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THE GEOMETRY OF FINITE RANK DIMENSION GROUPS

BY

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George Elliott has recently observed [6] that an important class of C^*-algebras, those that are "approximately finite dimensional" (see [2]), are essentially classified by certain countable (necessarily torsion free) ordered abelian groups, which he called dimension groups. To formulate the latter notion, let $\mathbf{Z}^k$ be k-tuples of integers, ordered in the usual way by the set $(\mathbf{Z}^k)^+$ of k-tuples of non-negative integers. The dimension groups are just the ordered direct limit groups $\varinjlim \mathbf{Z}^{k(n)}$ that arise when one is given a sequence of positive group homomorphisms $\mathbf{Z}^{k(1)} \to \mathbf{Z}^{k(2)} \to \mathbf{Z}^{k(3)} \to \cdots$. The dimension groups and the closely related Riesz groups of Fuchs [8] (see Section 1) have been carefully investigated in [4], [7], [11] for the ultimate purpose of classifying the AF algebras. Perhaps, as A. Connes has suggested to us, they will also prove useful in the study of certain C^*-algebras with AF prototypes.

In this paper we shall use methods well known among convexity theorists to give an elementary and complete geometric description of the divisible finite rank dimension groups. Since one may also take the divisible hull of a dimension group (see Lemma 2.1), this has many implications for more general dimension groups. Some of these are discussed in Section 2. In particular, we have found a necessary and sufficient condition for an order simple finite rank Riesz group to be imbeddable in $\mathbf{R}^p$ with the usual ordering (see Theorem 2.3).

1. Divisible finite rank dimension groups

We use the notation $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ for the positive integers, the integers, the rationals, and the reals, respectively, and $\mathbf{Q}^+$, $\mathbf{R}^+$ for the non-negative rationals and reals. If V is a real vector space, we let V^* denote its dual vector space.

By an *ordered abelian group* we mean an abelian group G together with a subset $P = G^+$ such that $P + P \subseteq P$, $P \cap (-P) = 0$, and $P - P = G$. We write $a \leq b$ if $b - a \in P$. (G, P) is a *Riesz group* if in addition

(a) $a \in G$, $na \in G^+$ $(n \in \mathbf{N})$ imply $a \in G^+$, and
(b) given $a_i, b_j \in G$ $(i, j = 1, 2)$ with $a_i \leq b_j$, there exists a $c \in G$ with $a_i \leq c \leq b_j$ $(i, j = 1, 2)$.

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(b) is equivalent to the *Riesz decomposition property*:

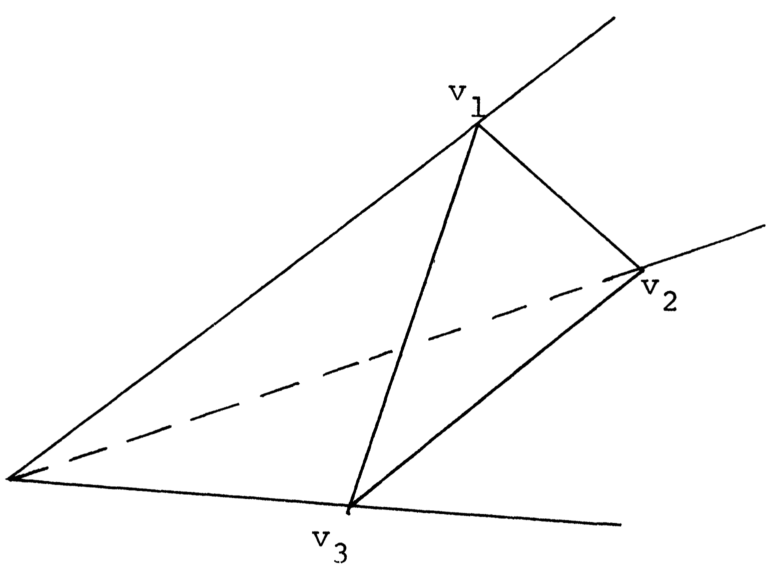
(b′) if $a_i, b_j \in G^+$ $(i, j = 1, 2)$ and $a_1 + a_2 = b_1 + b_2$, then there exist $c_{ij} \in G^+$ with $a_j = c_{1j} + c_{2j}$ and $b_i = c_{i1} + c_{i2}$ (see [8, Theorem 2.3]).

All dimension groups are Riesz groups [7, Section 2.7] and it is also known that countable divisible Riesz groups are dimension groups [11, Prop. 3.5]. Thus throughout this section we shall use (a) and (b) as an intrinsic characterization of the divisible dimension groups. If G is divisible and of finite rank r, we may identify G with the additive group of the rational vector space $\mathbf{Q}^r$. In this case $\alpha G^+ \subseteq G^+$ for $\alpha \in \mathbf{Q}^+$, since if $\alpha = p/q$, $p, q \in \mathbf{N}$ and $a \in G^+$, then $q\alpha a = pa \in G^+$. We regard $\mathbf{Q}^r$ as a subset of $\mathbf{R}^r$ and we give it the relative topology. We let e_i, $1 \leq i \leq r$, be the usual basis in $\mathbf{R}^r$.

