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POSITIVE OPERATORS ON SPACES OF BAIRE FUNCTIONS

BY

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The symbol θ will be used to denote the zero element of any vector space. Suppose L is a Riesz space (lattice ordered vector space). For notation and basic terminology concerning Riesz spaces, the reader is referred to Luxemburg and Zaanen [6]. The sequence $f_1, f_2, f_3, \ldots$ of L is said to be *order Cauchy* if there exists a sequence $y_1 \geq y_2 \geq \cdots \geq \theta$, $\bigwedge y_n = \theta$ such that, for $m \geq n$, $|f_m - f_n| < y_n$. If every order Cauchy sequence converges, then L is *order Cauchy complete.* It is *order separable* if each subset with a supremum has a countable subset with the same supremum. Also, L is *almost σ-complete* if it is Riesz isomorphic to a subspace $L^\sim$ of a σ-complete space M with the property that if $m \in M^+$, there is a sequence $\theta \leq u_1 \leq u_2 \leq \cdots$, $u_n \in L^\sim$ such that $\bigvee u_n = m$. In particular, if L is order separable, it is almost σ-complete (Aliprantis and Langford [2]). The Riesz space L is *universally complete* if it is complete and every disjoint subset of L^+ has a supremum.

Suppose X is a set and Ω is a collection of real valued functions defined on X. Then $B_1(\Omega)$ (the first Baire class of Ω) is the set of all pointwise limits of sequences of Ω, $B_2(\Omega) = B_1(B_1(\Omega))$, and in general if α is an ordinal, $\alpha > 0$, $B_\alpha(\Omega)$ is the family of pointwise limits of sequences from $\bigcup_{\alpha > \gamma} B_\gamma(\Omega)$. If ω_1 is the first uncountable ordinal then $B_{\omega_1}(\Omega) = B_{\omega_1 + 1}(\Omega)$ which will be denoted $B(\Omega)$. For a discussion of Baire spaces see Mauldin [7] or [8].

Let LS Ω (lower semi-Ω) be the set of pointwise limits of non-decreasing sequences from Ω, US Ω be the set of pointwise limits of non-increasing sequences in Ω, and Ω^* be the set of bounded functions in Ω.

Spaces of the form $B(\Omega)$ include the set of all A measurable functions for some σ-algebra A and the σ-laterally complete function spaces as discussed in Chapter 7 of Aliprantis and Burkinshaw [1].

A complete ordinary function system Ω is a linear lattice of functions containing the constants which is uniformly closed, which is a ring, and which is closed under inversion (if $f \in \Omega$ and $f > 0$, then $1/f \in \Omega$). In particular, each space $C(X)$ of all continuous functions on a topological space is a complete ordinary function system and if Ω is a linear lattice containing the constant functions, then $B_1(\Omega)$ is a complete ordinary function system (Mauldin, [7, Theorem 8]).

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THEOREM 1. *Suppose Ω is a complete ordinary function system and φ is a positive linear functional on $B_1(\Omega)$. Then φ is the sum of a finite number of Riesz homomorphisms into the real numbers.*

Proof. Let $Z(\Omega)$ denote the collection of zero sets of functions in Ω. Since Ω is uniformly closed, $Z(\Omega)$ is closed under countable intersection (see Gilman and Jerison [5, p. 16]). Let ω be the collection of subsets M of X such that χ_M, the characteristic function of M, belongs to $B_1(\Omega)$ and $\varphi(\chi_M) > 0$. Suppose $M \in \omega$. By a theorem of Sierpinski [10], there exists a sequence $f_1 \le f_2 \le f_3 \le \ldots$ of functions in US (Ω) converging pointwise to X_M. Replace each f_n with $2f_n \wedge 1$. Then $M = \bigcup f_n^{-1}(1)$. Each $f_n^{-1}(1)$ is the countable intersection of sets in $Z(\Omega)$ and is thus a zero set itself. Therefore M is the union of a countable collection of sets in $Z(\Omega)$. Now φ is sequentially continuous with respect to monotone pointwise convergence (see Example 2 and Theorem 3 of Tucker [11] and Theorem 1 and Lemma 3 of Tucker [12]). It follows that M contains a set in $Z(\Omega) \cap \omega$. Thus, if there exists a countable disjoint subcollection of ω, there exists a countable disjoint subcollection $\{Z_1, Z_2, Z_3, \ldots.\}$ of $Z(\Omega) \cap \omega$.

