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ON A GENERALIZATION OF TWO EXACT SEQUENCES OF STEINER

BY

PETER HILTON AND JOSEPH ROITBERG

1. Introduction

In [2], Steiner describes two exact sequences (or, rather, two families of exact sequences) associated with a short exact sequence of groups

$$K \overset{\alpha}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H \quad (\alpha \text{ may be regarded as an inclusion}). \tag{1.1}$$

Let $a \in K$; write $K(a)$ for the centralizer of a in K and $G(a)$ for the centralizer of a in G; write $[K]$, $[G]$, $[H]$ for the sets of conjugacy classes of elements of K, G, H respectively and use a similar notation $[x]$ for the conjugacy class containing x. Then Steiner's first sequence is

$$K(a) \overset{\alpha_1}{\rightarrowtail} G(a) \xrightarrow{\beta_1} H \xrightarrow{\partial} [K] \xrightarrow{\alpha_2} [G] \overset{\beta_2}{\twoheadrightarrow} [H]. \tag{1.2}$$

Here α_1, α_2 are induced by α; β_1, β_2 are induced by β; and ∂ is defined by

$$\partial \beta x = [xax^{-1}], \quad x \in G.$$

We particularly note: (i) if $[K]$, $[G]$, $[H]$ are furnished with base points $[a]$, $[a]$, $[1]$ respectively, then the sequence (1.2) is exact; (ii) the exactness at H takes the strong form that

$$\partial h_1 = \partial h_2 \Leftrightarrow h_1 = h_2(\beta_1 x), \quad \text{some } x \in G(a),\ h_1, h_2 \in H; \tag{1.3}$$

(iii) if $K \mid G$ stands for the set of conjugacy classes of elements of K under conjugation by elements of G, then $K \mid G = \beta_2^{-1}[1]$ and (1.2) may be shortened to the 5-term sequence

$$K(a) \overset{\alpha_1}{\rightarrowtail} G(a) \xrightarrow{\beta_1} H \xrightarrow{\partial} [K] \overset{\alpha_2}{\twoheadrightarrow} K \mid G. \tag{1.4}$$

Steiner's second sequence is defined when (1.1) is *central*. It then reads, with $a \in G$ now,

$$K \overset{\alpha_1}{\rightarrowtail} G(a) \xrightarrow{\beta_1} H(\beta a) \xrightarrow{\partial} K \xrightarrow{\rho} [G] \overset{\beta_2}{\twoheadrightarrow} [H]. \tag{1.5}$$

Here α_1 is induced by α; β_1, β_2 are induced by β; $\rho b = [ba]$, $b \in K$; and ∂ is defined by $\partial \beta x = xax^{-1}a^{-1}$. We particularly note: (i) if $[G]$, $[H]$ are furnished

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with base points $[a]$, $[\beta a]$ respectively, then the sequence (1.5) is exact; (ii) the exactness at the second occurrence of K takes the strong form that

$$\rho b_1 = \rho b_2 \Leftrightarrow b_1 = b_2\, \partial y, \quad \text{some } y \in H(\beta a),\ b_1, b_2 \in K;$$

(iii) if H operates on G by $\beta x \cdot x_1 = x x_1 x^{-1}$ and if $H(a)$ is the subgroup of H fixing a, then, in (1.5), coker $\alpha_1 = H(a)$ and (1.5) may be shortened to the 5-term sequence

$$(1.6) \qquad H(a) \rightarrowtail H(\beta a) \xrightarrow{\partial} K \xrightarrow{\rho} [G] \overset{\beta_2}{\twoheadrightarrow} [H].$$

It is our purpose in this paper to generalize (1.4) and (1.6) and to discuss their properties, especially with regard to the localization of nilpotent groups. Indeed, this programme was already carried out with regard to (1.6) in [1], although our generalization will be more substantial than in [1]. In our generalization we retain the short exact sequence (1.1) and consider two possible additional pieces of structure. In the first there is a group N on which G acts; in the second there is a group Q acting on the sequence (1.1). In the course of our study of these generalized sequences, we will have occasion to consider localizations of (right) coset spaces G/H, where H is a subgroup of the nilpotent group G, and a study of this situation will be found in the last section. Here, of course, we go beyond [1] since, for $h \in H$, the function $g \mapsto gh$, $g \in G$, is not an automorphism of G, and we only discussed in [1] representations of groups as groups of automorphisms.