An *order ideal* J in a Riesz group G is a subgroup such that $J = J^+ - J^+$ (where $J^+ = J \cap G^+$), and if $0 \leq a \leq b \in J^+$, then $a \in J^+$. If $G = \mathbf{Q}^r$, then J must be a subspace of the rational vector space $\mathbf{Q}^r$ since if $\alpha = p/q$, $p, q \in \mathbf{N}$, then for $a \in J^+$, $0 \leq \alpha a \leq pa \in J^+$. We say that a Riesz group G is *order simple* if $\{0\}$ and G are its only order ideals.

Turning to linear theory, we recall that a subset of a real vector space V is a *cone* if $C + C \subseteq C$ and $\alpha C \subseteq C$ for all $\alpha \in \mathbf{R}^+$. A cone must be convex. C is *proper* (resp. *generating*) if $C \cap (-C) = \{0\}$ (resp. $C - C = V$). If C is proper, we let $\leq$ be the corresponding *linear order* on V, i.e., $v \leq w$ if $w - v \in C$. C is *simplicial* if there are linearly independent elements $v_1, \ldots, v_d \in V$ which generate C, i.e.,

$$C = \{\textstyle\sum \alpha_i v_i \colon \alpha_i \geq 0\}.$$



Given a closed cone C in $\mathbf{R}^r$, we say that C is *cosimplicial* if the *dual cone*

$$C^* = \{f \in V^*: f \mid C \geq 0\}$$

is simplicial. From the Bipolar Theorem (see [3, p. 51]), it is equivalent to assume there exist linearly independent $f_1, \ldots, f_d \in (\mathbf{R}^d)^*$ such that

$$C = \{v: f_k(v) \geq 0, k = 1, \ldots, d\}.$$

LEMMA 1.1. *Suppose that* $(\mathbf{Q}^r, P)$ *is a Riesz group. Then the closure* $\bar{P}$ *is a cosimplicial cone in* $\mathbf{R}^r$.

Proof. We have that $P + P \subseteq P$, and from above, $\alpha P \subseteq P$ for $\alpha \in \mathbf{Q}^+$. It follows that $\bar{P}$ is a closed cone in $\mathbf{R}^r$. P^* is proper since if $f \in P^* \cap (-P^*)$, then $f \mid P = 0$, or since $\mathbf{Q}^r = P - P, f \mid \mathbf{Q}^r = 0$ and by continuity $f = 0$. We claim that P^* is a lattice cone, i.e., in the relative ordering on P^* defined by P any two elements $f, g \in P^*$ have least upper bound and greatest lower bound. We define $f \vee g$ on P by

$$(f \vee g)(a) = \sup\{f(a_1) + g(a_2): a = a_1 + a_2, a_i \in P\}.$$

This exists since if $a = a_1 + a_2, 0 \leq a_i \leq a$, then $f(a_1) + g(a_2) \leq (f + g)(a)$. We claim that $f \vee g$ is additive on P. If $a = b + c$, b, $c \in P$, let $b = b_1 + b_2$, $c = c_1 + c_2$. Then since $a = (b_1 + b_2) + (c_1 + c_2)$,

$$f(b_1) + g(b_2) + f(c_1) + g(c_2) = f(b_1 + c_1) + g(b_2 + c_2) \leq (f \vee g)(a),$$

i.e.,

$$(f \vee g)(b) + (f \vee g)(c) \leq (f \vee g)(a).$$

On the other hand if $a = a_1 + a_2$, then using the Riesz decomposition property, we may select $b_i, c_j \in P$ as described by the following table:

$$\begin{array}{c|cc} & a_1 & a_2 \\ \hline b & b_1 & b_2 \\ c & c_1 & c_2 \end{array}$$

(the rows add up to b and c, the columns to a_1, a_2). Then

$$\begin{aligned} f(a_1) + g(a_2) &= f(b_1) + g(b_2) + f(c_1) + g(c_2) \\ &\leq (f \vee g)(b) + (f \vee g)(c), \end{aligned}$$

i.e. $(f \vee g)(a) \leq (f \vee g)(b) + (f \vee g)(c)$, and equality follows. We may then extend $f \vee g$ to $\mathbf{Q}^d = P - P$ by letting

$$(f \vee g)(a_1 - a_2) = (f \vee g)(a_1) - (f \vee g)(a_2),$$

the non-ambiguity being a consequence of the additivity of $f \vee g$ on P. $f \vee g$ is clearly an additive homomorphism on $\mathbf{Q}^d$. Given $\alpha \in \mathbf{Q}$, $\alpha > 0$, and $a \in P$, we have $(f \vee g)(\alpha a) = \alpha(f \vee g)(a)$. To see this, note that if $a = a_1 + a_2$, $a_i \in P$, then $\alpha a = \alpha a_1 + \alpha a_2$, $\alpha a_i \in P$ and

$$\alpha[f(a_1) + g(a_2)] = f(\alpha a_1) + g(\alpha a_2) \leq (f \vee g)(\alpha a);$$

