thus, by Lemma 2 and (1.4), the number of such r is

$$\ll W(U^{\sigma-1/2}V_1^{-1}\log^{-1}T)^{-2}\sum_{n\leq W}|a^{*-1}(n)|^2 n^{-1}$$

$$\ll WU^{1-2\sigma}V_1^2\log^{3+M_3}T. \tag{3.5}$$

Thus, by (3.3) and (3.5), we have

$$R_2 \ll V_1^{-k}T^{\mu(k)}\log^{\nu(k)}T + WU^{1-2\sigma}V_1^2\log^{3+M_3}T. \tag{3.6}$$

Thus, by (3.2) and (3.6) we have

$$N(\sigma, T) \ll (U^{2-2\sigma} + V_1^{-k}T^{\mu(k)} + WU^{1-2\sigma}V_1^2)\log^{M_2(k)}T.$$

Choose W so that we have equality in (3.1) and U so that we have equality in (3.4). Choosing

$$V_1 = T^{((2\mu(k)+1)\sigma-(1+\mu(k))/(4-k+(2k-4)\sigma)}$$

gives the result and completes the proof of Theorem 2.

To prove the corollary to Theorem 2 we need only note that the function (of σ) $2(k+2\mu(k))/(k+4-4\sigma)$ is increasing, whereas the function (of σ) $(4\mu(k)+k)/(4-k+(2k-4)\sigma)$ is decreasing for $k \geq 2$. Since these two functions are equal at

$$\sigma = (8\mu(k)+3k-4)/(8\mu(k)+4k-4)$$

the result of the corollary follows.

4. Proof of Theorem 3

We state two lemmas that we need for the proof of Theorem 3.

LEMMA 3 (Montgomery-Vaughn). *Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty}|a_n|^2$ and $\sum_{n=1}^{\infty}n|a_n|^2$ both converge. Then, as $T \to +\infty$,*

$$\int_{-T}^{T}\left|\sum_{n=1}^{\infty}a_n n^{-it}\right|^2 dt \ll \sum_{n=1}^{\infty}(T+n)|a_n|^2.$$

This is Corollary 2 of [14].

LEMMA 4. *Let $\{c_n\}$ be a sequence of nonnegative numbers such that*

$$\sum_{n\leq x}c_n \ll x^c\log^d x,$$

as $x \to +\infty$. If $a > 0$, then as $U \to +\infty$,

$$\sum_{n=1}^{\infty}c_n n^{-1}e^{-an/U} \ll \begin{cases}\log^{d+1}U & \text{if } l = c\\ U^{c-1}\log^d U & \text{if } l \neq c.\end{cases}$$

This is easily proved by partial summation.

Let $\varepsilon > 0$. Then we have, if $s = 1/2 + it$,

$$\sum_{n=1}^{\infty}a(n)n^{-s}e^{-n/U} = \frac{1}{2\pi i}\int_{(1+\varepsilon)}f(s+z)U^z\Gamma(z)\,dz.$$

Let $-1 < \eta < 0$ and let λ denote a pole of the integrand. Then, since all the poles of $f(w)$ are in the strip $0 < \mathrm{Re}\,(w) \leq 1$, we have, for Q sufficiently small,

$$\begin{aligned}\sum_{n=1}^{\infty} a(n)n^{-s}e^{-n/U} &= \sum_{0 \leq \mathrm{Re}\,(\lambda) < 1/2} \frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)U^2\Gamma(z)\,dz \\ &\quad + \frac{1}{2\pi i}\int_{(\eta)} f(s+z)U^z\Gamma(z)\,dz \\ &= f(s) + \sum_{0 < \mathrm{Re}\,(\lambda) < 1/2} \frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)U^z\Gamma(z)\,dz \\ &\quad + \frac{1}{2\pi i}\int_{(\eta)} f(s+z)U^z\Gamma(z)\,dz,\end{aligned}$$

which can be rewritten as

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} a(n)n^{-s}e^{-n/U} - \frac{1}{2\pi i}\int_{(\eta)} f(s+z)U^z\Gamma(z)\,dz \\ &\quad - \sum_{0 < \mathrm{Re}\,(\lambda) < 1/2} \frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)U^z\Gamma(z)\,dz. \end{aligned} \tag{4.1}$$