2. The generalized sequences

We first give a generalization of (1.4). Let N be a group and let (1.1) again be a short exact sequence of groups and suppose that *G acts on N*; then, by restriction, K acts on N, and we allow $K(a)$, $G(a)$, for $a \in N$, to be the subgroup of K, G respectively, fixing a. Let $N \mid K$ be the orbit set of N under the K-action, and define $N \mid G$ similarly. We define $\partial\colon H \to N \mid K$ by $\partial \beta x = [xa]_K$, $x \in G$, where $[a']_K$ is the K-orbit of N containing $a' \in N$.

THEOREM 2.1. *There is an exact 5-term sequence, with base points $[a]_K \in N \mid K$, $[a]_G \in N \mid G$,*

$$(2.1) \qquad K(a) \overset{\alpha_1}{\rightarrowtail} G(a) \xrightarrow{\beta_1} H \xrightarrow{\partial} N \mid K \overset{\alpha_2}{\twoheadrightarrow} N \mid G$$

in which

(i) *α_1, α_2 are induced by α, and β_1 is induced by β;*
(ii) *the sequence is exact at $N \mid K$ in the sense that $\alpha_2^{-1}[a]_G = \partial H$;*
(iii) *the sequence is exact at H in the sense of* (1.3).

Proof. We will be content to prove (ii) and (iii). Plainly $\alpha_2[xa]_K = [xa]_G = [a]_G$. Conversely, if $\alpha_2[a']_K = [a]_G$, then $a' = xa$, for some $x \in G$, so that $[a']_K = \partial \beta x$. This proves (ii). To prove (iii), observe first that if $h_1 = h_2 \beta_1 x$, $x \in G(a)$, then $x_1 = bx_2 x$, where $\beta x_i = h_i$, $i = 1, 2$, and $b \in K$. Thus $\partial h_1 = [x_1 a]_K = [bx_2 xa]_K = [bx_2 a]_K = [x_2 a]_K = \partial h_2$. Conversely, if $[x_1 a]_K = [x_2 a]_K$, then $x_1 a = bx_2 a$, some $b \in K$, so that $(bx_2)^{-1} x_1 = x \in G(a)$, $x_1 = bx_2 x$, $\beta x_1 = (\beta x_2)(\beta x)$, and (iii) is proved.

Remark. We recover (1.4) by taking $N = K$ with G acting by conjugation.

COROLLARY 2.2. *If H is commutative, then we may give ∂H a unique group structure such that $\partial\colon H \twoheadrightarrow \partial H$ is a homomorphism. Then* (2.1) *breaks up into the exact sequence of groups*

$$(2.2) \qquad K(a) \overset{\alpha_1}{\rightarrowtail} G(a) \xrightarrow{\beta_1} H \overset{\partial}{\twoheadrightarrow} \partial H$$

and the exact sequence of pointed sets

$$(2.3) \qquad \partial H \rightarrowtail N \mid K \overset{\alpha_2}{\twoheadrightarrow} N \mid G$$

with base points $[a]_K$, $[a]_K$, $[a]_G$.

Remark. It would suffice for the conclusion that $\beta_1 G(a)$ be normal in H, but we will not insist on this generality. Of course, we could still break (2.1) up into (2.2), (2.3), even if H were not commutative, but then $\partial\colon H \twoheadrightarrow \partial H$ would not be a homomorphism.

COROLLARY 2.3. *Let H act on $N \mid K$ by the rule $\beta x \cdot [a]_K = [xa]_K$, $x \in G$. Then the orbit of $N \mid K$ under this action containing $[a]$ is in bijective correspondence with the right coset space $H/\beta_1 G(a)$. In particular, if H is commutative, then $N \mid K$ may be represented as a disjoint union of commutative groups, each a homomorphic image of H.*

The following result relates to the dependence of (2.1) on the choice of $a \in N$.