Let g_1 and g_2 be members of Ω such that $Z_1 = g_1^{-1}(0)$ and $Z_2 = g_2^{-1}(0)$. Let

$$f_1 = \frac{|g_2|}{|g_1| + |g_2|} - \left(\frac{|g_1|}{|g_1| + |g_2|} \wedge \frac{|g_2|}{|g_1| + |g_2|}\right)$$

and

$$f_2 = \frac{|g_1|}{|g_1| + |g_2|} - \left(\frac{|g_1|}{|g_1| + |g_2|} \wedge \frac{|g_2|}{|g_1| + |g_2|}\right).$$

Then f_1 is 1 on Z_1 and 0 on Z_2, f_2 is 1 on Z_2 and 0 on Z_1, and $f_1 \wedge f_2 = 0$. Let N_1 be the cozero set of f_1 and N_2 be the cozero set of f_2. For each positive integer n, each of $Z_n \cap N_1$, $Z_n \cap N_2$, $Z_n - Z_n \cap N_1$, and $Z_n - Z_n \cap N_2$ is a set whose characteristic function is in $B_1(\Omega)$. Since f_1 and f_2 are disjoint,

$$N_1 \cap N_2 = \phi \quad \text{and} \quad Z_n \cap N_2 \subset Z_n - (Z_n \cap N_1).$$

If φ evaluated at the characteristic function of $Z_n - (Z_n \cap N_1)$ is zero then φ evaluated at the characteristic function of $Z_n - (Z_n \cap N_2)$ can not be zero. Either there are infinitely many Z_n such that $Z_n - (Z_n \cap N_1) \in \omega$ or there are infinitely many Z_n such that $Z_n - (Z_n \cap N_2) \in \omega$. Suppose there are infinitely many Z_n such that $Z_n - (Z_n \cap N_1) \in \omega$. As before, if $Z_n - (Z_n \cap N_1) \in \omega$, it contains a subset in $Z(\Omega) \cap \omega$. Rename this subset Z_n, $n > 2$. After this and after possibly renumbering to close the gaps there is a disjoint sequence $\{Z_1, Z_2, Z_3, \ldots\}$ of sets in $Z(\Omega) \cap \omega$ such that Z_n, $n \geq 2$, is disjoint from N_1.

Let g_3 be a member of Ω such that $Z_3 = g_3^{-1}(0)$. Redefine f_2 by letting

$$f_2 = \frac{|g_3 g_1|}{|g_3 g_1| + |g_2|} - \left(\frac{|g_3 g_1|}{|g_3 g_1| + |g_2|} \wedge \frac{|g_2 g_1|}{|g_2 g_1| + |g_3|}\right)$$

and

$$f_3 = \frac{|g_2 g_1|}{|g_2 g_1| + |g_3|} - \left(\frac{|g_3 g_1|}{|g_3 g_1| + |g_2|} \wedge \frac{|g_2 g_1|}{|g_2 g_1| + |g_3|}\right).$$

Then f_2 and f_3 are disjoint members of Ω^+ such that f_2 is 1 on Z_2 and 0 on $Z_1 \cup Z_3$ and f_3 is 1 on Z_3 and 0 on $Z_1 \cup Z_2$. Let N_2 be the cozero set of f_2 and N_3 the cozero set of f_3. Either there are infinitely many Z_n such that $Z_n - (Z_n \cap N_2) \in \omega$ or there are infinitely many Z_n such that $Z_n - (Z_n \cap N_3) \in \omega$. Suppose there are infinitely many Z_n such that $Z_n - (Z_n \cap N_2) \in \omega$. Rename and renumber as before.

Proceeding in this fashion there results an infinite disjoint sequence $Z_1, Z_2, Z_3, \ldots$ of sets in $Z(\Omega) \cap \omega$ such that for each n there is a function f_n in Ω^+ such that f_n is 1 on Z_n and f_n is 0 on Z_i, $i \neq n$.

Let $f = \sum_{n=1}^{\infty} f_n/2^n$. As Ω is closed under uniform convergence, $f \in \Omega$. For each positive integer n, f is $1/2^n$ on Z_n.

For each n, let

$$h_n = \left(\left(f \wedge \left(\frac{1}{2^n} + \frac{1}{2^{n+2}}\right)\right) \vee \left(\frac{1}{2^n} - \frac{1}{2^{n+2}}\right)\right) - \left(\frac{1}{2^n} - \frac{1}{2^{n+2}}\right).$$

Let

$$k_n = \left(-\left|h_n - \frac{1}{2^{n+2}}\right|\right) + \frac{1}{2^{n+2}}.$$

Then $\{k_1, k_2, \ldots\}$ is a disjoint sub-collection of Ω^+. For each n, $k_n = 1/2^{n+2}$ on Z_n. Thus $\phi(k_n) > 0$. There exists a sequence $c_1, c_2, \ldots$ of positive numbers such that $\sum_{n=1}^{\infty} c_n \phi(k_n) = \infty$. But $\sum_{n=1}^{\infty} c_n k_n$ belongs to $B_1(\Omega)$. Therefore, there does not exist an infinite disjoint subcollection of ω. It follows that there exists a maximal disjoint subcollection $\{N_1, N_2, \ldots, N_k\}$ of ω with the property that for each member N_i, there do not exist two disjoint elements of ω which are subsets of N_i. Also, $\bigcup_{i=1}^{k} N_i = X$.