hence

$$\alpha(f \vee g)(a) \leq (f \vee g)(\alpha a).$$

Equality follows since $1/\alpha(f \vee g)(\alpha a) \leq f \vee g(a)$. Given $\alpha > 0$ and $a \in \mathbf{Q}^d$, $a = a_1 - a_2$, $a_i \in P$, we have $\alpha a = \alpha a_1 - \alpha a_2$ and

$$\begin{aligned}(f \vee g)(\alpha a) &= (f \vee g)(\alpha a_1) - (f \vee g)(\alpha a_2)\\ &= \alpha(f \vee g)(a_1) - \alpha(f \vee g)(a_2)\\ &= \alpha(f \vee g)(a).\end{aligned}$$

Finally if $\alpha < 0$, and $a = a_1 - a_2$, $a_i \in P$, then $\alpha a = (-\alpha)a_2 - (-\alpha)a_1$, and

$$\begin{aligned}(f \vee g)(\alpha a) &= (-\alpha)(f \vee g)(a_2) - (-\alpha)(f \vee g)(a_1)\\ &= \alpha(f \vee g)(a).\end{aligned}$$

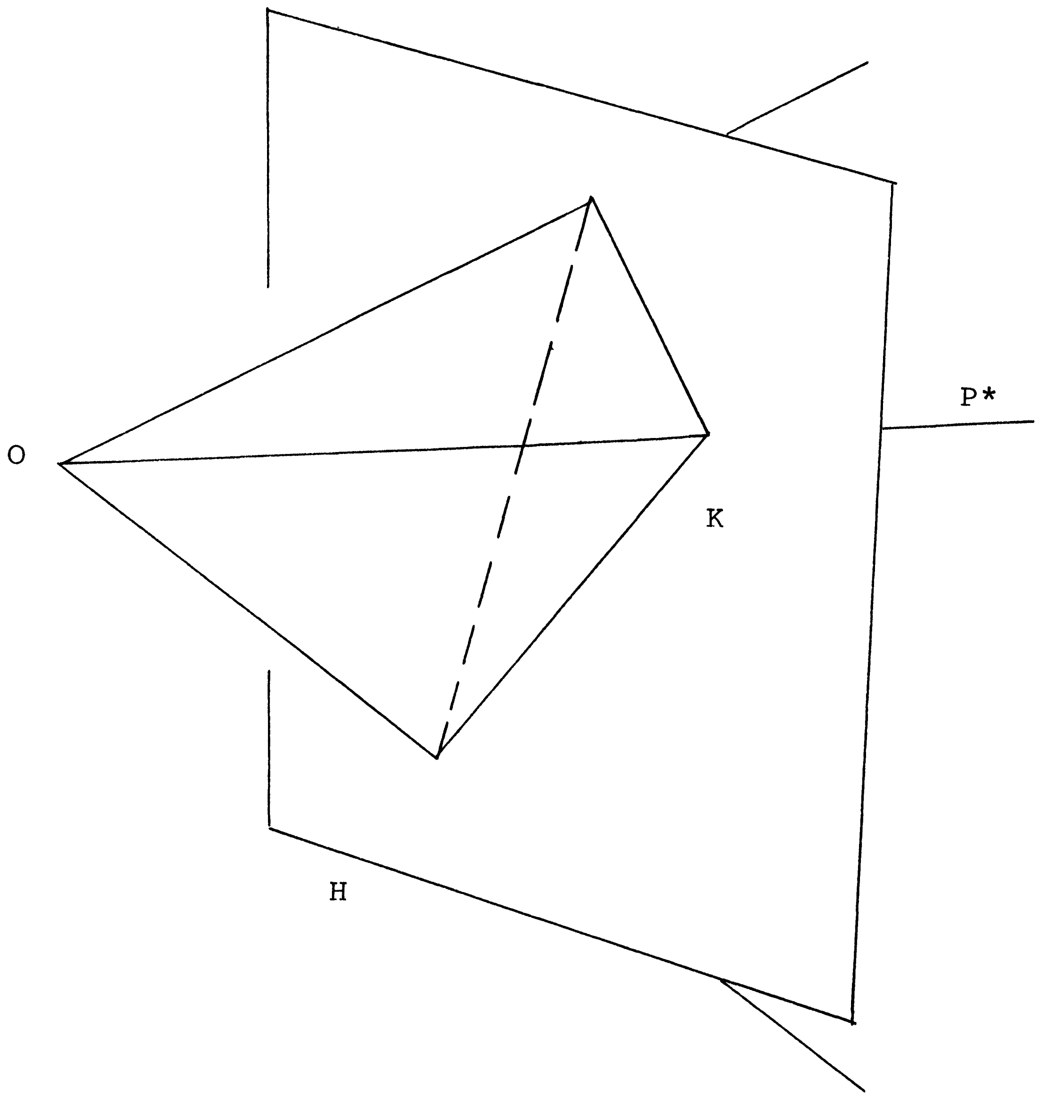
We conclude that $f \vee g \colon \mathbf{Q}^d \to \mathbf{R}$ is rationally linear, and letting $t_i = f \vee g(e_i)$ we may extend it to an element of $(\mathbf{R}^d)^*$ by letting $(f \vee g)(v) = \sum \alpha_i t_i$, $(v = \sum \alpha_i e_i \in \mathbf{R}^d)$. We have that $f \leq f \vee g$ since if $a \in P$, $a = a + 0$ implies $f(a) = f(a) + g(0) \leq f \vee g(a)$ and similarly, $g \leq f \vee g$. On the other hand, given f, $g \leq h \in P^*$, then $a = a_1 + a_2$, $a_i \in P$ implies