THEOREM 2.4. *With the data of Theorem* 2.1, *let $y \in G$, $a' = ya$. There is then a bijection of exact sequences (in obvious notation)*

$$(2.4) \qquad \begin{array}{ccccccccc} K(a) & \overset{\alpha_1}{\rightarrowtail} & G(a) & \xrightarrow{\beta_1} & H & \xrightarrow{\partial} & N \mid K & \overset{\alpha_2}{\twoheadrightarrow} & N \mid G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(a') & \overset{\alpha_1'}{\rightarrowtail} & G(a') & \xrightarrow{\beta_1'} & H & \xrightarrow{\partial'} & N \mid K & \overset{\alpha_2}{\twoheadrightarrow} & N \mid G \end{array} \quad \omega_y$$

where $\omega_y x = yxy^{-1}$, $x \in G(a)$ (or $K(a)$); $\omega_y h = (\beta y)h(\beta y)^{-1}$, $h \in H$; $\omega_y[c] = [yc]$, $[c] \in N \mid K$ or $N \mid G$.

Notice that $\omega_y\colon N \mid K \to N \mid K$ simply moves elements within their H-orbits; see the preceding remark.

We now turn attention to the generalization of (1.6); see also [1]. We again consider the short exact sequence of groups (1.1); but now we take a group Q

and suppose that *Q acts on G and H in such a way that $\beta\colon G \to H$ is a Q-homomorphism.* If $a \in G$, we define $\partial\colon Q(\beta a) \to K$ by $\partial x = a(xa)^{-1}$, $x \in Q(\beta a)$, and $\rho\colon K \to G\,|\,Q$ by $\rho b = [ba]$, $b \in K$.

THEOREM 2.5. *There is an exact 5-term sequence, with base points $[a] \in G\,|\,Q$, $[\beta a] \in H\,|\,Q$,*

$$(2.5) \qquad Q(a) \overset{\beta_1}{\rightarrowtail} Q(\beta a) \xrightarrow{\partial} K \xrightarrow{\rho} G\,|\,Q \xrightarrow{\beta_2} H\,|\,Q$$

in which

(i) *β_1, β_2 are induced by β;*
(ii) *∂ is a crossed homomorphism; that is, $\partial(xy) = \partial x(x\,\partial y)$, $x, y \in Q(\beta a)$;*
(iii) *the sequence is exact at $G\,|\,Q$ in the sense that $\beta_2^{-1}[\beta a] = \rho K$;*
(iv) *the sequence is exact at K in the sense that $\rho b_1 = \rho b_2 \Leftrightarrow x b_1 = b_2\,\partial x$, for some $x \in Q(\beta a)$, $b_1, b_2 \in K$.*

Proof. We are content to prove (ii), (iii), (iv). To prove (ii), note that

$$\partial(xy) = a(xya)^{-1} = a(xa)^{-1}(xa)(xya)^{-1} = a(xa)^{-1}x(a(ya)^{-1}) = \partial x(x\,\partial y).$$

To prove (iii), observe first that $\beta_2[ba] = [\beta(ba)] = [\beta a]$. Conversely, if $\beta_2[a'] = [\beta a]$, then $\beta a' = x\beta a = \beta x a$, $x \in Q$, so $a' = b(xa) = x(b'a)$, where $b \in K$, $xb' = b$. Thus $[a'] = [b'a] = \rho b'$. To prove (iv), first let $xb_1 = b_2\,\partial x$, $x \in Q(\beta a)$. Then $x(b_1 a) = (xb_1)(xa) = b_2 a$, so that $\rho b_1 = [b_1 a] = [x(b_1 a)] = [b_2 a] = \rho b_2$. Conversely, if $\rho b_1 = \rho b_2$, then $[b_1 a] = [b_2 a]$, so that $x(b_1 a) = b_2 a$ for some $x \in Q$. But then $\beta(x(b_1 a)) = x\beta a$, $\beta(b_2 a) = \beta a$, so that $x\beta a = \beta a$, $x \in Q(\beta a)$, and $xb_1 = b_2\,\partial x$.