Let φ_i be defined by $\varphi_i(f) = \varphi(f \cdot \chi_{N_i})$. Clearly $\varphi = \sum_{i=1}^{k} \varphi_i$.

Suppose f and g are bounded functions in $B_1(\Omega)$ such that $f \wedge g = \theta$. For each positive integer n, there exists a finite partition $G_1, G_2, \ldots, G_p$ of X such that χ_{G_i} is in $B_1(\Omega)$, $\sum_{i=1}^{p} c_i \chi_{G_i}$ uniformly approximates f within $1/n$, and $\sum_{i=1}^{p} k_i \chi_{G_i}$ uniformly approximates g within $1/n$. (See Theorem 7 of Tucker [13].)

Let H_n be the union of all G_i such that $c_i > 1/n$ and Q_n be the union of all G_i such that $k_i > 1/n$. Then $f_n = f \cdot \chi_{H_n}$ is a function in $B_1(\Omega)$ which agrees with f for every x such that $f(x) > 2/n$ and is zero for every x such that $f(x) = 0$. Define g_n similarly. Since $H_p \cap Q_j = \phi$, $\varphi_i(\chi_{H_p}) \wedge \varphi_i(\chi_{Q_j}) = 0$ which implies $\varphi_i(f_p) \wedge \varphi_i(g_j) = 0$. Since $\{f_p\}$ converges to f uniformly and $\{g_j\}$ converges to g uniformly $\varphi_i(f) \wedge \varphi_i(g) = 0$.

Now suppose f and g are points of $B_1(\Omega)$ such that $f \wedge g = \theta$. Then for each positive integer n and positive integer m, $\varphi_i(f \wedge \wedge n) \wedge \varphi_i(g \wedge m) = 0$. Since φ_i is sequentially continuous, $\varphi_i(f) \wedge \varphi_i(g \wedge m) = 0$. Similarly $\varphi_i(f) \wedge \varphi_i(g) = 0$, so that φ_i is a Riesz homomorphism.

Compare this theorem with Proposition 1.15 of Fremlin [4].

It follows from this theorem that Corollary 5 of Tucker [12] can be generalized by removing the boundedness condition.

If φ is a positive operator on $B_1(\Omega)$ the statement that *φ preserves pointwise convergence* means that if $f_1, f_2, f_3, \ldots$ is a pointwise convergent sequence of functions in $B_1(\Omega)$ then $\varphi(f_1), \varphi(f_2), \varphi(f_3), \ldots$ converges pointwise and further that if $f_1, f_2, f_3, \ldots$ converges pointwise to a function f in $B_1(\Omega)$ then $\varphi(f_1)$, $\varphi(f_2)$, $\varphi(f_3), \ldots$ converges to $\varphi(f)$.

COROLLARY 2. *If φ is a positive operator on $B_1(\Omega)$ into a function space, then φ preserves pointwise convergence.*

Proof. Suppose the range of φ is a set of functions defined on a set K. For each k in K let φ_k be φ followed by a point evaluation at k. By Theorem 1, φ_k is the sum of a finite number of Riesz homorphisms, each of which preserves pointwise convergence by Theorem 3 of Tucker [13].

COROLLARY 3. *If φ is a positive linear functional defined on B_1 (Ω) then φ can be extended to $B_2(\Omega)$.*

Proof. If f is in $B_2(\Omega)$ it is the pointwise limit of a sequence $f_1, f_2, f_3, \ldots$ of functions in $B_1(\Omega)$. Define $\varphi(f)$ to be the limit of $\varphi(f_1)$, $\varphi(f_2)$, $\varphi(f_3), \ldots$. This limit exists and $\varphi(f)$ is unique by Corollary 2. It can be verified that this extension is linear and positive.

Of course this corollary would follow immediately if $B_\alpha(\Omega)$ is cofinal in $B_{\alpha+1}(\Omega)$, but this is in general not the case as Theorem 5 shows.

Regoli [9] shows that there are spaces $C(X)$ which are σ-complete but not closed under pointwise convergence. This can not occur in $B_1(\Omega)$.

In the following Ω is only assumed to be a linear lattice containing the constant functions.