Let $H(s) = C^{\theta s+\delta}\Delta(1-s)/\Delta(s)$ and assume $H(s)$ has no poles in $[-1, 0)$. Then, by the functional equation (1.1), we have

$$f(s+z) = H(s+z)g(1-s-z).$$

Thus, if $-1 < \eta < -1/2$ and $-1 < \eta_1 < 0$, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(\eta)} f(s+z)U^z\Gamma(z)\,dz \\ &\qquad = \frac{1}{2\pi i}\int_{(\eta)} H(s+z)g(1-s-z)U^z\Gamma(z)\,dz \\ &\qquad = \frac{1}{2\pi i}\int_{(\eta)} H(s+z)\left(\sum_{n>u} b(n)n^{s+z-1}\right)U^z\Gamma(z)\,dz \\ &\qquad\quad + \frac{1}{2\pi i}\int_{(\eta_1)} H(s+z)\left(\sum_{n\leq U} b(n)n^{s+z-1}\right)U^z\Gamma(z)\,dz. \end{aligned} \tag{4.2}$$

By Stirling's formula we have

$$\begin{aligned} |H(s+z)U^z| &= |C^{\theta(s+z)+\delta}\Delta(1-s-z)U^z/\Delta(s+z)| \\ &= C^{\theta\mathrm{Re}\,(s+z)+\mathrm{Re}\,(\delta)}U^{\mathrm{Re}\,(z)}|\Delta(1-s-z)/\Delta(s+z)| \\ &\ll C^{\theta\mathrm{Re}\,(s+z)}U^{\mathrm{Re}\,(z)}D^{-2\,\mathrm{Re}\,(s+z)}T^{A(1-2\,\mathrm{Re}\,(s+z))}, \end{aligned}$$

where

$$D = \exp\left\{\sum_{j=1}^{N} \alpha_j \log \alpha_j\right\}.$$

Choose $\eta = -1/2 - 1/\log T$, $\eta_1 = -1/\log T$ and $U = T$. Then on $\operatorname{Re}(z) = \eta$ we have

$$|H(s+z)U^z| \ll C^{-\theta/\log T} T^{-1/2-1/\log T} D^{-2/\log T} T^{A(1-2/\log T)} \ll T^{A-1/2}$$

and on $\operatorname{Re}(z) = \eta_1$ we have

$$|H(s+z)U^z| \ll C^{\theta(1/2-1/\log T)} T^{-1/\log T} D^{-2(1/2-1/\log T)} T^{-A/\log T} \ll 1.$$

With the notation as in the proof of Theorem 1 we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{|z-(\lambda-s)|=Q} f(s+z)U^z\Gamma(z)\,dz \\ &= U^{\lambda-s}\Gamma(\lambda-s)\sum_{j=1}^{n(\lambda)} a_{-j}(\lambda) \sum_{e+f=j-1} \frac{(\Gamma^{(e)}/\Gamma)(\lambda-s)\log^f U}{e!\,f!} \qquad (4.3)\\ &\ll U^{\operatorname{Re}(\lambda-s)}|\Gamma(\lambda-s)|\log^{n(\lambda)-1} U. \end{aligned}$$

Thus, by (4.1)–(4.3),

$$\begin{aligned} f(\tfrac{1}{2}+it) &= \sum_{n=1}^{\infty} a(n)n^{-1/2-it}e^{-n/T} \\ &\quad + O\left\{\sum_{0<\operatorname{Re}(\lambda)<1/2} T^{\operatorname{Re}(\lambda)-1/2}|\Gamma(\lambda-\tfrac{1}{2}-it)|\log^{n(\lambda)-1} T\right\} \\ &\quad - \frac{1}{2\pi i}\int_{(\eta)} H(s+z)\left(\sum_{n>T} b(n)n^{s+z-1}\right)T^z\Gamma(z)\,dz \\ &\quad - \frac{1}{2\pi i}\int_{(\eta_1)} H(s+z)\left(\sum_{n\le T} b(n)n^{s+z-1}\right)T^z\Gamma(z)\,dz. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{-T}^{T} |f(1/2+it)|^2\,dt \\ &\ll \int_{-T}^{T}\left|\sum_{n=1}^{\infty} a(n)n^{-1/2-it}e^{-n/T}\right|^2 dt \\ &\quad + \int_{-T}^{T}\left(\sum_{0<\operatorname{Re}(\lambda)<1/2} T^{\operatorname{Re}(\lambda)-1/2}|\Gamma(\lambda-\tfrac{1}{2}-it)|\log^{n(\lambda)-1} T\right)^2 dt \\ &\quad + \int_{-T}^{T} T^{2A-1}\left(\int_{-\infty}^{+\infty}\left|\sum_{n>T} b(n)n^{-1-1/\log T+i(t+v)}\Gamma(\eta+iv)\right|dv\right)^2 dt \qquad (4.4)\\ &\quad + \int_{-T}^{T}\left(\int_{-\infty}^{+\infty}\left|\sum_{n\le T} b(n)n^{-1/2-1/\log T+i(t+v)}\Gamma(\eta_1+iv)\right|dv\right)^2 dt \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

say.