Remark. To obtain (1.6) from (2.5) we take $Q = H$, K central in G, H acting on G as described in (1.6) and H acting on itself by conjugation. There is then an insignificant difference between $\partial\colon H(\beta a) \to K$ as described in Theorem 2.5 and in Steiner's sequence. If we call Steiner's definition ∂^*, then $\partial^* x = \partial x^{-1}$. Since, in Steiner's situation, K is commutative, there is clearly no essential difference between the roles of ∂^* and ∂ in the sequence. Of course, ∂ is a homomorphism if Q acts trivially on K; this is (with $Q = H$) precisely the condition, in Steiner's case, that K be central.

COROLLARY 2.6. *If K is commutative and Q acts trivially on K, then we may give ρK a unique group structure such that $\rho\colon K \twoheadrightarrow \rho K$ is a homomorphism. Then (2.5) breaks up into the exact sequence of groups*

$$(2.6) \qquad Q(a) \overset{\beta_1}{\rightarrowtail} Q(\beta a) \xrightarrow{\partial} K \overset{\rho}{\twoheadrightarrow} \rho K$$

and the exact sequence of pointed sets

$$(2.7) \qquad \rho K \rightarrowtail G\,|\,Q \twoheadrightarrow H\,|\,Q$$

with base points $[a]_G$, $[a]_G$, $[\beta a]_H$.

COROLLARY 2.7. *Let Q act trivially on K. Then K acts on $G\,|\,Q$ by the rule $b\cdot[a]=[ba]$, $b\in K$. Then the orbit of $G\,|\,Q$ under this action containing $[a]$ is in bijective correspondence with the right coset space $K/\partial Q(\beta a)$. In particular, if K is commutative, then $G\,|\,Q$ may be represented as a disjoint union of commutative groups, each a homomorphic image of K.*

The following result relates to the dependence of (2.5) on the choice of $a\in G$.

THEOREM 2.8. *Let Q act trivially on K, let $b\in K$, $a'=ba$. There is then a bijection of exact sequences (in obvious notation)*

$$
\begin{tikzcd}
Q(a) \arrow[r, tail, "\beta_1"] \arrow[d, "\omega_b"'] & Q(\beta a) \arrow[r, "\partial"] \arrow[d] & K \arrow[r, "\rho"] \arrow[d] & G\,|\,Q \arrow[r, two heads, "\beta_2"] \arrow[d] & H\,|\,Q \arrow[d] \\
Q(a') \arrow[r, tail, "\beta_1'"] & Q(\beta a') \arrow[r, "\partial'"] & K \arrow[r, "\rho'"] & G\,|\,Q \arrow[r, two heads, "\beta_2"] & H\,|\,Q
\end{tikzcd}
\tag{2.8}
$$

where $\omega_b=1$ on $Q(a)$ and $Q(\beta a)$; $\omega_b b'=bb'b^{-1}$, $b'\in K$; $\omega_b[c]=[bc]$, $c\in G$; $\omega_b=1$ on $H\,|\,Q$.

Notice that $\omega_b\colon G\,|\,Q\to G\,|\,Q$ simply moves elements within their K-orbits; see the preceding corollary.

With a view to a remark we wish to make at the end of the next section, we study the effect on the sequences (2.1), (2.5) of replacing a by a^n. We first note that if the group G acts on the group N and if n is any positive integer, then the orbit set $N\,|\,G$ admits the *nth* power map $\pi^n\colon N\,|\,G\to N\,|\,G$, given by the rule $[c]\mapsto[c^n]$; for, if $x\in G$, then $(xc)^n=xc^n$. The following is then easy to see in relation to (2.1):