$$f(a_1) + g(a_2) \leq h(a_1) + h(a_2) = h(a);$$

hence $f \vee g \leq h$. It is now a simple matter to verify that $f \wedge g = (f + g) - (f \vee g)$ is the greatest lower bound for f and g in P^*.

It is well known to convexity theorists that a proper closed lattice cone in a finite-dimensional space must be simplicial. In convexity terminology, there must exist a geometric simplex K on a hyperplane H not passing through 0 which generates the cone P^* (for some highly instructive pictures, see [3, p. 159]). Fortunately, Phelps has given a completely elementary proof of this result [10]. To begin with, let v be a linear functional on $(\mathbf{R}^r)^*$ (equivalently, an element of $\mathbf{R}^r$) which is strictly positive on $P^* \backslash \{0\}$. The existence of v is guaranteed by the fact that P^* is closed and thus locally compact, and one may appeal to a theorem of Klee (see [1, p. 83]). We let

$$H = \{f \in (R^r)^* \colon v(f) = 1\} \quad \text{and} \quad K = H \cap P^*.$$



Without appealing to Choquet theory, Phelps used the lattice ordering of **P*** to prove that K is a geometric simplex [10, pp. 58–62, 75–76 (note the last remark)]. The extreme points of K provide the desired functions f_k, $k = 1, \ldots, d$. Q.E.D.

LEMMA 1.2. *If* $(\mathbf{Q}^r, P)$ *is a dimension group, then* P *has interior in* $\mathbf{Q}^r$.

Proof. We may let $e_i = a_i - b_i$ $(1 \leq i \leq r)$, a_i, $b_j \in P$. If $a = \sum b_i \in P$ then, for each i, $a + e_i \in P$. But P must be rationally convex since if $p, q \in \mathbf{N}$, $a, b \in P$ implies $pa + qb \in P$ and hence

$$(pa + qb)/(p + q) \in P.$$

Thus the rational convex hull of $\{a, a + e_1, \ldots, a + e_r\}$ lies in P. The latter is just the intersection of a geometric simplex in $\mathbf{R}^r$ with $\mathbf{Q}^r$, and must have interior in $\mathbf{Q}^r$. Q.E.D.

THEOREM 1.3. *If* $(\mathbf{Q}^r, P)$ *is a dimension group, then there exist linearly independent elements* $f_1, \ldots, f_d \in (\mathbf{R}^d)^*$ *such that*

$$\operatorname{int} P = \{a \in \mathbf{Q}^r : f_k(a) > 0,\ k = 1, \ldots, d\}$$

(the interior is taken relative to $\mathbf{Q}^r$*).*

Proof. From Lemma 1.1, there exist $f_1, \ldots, f_d \in (\mathbf{R}^r)^*$ with

$$\bar{P} = \{a \in \mathbf{Q}^r : f_k(a) \geq 0,\ k = 1, \ \ldots, d\}. \tag{1.1}$$

It is evident that if $a \in \operatorname{int} P$, then $f_k(a) > 0$ for all k. Conversely suppose that $f_k(a) > 0$ for all k, but $a \notin \operatorname{int} P$. Since P has interior (Lemma 1.2), we may choose a bounded open set $B \subseteq P$. Choosing $\varepsilon \in \mathbf{Q}$, $\varepsilon > 0$, sufficiently small, we may assume that $f_k \mid a - \varepsilon B > 0$ for all k. But we have that $(a - \varepsilon B) \cap P = \emptyset$ since otherwise given $c \in \varepsilon B$ $(\subseteq P)$ with $a - c \geq 0$, we will have that $a \geq 0$, a contradiction. But $a - \varepsilon B$ is open, hence $(a - \varepsilon B) \cap \bar{P} = \emptyset$. This contradicts the fact that from (1.1), $a - \varepsilon B \subseteq \bar{P}$. Q.E.D.