By Lemmas 3 and 4 and (1.2), we have

$$
\begin{aligned}
I_1 &\ll \sum_{n=1}^{\infty} |a(n)n^{-1/2}e^{-n/T}|^2(T+n) \\
&= \sum_{n=1}^{\infty} |a(n)|^2 e^{-2n/T} + T\sum_{n=1}^{\infty} |a(n)|^2 n^{-1} e^{-2n/T} \qquad (4.5)\\
&\ll T\log^{M_1} T + T\log^{M_1+1} T \\
&\ll T\log^{M_1+1} T.
\end{aligned}
$$

Let $n^* = \max\{n(\lambda)\colon 0 < \mathrm{Re}\,(\lambda) \le 1/2\}$. Then, by the Cauchy-Schwarz inequality,

(4.6)

$$
\begin{aligned}
I_2 &\ll \sum_{0<\mathrm{Re}\,(\lambda)<1/2} T^{2\,\mathrm{Re}\,(\lambda)-1}\log^{2n(\lambda)-2} T \int_{-T}^{T} \sum_{0<\mathrm{Re}\,(\lambda)<1/2} |\Gamma(\lambda - \tfrac{1}{2} - it)|^2\,dt \\
&\ll \log^{2n\mathrm{A}-2} T \int_{1}^{T} \sum_{0<\mathrm{Re}\,(\lambda)<1/2} t^{2\,\mathrm{Re}\,(\lambda)-2-t}\,dt \\
&\ll \log^{2n\mathrm{A}-2} T,
\end{aligned}
$$

since $\mathrm{Re}\,(\lambda) < 1/2$.

We have, by Lemmas 3 and 4, (1.3) and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
I_3 &\ll T^{2A-1}\int_{-T}^{T}\int_{-\infty}^{+\infty}\left|\sum_{n>T} b(n)n^{-1-1/\log T + i(t+v)}\right|^2 |\Gamma(\eta+iv)|\,dv \\
&\quad\times \int_{-\infty}^{+\infty} |\Gamma(\eta+iv)|\,dv\,dt \\
&\ll T^{2A-1}\sum_{n>T} |b(n)|^2 n^{-2-2/\log T}(n+T) \qquad (4.7)\\
&\ll T^{2A-1}(T^{-2/\log T}\log^{M_2} T + T\cdot T^{-1-2/\log T}\log^{M_2} T) \\
&\ll T^{2A-1}\log^{M_2} T.
\end{aligned}
$$

Finally, as for the estimate of I_3, we have

$$
I_4 \ll \sum_{n\le T} |b(n)|^2 n^{-1-1/\log T}(n+T) \ll T\log^{M_2} T. \qquad (4.8)
$$

Thus, by (4.4)–(4.8), we have

$$
\begin{aligned}
\int_{-T}^{T} |f(1/2+it)|^2\,dt &\ll T\log^{M_1+1} T + \log^{2n^*-2} T + T^{2A-1}\log^{M_2} T + T\log^{M_2} T \\
&\ll T^{\mu(2)}\log^{\nu(2)} T,
\end{aligned}
$$

where $\mu(2) = \max(1, 2A - 1)$ and $\nu(2) = \max(1 + M_1, M_2)$, which proves the first part of Theorem 3.

The second part follows easily from the first part if we note that

$$f^2(s) = \sum_{n=1}^{\infty} (a^*a)(n)n^{-s}, \qquad g^2(s) = \sum_{n=1}^{\infty} (b^*b)(n)n^{-s}$$

and

$$\Delta^2(s)f^2(s) = C^{2\theta s + 2\delta}\,\Delta^2(1-s)g^2(1-s).$$

This completes the proof of Theorem 3.

5. Examples

Example 1. The Riemann zeta function. Here $f(s) = g(s) = \zeta(s)$, $a(n) = b(n) = 1$, $\Delta(s) = \Gamma(s/2)$, $C = \pi$, $\theta = 1$ and $\delta = -1/2$. Also $a^{*-1}(n) = \mu(n)$, the Möbius function. Thus we can take $M_1 = M_2 = M_3 = 0$. By the remark of (1.5) we see that $|c(n)| \le d(n)$ and so we have $M_4 = 3$. By Theorem 3 we have $\mu(2) = \mu(4) = \nu(2) = 1$ and $\nu(4) = 4$.