PROPOSITION 2.9. *Under the hypotheses of Theorem 2.1 there is a map of exact sequences, with $a\in N$,*

$$
\begin{tikzcd}
K(a) \arrow[r, tail, "\alpha_1"] \arrow[d] & G(a) \arrow[r, "\beta_1"] \arrow[d] & H \arrow[r, "\partial"] \arrow[d, equal] & N\,|\,K \arrow[r, two heads, "\alpha_2"] \arrow[d, "\pi^n"] & N\,|\,G \arrow[d, "\pi^n"] \\
K(a^n) \arrow[r, tail, "\alpha_1'"] & G(a^n) \arrow[r, "\beta_1'"] & H \arrow[r, "\partial'"] & N\,|\,K \arrow[r, two heads, "\alpha_2"] & N\,|\,G
\end{tikzcd}
\tag{2.9}
$$

The situation with regard to (2.5) is a little more complicated:

PROPOSITION 2.10. *Under the hypotheses of Theorem 2.5 and the additional hypothesis that K be central in G, there is a map of exact sequences, with $a\in G$,*

$$
\begin{tikzcd}
Q(a) \arrow[r, tail, "\beta_1"] \arrow[d] & Q(\beta a) \arrow[r, "\partial"] \arrow[d] & K \arrow[r, "\rho"] \arrow[d, "\pi^n"] & G\,|\,Q \arrow[r, two heads, "\beta_2"] \arrow[d, "\pi^n"] & H\,|\,Q \arrow[d, "\pi^n"] \\
Q(a^n) \arrow[r, tail, "\beta_1'"] & Q(\beta a^n) \arrow[r, "\partial'"] & K \arrow[r, "\rho'"] & G\,|\,Q \arrow[r, two heads, "\beta_2"] & H\,|\,Q.
\end{tikzcd}
\tag{2.10}
$$

Proof. Now, in any group G, if $u, v \in G$ and uv is in the centre of G, then, for any n, $u^n v^n = (uv)^n$. For if $uv = w$, and $g \in G$, then $(wg)^n = w^n g^n$ and we set $g = v^{-1}$. Thus if $x \in Q(\beta a)$, $\pi^n \, \partial x = (a(xa)^{-1})^n = a^n(xa)^{-n}$, since ∂x is in the centre of G. Hence $\pi^n \, \partial x = a^n(xa^n)^{-1} = \partial' x$. Also if $b \in K$, $\pi^n \rho b = \pi^n[ba] = [(ba)^n] = [b^n a^n] = \rho' b^n$. The remaining assertions of the proposition are trivial.

COROLLARY 2.11. *If, in Proposition* 2.9, *N admits unique nth roots, then* (2.9) *is a bijection.*

COROLLARY 2.12. *If, in Proposition* 2.10, *G and K* (*and hence H*) *admit unique nth roots, then* (2.10) *is a bijection. In particular, if also Q operates trivially on K, then* π^n: coker $\partial \cong$ coker ∂'.

3. Localization

We again consider a short exact sequence of groups (1.1)

$$K \overset{\alpha}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H, \tag{3.1}$$

but will insist in this section that the groups be nilpotent. We further suppose given a group N on which G acts nilpotently, so that N is itself a nilpotent group. Then we consider a family of primes P and let e stand for the localization map at the family P. There is then an induced action of G_P on N_P. Let $a \in N$. Then we know (see Theorem 1.1 of [1]) that

$$G(a)_P = G_P(ea) \quad \textit{for } e\colon N \to N_P. \tag{3.2}$$

Plainly localization induces a map of exact sequences (2.1)

$$\begin{tikzcd} K(a) \arrow[r, tail, "\alpha_1"] \arrow[d, "e_1"] & G(a) \arrow[r, "\beta_1"] \arrow[d, "e_2"] & H \arrow[r, "\partial"] \arrow[d, "e_3"] & N\,|\,K \arrow[r, two heads, "\alpha_2"] \arrow[d, "e_4"] & N\,|\,G \arrow[d, "e_5"] \\ K_P(ea) \arrow[r, tail, "\alpha_{P1}"] & G_P(ea) \arrow[r, "\beta_{P1}"] & H_P \arrow[r, "\partial_P"] & N_P\,|\,K_P \arrow[r, two heads, "\alpha_{P2}"] & N_P\,|\,G_P \end{tikzcd} \tag{3.3}$$