THEOREM 1.4. *If* $(\mathbf{Q}^r, P)$ *is an order simple dimension group, then there exist linearly independent elements* $f_1, \ldots, f_d \in (\mathbf{R}^r)^*$ *such that*

$$P = \{a \in \mathbf{Q}^r : f_k(a) > 0,\ k = 1, \ldots, d\} \cup \{0\}.$$

Conversely given such a set P, $(\mathbf{Q}^r, P)$ *must be a simple Riesz group.*

Proof. Assume that $(\mathbf{Q}^r, P)$ is order simple. From Theorem 1.3, it suffices to show that if $a \in P$, then $f_k(a) > 0$ for all k. Letting

$$H_k = \{a \in P : f_k(a) = 0\}, \tag{1.2}$$

it thus suffices to show that $H_k \cap P = \{0\}$ for all k. Note that if $0 \leq a \leq b \in H_k \cap P$, then $a \in H_k \cap P$, since a, $b - a \in P$ imply that $f_k(a), f_k(b - a) \geq 0$, i.e., $f_k(b) \geq f_k(a) \geq 0$. It follows that $J = H_k \cap P - H_k \cap P$ is an order ideal in $\mathbf{Q}^r$ (see [8, Prop. 5.1]). Now $J \neq \mathbf{Q}^r$, since $J = \mathbf{Q}^r$ would imply $f_k = 0$, a contradiction. Thus $J = \{0\}$, and $H_k \cap P = \{0\}$.

Conversely given such a set P, it is evident that $P + P \subseteq P$ and $P \cap (-P) = \{0\}$. To see that $P - P = \mathbf{Q}^r$, extend $f_1, \ldots, f_d$ to a vector basis $f_1, \ldots, f_d, \ldots, f_r$ for

$(\mathbf{R}^r)^*$, and let $v_1, \ldots, v_r$ be the dual basis in $\mathbf{R}^r$ (i.e., $f_k(v_j) = \delta_{jk}$). The set $A = \{\sum_{j=1}^r \alpha_j v_j \colon \alpha_j > 0\}$ is open in $\mathbf{R}^r$ and contains elements arbitrarily close to each v_i, and the latter is therefore also true for $A \cap \mathbf{Q}^r$. If we slightly perturb the v_i, we will still have a basis, and we thus obtain a basis v_i' for $\mathbf{R}^r$ lying in $A \cap \mathbf{Q}^r$. Since the latter is a subset of P, $P - P = \mathbf{Q}^r$. Given $a \in \mathbf{Q}^r$ with $na \in P$ $(n \in \mathbf{N})$, it is immediate that $a \in P$. Finally, suppose that relative to P, $a_i \leq b_j$ $(i, j = 1, 2)$. If $a_1 = b_1$, we will have $a_i \leq a_1 \leq b_j$. Thus we may assume that $b_j - a_i \in P \backslash \{0\}$. Then for all k, $f_k(a_i) < f_k(b_j)$. Letting $\alpha_k = \max\{f_k(a_1), f_k(a_2)\}$, and $\beta_k = \min\{f_k(b_1), f_k(b_2)\}$, we have $\alpha_k < \beta_k$. It follows that

$$\begin{aligned} B &= \{v \in \mathbf{R}^r \colon \alpha_k < f_k(v) < \beta_k,\ k = 1, \ldots, r\} \\ &= \{\textstyle\sum \gamma_k v_k \colon \alpha_k < \gamma_k < \beta_k,\ k = 1, \ldots, r\} \end{aligned}$$

is non-empty and open, and thus $B \cap \mathbf{Q}^r$ is non-empty. If $c \in B \cap \mathbf{Q}^r$, then $a_i \leq c \leq b_j$, and we are done. Q.E.D.

The situation for non-simple dimension groups $(\mathbf{Q}^r, P)$ is now reasonably clear. Since it is somewhat cumbersome, we will only sketch the details. The maximal order ideals of $(\mathbf{Q}^r, P)$ are generated by the "facial intersections" $P_k = H_k \cap P$ (see the proof of Theorem 1.4). Then $J_k = P_k - P_k$ is a rational subspace of $\mathbf{Q}^r$, and (J_k, P_k) is again a dimension group. P_k will itself have facial intersections $P_{kl} = H_{kl} \cap P_k$. In this manner we obtain successive decompositions

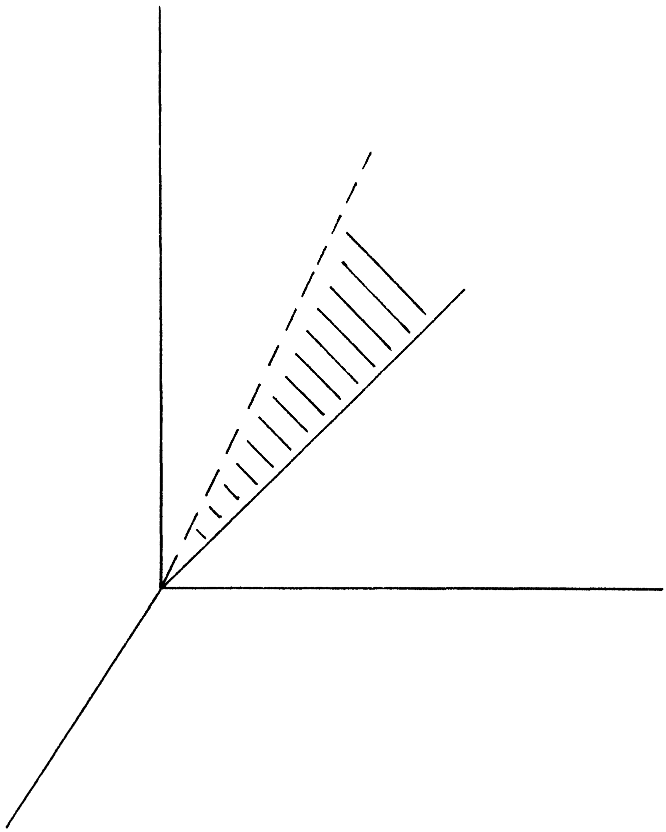
$$P = (\operatorname{int} P) \cup (\bigcup P_k) = (\operatorname{int} P) \cup \bigcup_k (\operatorname{int} P_k) \cup \bigcup_{k,l} P_{kl} = \cdots$$