Thus, for $k = 2$, we have from Theorem 1,

$$N(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)} \log^{13} T$$

and for $k = 4$ we have

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} \log^{13} T.$$

The first result is due to Titchmarsh [17] and the second is due to Ingham [9].

By the corollary to Theorem 2 these results may be improved to

$$N(\sigma, T) \ll T^{3(1-\sigma)} \log^{13} T \quad \text{and} \quad N(\sigma, T) \ll T^{5(1-\sigma)/2} \log^{13} T,$$

respectively, for $\frac{1}{2} \le \sigma \le 1$. The second result is due to Montgomery [13].

Example 2. Cusp forms of weight k with Euler product. Let

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$

be a cusp form of weight k with Euler product. Then it is known [6] that

$$\Gamma(s)f(s) = (2\pi)^{2s-k}\Gamma(k-s)f(k-s)$$

and that $f(s)$ is absolutely convergent for $\mathrm{Re}\,(s) > (k+1)/2$.

Let $a_1(n) = a(n)n^{-(k-1)/2}$ and

$$F(s) = \sum_{n=1}^{\infty} a_1(n)n^{-s} = \sum_{n=1}^{\infty} a(n)n^{-(k-1)/2-s} = f(s + (k-1)/2).$$

Then we see that $F(s)$ is absolutely convergent for $\mathrm{Re}\,(s) > 1$ and satisfies the functional equation

$$\Gamma(s + (k-1)/2)F(s) = (2\pi)^{2s-1}\Gamma(1 - s + (k-1)/2)F(1-s).$$

Here we have $a(n) = b(n) = a_1(n)$, $\Delta(s) = \Gamma(s + (k-1)/2)$, $C = 2\pi$, $\theta = 2$ and $\delta = -1$.

By a result of Hecke [7] we know that $\sum_{n \le x} |a(n)|^2 \ll x^k$ and so

$$\sum_{n \le x} |a_1(n)|^2 \ll x.$$

Thus we have $M_1 = M_2 = 0$.

Goldstein [4] has shown that, for every prime p,

$$a^{*-1}(p^j) = \begin{cases} 1, & j = 0, \\ -a(p), & j = 1, \\ p^{k-1}, & j = 2, \\ 0, & j > 3, \end{cases}$$

and is defined on the integers by multiplicativity. From this it is easy to show that $x^k \ll \sum_{n \le x} |a^{*-1}(n)|^2 \ll x^k$ and so

$$x \ll \sum_{n \le x} |a_1^{*-1}(n)|^2 \ll x.$$

Thus $M_3 = 0$.

By the Ramanujan-Petersson conjecture (see Deligne [3])

$$|a(n)| \le d(n) n^{(k-1)/2}.$$

From this it is easy to show that $|c(n)| \le d_4(n) n^{(k-1)/2}$. Thus

$$\sum_{n \le x} |c(n)|^2 \ll x^k \log^{15} x$$

and so

$$\sum_{n \le x} |c_1(n)|^2 \ll x \log^{15} x.$$

Thus we have $M_4 = 15$.

By Theorem 3 we have $\mu(2) = \nu(2) = 1$. This gives, by Theorem 1,

$$N(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)} \log^{25} T,$$

which improves the result obtainable from the theorem of Sokolovskii [16, Theorem 2]. By the corollary to Theorem 2 we have

$$N(\sigma, T) \ll T^{3(1-\sigma)} \log^{25} T, \tag{5.1}$$

for $\frac{1}{2} \le \sigma \le 1$, which we believe to be new.

One can use part (2) of Theorem 3 to show that $\mu(4) = 3$ and $\nu(4) = 16$, but this does not lead to a better result than (5.1).

If we translate (5.1) back to the cusp form $f(s)$ we have

$$N(\sigma, T) \ll T^{3((k+1)/2-\sigma)} \log^{25} T.$$

Example 3. The Dedekind zeta function. Let K be an algebraic number field of degree $n \geq 2$ over the rationals and let $\zeta_K(s)$ be the associated Dedekind zeta function. For Re $(s) > 1$ we have $\zeta_K(s) = \sum_{m=1}^{\infty} a_K(m)m^{-s}$ where $a_K(m)$ is the number of integral ideals of K with norm exactly m. Then it is known [11, p. 75] that $\zeta_K(s)$ satisfies the functional equation

$$\Gamma^{r_1}(s/2)\Gamma^{r_2}(s)\zeta_K(s) = B^{2s+1}\Gamma^{r_1}((1-s)/2)\Gamma^{r_2}(1-s)\zeta_K(1-s),$$

where B is a constant depending on the field K, r_1 is the number of real conjugates and r_2 is the number of imaginary conjugates of K so that $r_1 + 2r_2 = n$.