THEOREM 3.1. *In* (3.3), e_1, e_2, e_3 *are localization maps, so that* $\alpha_{P1} = \alpha_{1P}$, $\beta_{P1} = \beta_{1P}$, *and* $(\beta_1 \, G(a))_P = \beta_{P1} \, G_P(ea)$. *Moreover, if H is commutative, then, in the disjoint unions, over the H-orbits of* $N\,|\,K$ *and the* H_P*-orbits of* $N_P\,|\,K_P$,

$$N\,|\,K = \amalg\, H/\beta_1 G(a),\ a \in N, \qquad N_P\,|\,K_P = \amalg\, H_P/\beta_{P1} G_P(a_P),\ a_P \in N_P,$$

e_4, *restricted to* $H/\beta_1 \, G(a)$, *is the localization map to* $H_P/\beta_{P1} \, G_P(ea)$.

Proof. It is only necessary to invoke (3.2) and the exactness of localization.

With regard to the second sequence, we revert to (3.1) but β is now a Q-map, where Q acts nilpotently on G. Moreover we assume that *Q acts trivially on K*.

Let $a \in G$. Then, if Q is nilpotent, localization induces a map of exact sequences (2.5)

$$
\begin{array}{ccccccccc}
Q(a) & \xrightarrow{\beta_1} & Q(\beta a) & \xrightarrow{\partial} & K & \xrightarrow{\rho} & G|Q & \xrightarrow{\beta_2} & H|Q \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_4} & & \downarrow{\scriptstyle e_5} \\
Q_P(ea) & \xrightarrow{\beta_{P1}} & Q_P(\beta_P ea) & \xrightarrow{\partial_P} & K_P & \xrightarrow{\rho_P} & G_P|Q_P & \xrightarrow{\beta_{P2}} & H_P|Q_P.
\end{array}
\tag{3.4}
$$

THEOREM 3.2. *In* (3.4) e_1, e_2, e_3 *are localization maps, so that* $\beta_{P1} = \beta_{1P}$, *and*

$$(\partial Q(\beta a))_P = \partial_P Q_P(\beta_P ea).$$

Moreover, if K is commutative, then, in the disjoint unions, over the K-orbits of $G|Q$ and the K_P-orbits of $G_P|Q_P$,

$$G|Q = \amalg\, K/\partial Q(\beta a),\ a \in G, \qquad G_P|Q_P = \amalg\, K_P/\partial_P Q_P(\beta_P a_P),\ a_P \in G_P,$$

e_4, restricted to $K/\partial Q(\beta a)$, is the localization map to $K_P/\partial_P Q_P(\beta_P ea)$.

If we wish to generalize these results further to the case when H, or K, is not commutative, we will have to consider the *localization of coset spaces*, the topic of the next section.

Remark. We note that, in Theorem 3.1, although not every H_P-orbit of $N_P|K_P$ is in the target of an H-orbit of $N|K$ under e_4, nevertheless, by Corollary 2.11, every H_P-orbit is isomorphic to a target orbit if H is commutative. For, given $a_P \in N_P$, there exist $a \in N$ and n prime to P such that $ea = a_P^n$. Then

$$
\begin{aligned}
H_P/\beta_{P1} G_P(a_P) &= H_P/\beta_{P1} G_P(a_P^n) \quad \text{(by Corollary 2.11)} \\
&= H_P/\beta_{P1} G_P(ea),
\end{aligned}
\tag{3.5}
$$

and the latter is the target of e_4 restricted to $H/\beta_1 G(a)$. Note, however, that the first equality in (3.5) does *not* justify us in asserting that $H_P/\beta_{P1} G_P(a_P)$ is itself a target orbit—we are dealing here with several copies of the same group, some in the target, some not.