which must terminate in at most r steps. Conversely by taking an open cosimplicial cone intersected with $\mathbf{Q}^r$ and "decorating" its faces with smaller cosimplicial cones, we again obtain dimension groups. Some examples for $\mathbf{Q}^2$ are

$$\begin{aligned} P_1 &= \{(\alpha, \beta) \colon \alpha, \beta > 0\} \cup \{0\}, \\ P_2 &= \{(\alpha, \beta) \colon \alpha > 0, \beta > 0\} \cup \{(0, \beta) \colon \beta > 0\} \cup \{0\}, \\ P_3 &= \{(\alpha, \beta) \colon \alpha > 0, \beta > 0\} \cup \{(0, \beta) \colon \beta > 0\} \\ &\quad \cup \{(\alpha, 0) \colon \alpha > 0\} \cup \{0\}. \end{aligned}$$

We have that $(\mathbf{Q}^2, P_1)$ is simple, $(\mathbf{Q}^2, P_2)$ has one non-trivial order ideal, whereas $(\mathbf{Q}^2, P_3) = \mathbf{Q} \oplus_{\text{ord}} \mathbf{Q}$. An interesting rank 3 example is $(\mathbf{Q}^3, P)$, where

$$\begin{aligned} P = {} & \{(\alpha, \beta, \gamma) \colon \alpha > 0, \beta > 0, \gamma > 0\} \\ & \cup \{(0, \beta, \gamma) \colon \beta < \gamma < 2\beta;\ \beta, \gamma > 0\} \\ & \cup \{(0, \beta, \beta) \colon \beta > 0\} \cup \{0\}. \end{aligned}$$



2. Non-divisible finite rank Riesz groups

Let us suppose that (G, G^+) is a Riesz group of rank r. Then we may regard G as a subgroup of $\mathbf{Q}^r$, where $\mathbf{Q}^r$ is the *divisible hull* of G, i.e., for all $a \in \mathbf{Q}^r$, there exists an $n \in \mathbf{N}$ such that $na \in G$ (see [9, Section 19]). We define

$$P = \{a \in \mathbf{Q}^r\colon na \in G^+ \text{ for some } n \in \mathbf{N}\}.$$

It is immediate that

$$P \cap G = G^+ \tag{2.1}$$

(this will imply below that if $a, b \in G$ and $a \leq b$ in $\mathbf{Q}^r$, then $a \leq b$ in G).

LEMMA 2.1. *$(\mathbf{Q}^r, P)$ is a Riesz group, and the map $J^+ \mapsto J^+ \cap G$ determines a one-to-one correspondence between the order ideals of $(\mathbf{Q}^r, P)$ and of (G, G^+).*

Proof. We have $P + P \subseteq P$ since if $ma, nb \in G^+$ for $a, b \in \mathbf{Q}^r$; $m, n \in \mathbf{N}$, then $mn(a + b) \in G^+$ implies $a + b \in P$. P is proper, i.e., $P \cap (-P) = \{0\}$ because given $ma \in G^+$ and $na \in -G^+$ we have $mna \in G^+ \cap (-G^+) = \{0\}$, i.e., $a = 0$. $P - P = G$ since given $na \in G$, $na = g_1 - g_2, g_i \in P$ implies $a = a_1 - a_2$ where $a_i = n^{-1}g_i \in P$. If $na \in P$ for $n \in \mathbf{N}$, then $mna \in G^+$ for some $m \in \mathbf{N}$ and $a \in P$. Finally, if $a_i, b_j \in P$ $(i, j = 1, 2)$, and $a_1 + a_2 = b_1 + b_2$, choose $n \in \mathbf{N}$ with $na_i, nb_j \in G^+$ $(i, j = 1, 2)$. Then $na_1 + na_2 = nb_1 + nb_2$ and we may select $c_{ij} \in G^+$ with $na_j = \sum_i c_{ij}$, $nb_i = \sum_j c_{ij}$. Then we have $a_j = \sum_i c'_{ij}$, $b_i = \sum_j c'_{ij}$, where $c'_{ij} = n^{-1}c_{ij} \in P$.