In [2] it is shown that

$$\int_{-T}^{T} |\zeta_K(1/2 + it)|^2 \, dt \ll T^{n/2} \log^n T, \qquad \sum_{m \leq x} |a_K(m)|^2 \ll x \log^{n-1} x$$

and $a_K(m) \leq d_n(m)$.

Since $a_K(m) \geq 0$ we see that $|a_K^{*-1}(m)| \leq a_K(m)$. Thus

$$x \log^{n-1} x \ll \sum_{m \leq x} |a_K^{*-1}(m)|^2 \ll x \log^{n-1} x.$$

We have

$$\begin{aligned} |c(m)| &\leq \sum_{\substack{d \leq W \\ d \mid m}} |a_K(d)| * |a_K^{*-1}(m/d)| \\ &\leq (a_K * a_K)(m) \\ &\leq d_{2n}(m). \end{aligned}$$

Thus

$$\sum_{m \leq x} |c(m)|^2 \leq \sum_{m \leq x} d_{2n}^2(m) \ll x \log^{4n^2-1} x.$$

Thus, here, we have $M_1 = M_2 = M_3 = n - 1$, $M_4 = 4n^2 - 1$, $\mu(2) = n/2$ and $\nu(2) = n$.

Thus by Theorem 1 and the corollary to Theorem 2 we have

$$N(\sigma, T) \ll T^{(n+2)(1-\sigma)/(3-2\sigma)} \log^{4n^2+9} T$$

and

$$N(\sigma, T) \ll T^{(n+1)(1-\sigma)} \log^{4n^2+9} T,$$

respectively, for $\frac{1}{2} \leq \sigma \leq 1$. Both of these results better those of Sokolovskii [16, corollary to Theorem 2]. In [5] Heath-Brown has improved these results even more by showing that if $n \geq 3$, then, for any $\varepsilon > 0$, $N(\sigma, T) \ll T^{(n+\varepsilon)(1-\sigma)}$. The result for $n = 2$, is somewhat complicated, but it too shows that $N(\sigma, T) \ll T^{(2+\varepsilon)(1-\sigma)}$, for any $\varepsilon > 0$. His method was to use the later improvements of Huxley on large values of Dirichlet polynomials in the method of Montgomery that we have used in this paper.

6. The more general functional equation

In this section we simply indicate the results that can be obtained if we assume a more general functional equation. The method used is that of Section 2, though the details are more complicated.

We assume a functional equation of the form (under the notation as above)

$$\Delta(s)f(s) = C^{\theta s+\delta}\, \Delta(r-s)g(r-s),$$

where r is a positive real number, $f(s)$ and $g(s)$ converge absolutely for Re $(s) > r$ and $f(s)$ has as its singularities only a finite number of poles in the strip $0 < \text{Re}\,(s) \leq r$. We assume the more general estimates on the coefficients:

$$\sum_{n\leq x} |a^{*-1}(n)|^2 \ll x^a \log^b x \quad \text{and} \quad \sum_{n\leq x} |c(n)|^2 \ll x^{a_1} \log^{b_1} x.$$

Let $N(\sigma, T)$ be the number of zeros, ρ, of $f(s)$ in the region $r \geq \text{Re}\,(\rho) \geq \sigma$, $|\text{Im}\,(\rho)| \leq T$, with $\sigma \geq r/2$ and $T \geq 1$. Then we have

$$N(\sigma, T) \ll T^{(a_1+1-2\sigma)} + T^{(a_2+1+a')(a_1+1-2\sigma)/(2(a_1+1)-r-2\sigma))} \log^M T,$$

where $M = \max\,(b_1 + 10, 6 + b_2/2 + b'/2)$,

$$a' = \begin{cases} 0 & \text{if } a \leq r \\ a-r & \text{if } a \geq r \end{cases} \quad \text{and} \quad b' = \begin{cases} 0 & \text{if } a < r \\ b+1 & \text{if } a = r, \\ b & \text{if } a > r \end{cases}$$

if we assume that

$$\int_{-T}^{T} |f(r/2+it)|^2\, dt \ll T^{a_2} \log^{b_2} T.$$

There are also results for other power means and results corresponding to Theorems 2 and 3 and the corollary to Theorem 2.

Acknowledgement. I would like to thank Carlos Moreno for his help and suggestions.

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