A similar situation obtains with regard to Theorem 3.2. We assume that Q acts trivially on K and that K is central in G. Then, although not every K_P-orbit of $G_P|Q_P$ is in the target of a K-orbit of G/Q under e_4, nevertheless, by Corollary 2.12, every K_P-orbit is isomorphic to a target orbit. For, given $a_P \in G_P$, there exist $a \in G$ and n prime to P such that $ea = a_P^n$ and

$$
\begin{aligned}
K_P/\partial_P Q_P(\beta_P a_P) &\cong K_P/\partial_P Q_P(\beta_P a_P^n) \quad \text{(by Corollary 2.12)} \\
&= K_P/\partial_P Q_P(\beta_P ea),
\end{aligned}
$$

and the latter is the target of e_4 restricted to $K/\partial Q(\beta a)$.

Note that the paragraph above applies to the situation dealt with in [1]; for then Q is nilpotent and operates nilpotently on G, so that $\operatorname{nil}_Q G = c$. We then take $K = \Gamma_Q^c G$, so that Q operates trivially on K and K is in the centre of G.

4. Localizing coset spaces

To extend the study of the last section beyond the point at which H, in (2.1), or K, in (2.5), is commutative, it is necessary to consider the *localization of coset spaces*. Thus we consider in this section a nilpotent group G and subgroup H and we construct the right coset space G/H. Then the P-localization map $e\colon G \to G_P$ induces $e_*\colon G/H \to G_P/H_P$ and we propose to study e_* in this section.[1] The questions we will consider will be modelled on those of [1]. We first make a basic group-theoretical observation.

Let G be a group, K a subgroup of G, and L a normal subgroup of K. Then the group K/L acts on the right coset space G/L on the right by the rule

$$gL \cdot kL = gkL, \quad g \in G, \quad k \in K. \tag{4.1}$$

For if $g' = gl_1$, $k' = kl_2$, l_1, $l_2 \in L$, then $g'k' = gk(k^{-1}l_1k)l_2 = gkl$, $l \in L$, and

$$(gL \cdot k_1L) \cdot k_2L = gk_1k_2L = gL \cdot ((k_1L)(k_2L)), \; gL \cdot L = gL.$$

PROPOSITION 4.1. *The sequence*

$$K/L \rightarrowtail G/L \xrightarrow{\pi} G/K$$

is exact in the sense that K/L operates faithfully on G/L and the orbit set is in natural bijective correspondence with G/K.

Proof. The operation is faithful; for if $gL \cdot k_1L = gL \cdot k_2L$ then $gk_1 = gk_2l$, $l \in L$, so that $k_1 = k_2l$ and $k_1L = k_2L$. Moreover $\pi(g_1L) = \pi(g_2L) \Leftrightarrow g_1K = g_2K \Leftrightarrow g_1 = g_2k$, some $k \in K \Leftrightarrow g_1L = g_2L \cdot kL$, some $k \in K$.

This elementary proposition forms the basis for our further arguments when allied with the observation that if G is nilpotent and H is a subgroup of G then there is a normal series

$$H \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_r \lhd G; \tag{4.2}$$

indeed we may take N_i to be the normal closure of H in N_{i+1}, $i = 1, \ldots, r$, with $N_{r+1} = G$, $N_0 = H$.

THEOREM 4.2 (Cf. Theorem 1.2 of [1]). *Let G be finitely generated nilpotent, $H \subseteq G$, and let $S \subseteq T$ be families of primes. Then $G_T/H_T \to G_S/H_S$ is finite-one.*

Proof. We argue from (4.2), by backward induction on i, that $G_T/N_{iT} \to G_S/N_{iS}$ is finite-one, this being known in the case $i = r$. Thus if we

[1] It is plain that, in an appropriate category, e_* can be regarded as a P-localization, but this is not the aspect we wish to emphasize here.