A subset S of a Riesz group is the positive part of a (necessarily unique) order ideal if and only if $0 \in S$, $S + S \subseteq S$ and $0 \leq a \leq b \in S$ implies $a \in S$ (see [8, Section 5]). Given an order ideal J in $\mathbf{Q}^r$, it is evident that $J^+ \cap G$ has these properties in G. On the other hand if I is an order ideal in G,

$$S(I^+) = \{a \in \mathbf{Q}^d\colon na \in I^+ \text{ for some } n \in \mathbf{N}\}$$

is the positive part of an order ideal in $\mathbf{Q}^r$. Since we have

$$S(J^+ \cap G) = J^+ \quad \text{and} \quad S(I^+) \cap G = I^+,$$

we have the desired one-to-one correspondence. Q.E.D.

We shall say that $(\mathbf{Q}^r, P)$ is the *divisible hull* of (G, G^+).

THEOREM 2.2. *Suppose that (G, G^+) is a finite rank order simple Riesz group with divisible hull $(\mathbf{Q}^r, P)$. Then there exist linearly independent elements $f_1, \ldots, f_d \in (\mathbf{R}^r)^*$ such that*

$$G^+ = \{a \in G\colon f_k(a) > 0, k = 1, \ldots, d\} \cup \{0\}. \tag{2.2}$$

Proof. From Lemma 2.1, $(\mathbf{Q}^r, P)$ is order simple, and thus there exist $f_1, \ldots, f_d \in (\mathbf{R}^r)^*$ satisfying Theorem 1.4. (2.2) is then a consequence of (2.1). Q.E.D.

Let G be a finite rank Riesz group with divisible hull $(\mathbf{Q}^r, P)$ and let $f_1, \ldots, f_d \in (\mathbf{R}^r)^*$ be as in Theorem 1.3. We define H_k by (1.2) and for each k we let

$$H_k^+ = \{a \in H_k\colon f_j(a) \geq 0, j \neq k\}.$$

We will say that G is *positively irrational* if

$$H_k^+ \cap G = \{0\}(k = 1, \ldots, d) \quad \text{and} \quad \bigcap_k H_k \cap G = \{0\}.$$

If $a \in H_k^+ \cap \mathbf{Q}^r$, then selecting $n \in \mathbf{N}$ with $na \in G$, we have $na \in H_k^+ \cap G$. Thus it is equivalent to assume $H_k^+ \cap \mathbf{Q}^r = \{0\}$. We note that from (1.1), $H_k^+ = H_k \cap \bar{P}$.

Letting $(\mathbf{R}^p)^+ = (\mathbf{R}^+)^p$, we have:

THEOREM 2.3. *A finite rank Riesz group G is order isomorphic to an order simple subgroup of $(\mathbf{R}^p, (\mathbf{R}^p)^+)$ for some $p \in \mathbf{N}$ if and only if it is positively irrational.*

Proof. Since $H_k \cap P = H_k^+ \cap P = \{0\}$, $(\mathbf{Q}^r, P)$ is order simple, and from Lemma 2.1, so is G. We define $\theta: G \to \mathbf{R}^d$ by

$$\theta(a) = (f_1(a), \ldots, f_d(a)).$$

θ is an algebraic injection since $\ker \theta = \bigcap_k (H_k \cap G)$. On the other hand, since $H_k^+ \cap \mathbf{Q}^r = \{0\}$ for all k, $\theta(a) \geq 0$ if and only if $a = 0$ or $f_k(a) > 0$ for all k, hence from Theorem 2.2, if and only if $a \in P$.

Conversely suppose that G is a finite rank additive subgroup of $\mathbf{R}^p$ which is an order simple Riesz group in the relative ordering

$$G^+ = G \cap (\mathbf{R}^p)^+. \tag{2.3}$$

We let $g_1, \ldots, g_p \in (\mathbf{R}^p)^*$ be the co-ordinate maps, i.e., the dual basis to $\{e_j\}$. We may assume that G is not contained in any of the co-ordinate hyperplanes

$$K_i = \{w \in \mathbf{R}^p: g_i(w) = 0\}$$

(otherwise replace $\mathbf{R}^p$ by $\mathbf{R}^{p-1}$, etc.). Letting $K_i^+ = K_i \cap (\mathbf{R}^p)^+$, it is evident that $K_i^+ \cap G = K_i \cap G^+$ is the positive part of an order ideal. If $G^+ \subseteq K_i$, then $G \subseteq K_i$, a contradiction. Thus

$$K_i^+ \cap G = \{0\}. \tag{2.4}$$

The inclusion map $G \hookrightarrow \mathbf{R}^p$ has a unique rational linear extension $\phi: \mathbf{Q}^r \to \mathbf{R}^p$, where $(\mathbf{Q}^r, P)$ is the divisible hull. We let $\phi(e_i) = w_i$, and extend ϕ to a real linear map $\bar{\phi}: \mathbf{R}^r \to \mathbf{R}^p$ by letting

$$\bar{\phi}\left(\sum_i \alpha_i e_i\right) = \sum_i \alpha_i w_i, \quad \alpha_i \in \mathbf{R}.$$