write $L = N_{i-1}$, $K = N_i$, we are in the situation of Proposition 4.1 and have a map of exact sequences

$$
\begin{array}{ccccc}
K_T/L_T & \rightarrowtail & G_T/L_T & \xrightarrow{\pi_T} & G_T/K_T \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} \\
K_S/L_S & \rightarrowtail & G_S/L_S & \xrightarrow{\pi_S} & G_S/K_S
\end{array}
\tag{4.3}
$$

where e_1 is known to be finite-one and e_3 may be assumed finite-one by the inductive hypothesis. We want to prove that e_2 is finite-one. Let $a \in \operatorname{im} e_2$. Then $\pi_S a \in \operatorname{im} e_3$ so that $\pi_S a$ has finitely many counterimages in G_T/K_T. It thus suffices to prove that, for each $b \in G_T/K_T$ with $e_3 b = \pi_S a$, there exist only finitely many elements $c \in G_T/L_T$ with $\pi_T c = b$, $e_2 c = a$. Let us suppose there is such an element c and let c' be another such element. Since $\pi_T c' = \pi_T c$, it follows that $c' = c \cdot d$ with $d \in K_T/L_T$. Since $e_2 c' = e_2 c$, it follows that $e_2 c = e_2 c \cdot e_1 d$, so that $e_1 d = 1 \in K_S/L_S$. Since e_1 is finite-one, d belongs to a finite set, so that we have only finitely many choices for c'. This completes the inductive step.

THEOREM 4.3 (Cf. Example 1.6 of [1]). *Under the hypotheses of Theorem* 4.2, *there exists a cofinite family of primes P such that $G_T/H_T \to G_0/H_0$ is injective for $T \subseteq P$.*

Proof. We again employ backward induction, preserving the notation of the proof of Theorem 4.2. We find cofinite families P_1, P_3 such that $K_T/L_T \to K_0/L_0$ is injective if $T \subseteq P_1$ and $G_T/K_T \to G_0/K_0$ is injective if $T \subseteq P_3$. Then if $P_2 = P_1 \cap P_3$ and $T \subseteq P_2$ we have a map of exact sequences

$$
\begin{array}{ccccc}
K_T/L_T & \rightarrowtail & G_T/L_T & \twoheadrightarrow & G_T/K_T \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} \\
K_0/L_0 & \rightarrowtail & G_0/L_0 & \twoheadrightarrow & G_0/K_0
\end{array}
\tag{4.4}
$$

with e_1, e_3 injective. It quickly follows that e_2 is injective. Since P_2 is cofinite, the inductive step is complete.

Remark. Example 1.6 of [1] showed that the corresponding statement for orbit sets was *false*.

The strategy of proof now being evident, we are content to state our remaining results without proof.

THEOREM 4.4 (Local Hasse Principle for coset spaces; cf. Section 3 of [1]). *Let G be a nilpotent group, $H \subseteq G$, and, for each prime p, write e_p: $G/H \to G_p/H_p$ for the map induced by p-localization. Then:*

(i) *if $a, a' \in G/H$ and $e_p a = e_p a'$ for all primes p, then $a = a'$;*

(ii) *if G is finitely generated and if $a_p \in G_p/H_p$, for each prime p, such that $r_p a_p$ is independent of p, where r_p: $G_p/H_p \to G_0/H_0$ is induced by rationalization, then there exists $a \in G/H$ with $e_p a = a_p$ for each p.*

THEOREM 4.5 (Cf. Theorem 4.3 of [1]). *Let G be a nilpotent group, $H \subseteq G$, and let G be the local expansion* [1] *of $\check{G}$, thus $\check{G} = \prod_p G_p$ (and, similarly, $\check{H} = \prod_p H_p$). Then the square*

$$\begin{array}{ccc} G/H & \rightarrowtail & \check{G}/\check{H} \\ \downarrow & & \downarrow \\ G_0/H_0 & \longrightarrow & \check{G}_0/\check{H}_0 \end{array} \tag{4.5}$$

is cartesian.

Remark. Recall that $\check{G}_0$ means $(\check{G})_0$. Note, further, that in fact the lower horizontal arrow in (4.5) is injective. To see this, we may proceed, as usual, by observing that this is true when H is normal in G and then arguing by induction based on (4.2).

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