Then $h_j = g_j \circ \bar{\phi} \in (\mathbf{R}^r)^*$ and from (2.3),

$$G^+ = \{a \in G: g_j(a) \geq 0,\ 1 \leq j \leq p\} = \{a \in G: h_j(a) \geq 0,\ 1 \leq j \leq p\},$$
$$P = \{a \in \mathbf{Q}^r: h_j(a) \geq 0,\ 1 \leq j \leq p\},$$

and thus $\bar{P} = \{v \in \mathbf{R}^r: h_j(v) \geq 0,\ 1 \leq j \leq p\}$. From the Bipolar Theorem [3, p. 51], the dual cone P^* is the smallest cone containing $h_1, \ldots, h_p$. However P^* is also generated by the linearly independent elements $f_1, \ldots, f_d$ described in Theorem 1.3. Letting H be a hyperplane containing $f_1, \ldots, f_d$, $K = H \cap P^*$ is a geometric simplex with extreme points $f_1, \ldots, f_d$. Multiplying the h_j by positive constants, we may assume the h_j lie in H and thus in K. Since P^* is generated by the h_j, K is the convex hull of the h_j. It follows that (see [3, Section 25.14])

$$\{f_1, \ldots, f_d\} \subseteq \{h_1, \ldots, h_p\} \subseteq \text{convex hull } \{f_1, \ldots, f_d\}.$$

Suppose that $a \in G$ is such that $f_k(a) = 0$ and $f_j(a) \geq 0$ $(j \neq k)$. Choosing i with $f_k = h_i$, we have $g_i(a) = h_i(a) = f_k(a) = 0$, and for $l \neq i$, $g_l(a) = h_l(a) \geq 0$ since h_l is a convex combination of the $f_1, \ldots, f_d$. From (2.4), $a = 0$. On the other hand, if $a \in G$ is such that $f_j(a) = 0$ for all j, then $g_l(a) = h_l(a) = 0$ for all l, i.e., $a \in \bigcap_l K_l = \{0\}$. Thus G is positively irrational. Q.E.D.

The *construction* of non-divisible Riesz groups seems to be much more subtle. One can no longer choose an arbitrary cosimplicial cone C in $\mathbf{R}^r$ and expect $C \cap G$ to determine a simple Riesz ordering (see the classification of the Riesz groups $(\mathbf{Z}^2, P)$ in [4]). One has, for example:

PROPOSITION 2.4. *Suppose that (G, G^+) is a finitely generated order simple Riesz group, and that $f_1, \ldots, f_d \in (\mathbf{R}^r)^*$ are selected as in Theorem 2.2. Then one must have $d \leq r - 1$.*

Proof. We may assume that $G = \mathbf{Z}^r \subseteq \mathbf{Q}^r$. Suppose that to the contrary, $d = r$. Then $f_1, \ldots, f_r$ will be a basis for $(\mathbf{R}^r)^*$ and we may select a dual basis $v_1, \ldots, v_r \in \mathbf{R}^r$. Fixing an element $a_0 \in G^+$, we have $a_0 = \sum \alpha_i v_i$, where $\alpha_i = f_i(a_0) > 0$. The set $D = \{a \in G\colon 0 \leq a \leq a_0\}$ is contained in the compact set

$$K = \{v \in \mathbf{R}^r\colon 0 \leq f_k(a) \leq \alpha_i\} = \{\textstyle\sum \beta_i v_i\colon 0 \leq \beta_i \leq \alpha_i\}.$$

But K can contain only finitely many of the lattice points $G = \mathbf{Z}^r$, hence D is a finite set. It follows that D and thus G^+ must contain minimal elements, contradicting the simplicity of G (see [11, Section 2]). Q.E.D.

Another restriction for finitely generated simple Riesz groups (G, P) may be discovered in the proof of [4, Theorem 2.1]. Suppose that $G = \mathbf{Z}^n$ and

$$P = \{a \in G\colon f_1(a) > 0\} \cup \{0\}, f_1 \in (\mathbf{R}^r)^*,$$

and let us identify $(\mathbf{R}^r)^*$ with $\mathbf{R}^r$ by using the pairing

$$(\alpha, \beta) \mapsto \textstyle\sum \alpha_i \beta_i.$$

Then if $f = (\beta_1, \ldots, \beta_r)$, not all of the β_i can be rational, since that would imply the existence of minimal elements. On the other hand the β_i need not be independent over $\mathbf{Q}$ since $(\mathbf{Z}^3, P)$ with

$$P = \{(k, l, m) \in \mathbf{Z}^3\colon k + l + \sqrt{2}\, m > 0\} \cup \{(0, 0, 0)\}$$

is a simple (non-totally ordered) Riesz group.

Of course after one succeeds in geometrically constructing a Riesz group, one must then determine if it is a dimension group before one has found a new operator algebra. Methods for solving this problem will be explored in a subsequent paper.

We conclude by remarking that Choquet has formulated a version of simplex theory that should be appropriate for countable Riesz groups of infinite rank (see [3, Section 30]).

Added in proof. It is shown in [5] that all Riesz groups are dimension groups, and a more complete analysis of the finite rank case is given there.

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