LEMMA 4. *If $B_1(\Omega)$ is σ-complete, then it is closed under pointwise convergence.*

Proof. Since bounded monotone pointwise convergence is the same as order convergence in $B_1(\Omega)$ (Tucker [12, Lemma 3]) and $B_1(\Omega)$ is σ-complete then every bounded function in LS $B_1(\Omega)$ is in $B_1(\Omega)$. But every function of $B_2^*(\Omega)$ is the pointwise limit from below of a non-decreasing sequence of functions of (US $B_1(\Omega))^*$ and the pointwise limit from above of a non-increasing sequence of points of (LS $B_1(\Omega))^*$ and thus belongs to $B_1^*(\Omega)$. Suppose f belongs to LS $B_1(\Omega)$. It may be assumed that $f \geq 1$. Since $B_2(\Omega)$ is closed under inversion, $1/f$ belongs to $B_2^*(\Omega)$ and thus to $B_1^*(\Omega)$. Therefore f belongs to $B_1(\Omega)$ and $B_1(\Omega)$ is closed with respect to pointwise convergence.

THEOREM 5. *The space $B_1(\Omega)$ is not cofinal in $B_2(\Omega)$ unless $B_1(\Omega) = B_2(\Omega)$.*

Proof. Suppose $B_1(\Omega)$ is cofinal in $B_2(\Omega)$. Let $f_1, f_2, f_3, \ldots$ be a disjoint sequence of points of $B_1^+(\Omega)$. Let $f = \sum_{n=1}^{\infty} nf_n$. The point f is in $B_2(\Omega)$. Since $B_1(\Omega)$ is cofinal in $B_2(\Omega)$ there is a point g in $B_1(\Omega)$ such that $g \geq f$. It follows that since $B_1(\Omega)$ is relatively uniformly complete that $\vee f_i$ belongs to $B_1(\Omega)$. Thus, as every disjoint sequence of points of $B_1^+(\Omega)$ has a supremum then $B_1^+(\Omega)$ has the principal projection property (Veksler and Geiler [14, Theorem 8]). As $B_1^+(\Omega)$ is also relatively uniformly complete, it is σ-complete, by Theorem 42.5 of Luxemburg and Zaanen [6]. By Lemma 4, $B_1(\Omega) = B_2(\Omega)$.

A real valued Riesz homomorphism φ on $B_1(\Omega)$ will be said to be *fixed* if there exists a point x of X and a number c such that $\varphi(f) = cf(x)$ for each f in $B_1(\Omega)$.

Positive operators on $B_1(\Omega)$ are sequentially continuous for both order convergence and pointwise convergence. Every positive operator on $B_1(\Omega)$ to an Archimedean Riesz space (even an Archimedean, directed, partially ordered vector space) preserves order convergence of sequences (Example 2 and Theorem 3, Tucker [11] and Theorem 1, Tucker [12]). Also, if Ω is assumed to be a complete ordinary function system every positive operator on $B_1(\Omega)$ to a function space preserves pointwise convergence of sequences (Corollary 2).

On the other hand, the situation with respect to net continuity is mixed. If every positive operator on $B_1(\Omega)$ to an Archimedean Riesz space preserved order convergence of nets, then $B_1(\Omega)$ would be order separable. (Theorems 18.13 and 29.3, Luxemburg and Zaanen [6]). Theorem 7 shows that there is only a very special class of order separable spaces of Baire functions. In a positive direction, when $\Omega = C(X)$, X realcompact, then every positive operator from $B_1(C(X))$ to a function space preserves pointwise convergence of nets (Theorem 8). But this theorem fails if pointwise convergence is replaced by order convergence (Example 9).

Proposition 6 is included for the sake of completeness. Proposition 6 is essentially proved in Theorem 5 of Mauldin [7]. The argument given there is only slightly changed for the proof of Proposition 6. It was also proved in Dashiell [3] with the restriction that the functions in Ω be bounded.

PROPOSITION 6. *The space $B_1(\Omega)$ is order Cauchy complete.*

Proof. By Theorem 7 of Mauldin [7] $B_1(\Omega) = \text{US } B_1(\Omega) \cap \text{LS } B_1(\Omega)$. Suppose $y_1 \geq y_2 \geq \cdots \geq \theta$, $\bigwedge y_n = \theta$, and $f_1, f_2, \ldots$ is a sequence such that $|f - f_n| \leq y_n$ when $m > n$. By Lemma 3 of Tucker [12], the y_n's converge pointwise to θ. Thus $f_1, f_2, f_3, \ldots$ converges pointwise to a function f. Then $|f - f_n| \leq y_n$ and $f \leq y_n + f_n$. So that $f \leq \alpha_n = \bigwedge_{i=1}^{n} y_i + f_i$ and f is in LS $B_1(\Omega)$. Similarly f is in US $B_1(\Omega)$.

THEOREM 7. *If $B_1(\Omega)$ is order separable then $B_1(\Omega)$ is Riesz isomorphic to R^x for X countable.*

Proof. By Proposition 6, $B_1(\Omega)$ is order Cauchey complete. Also, since it is order separable, it is almost σ-complete. Thus by Theorem 1 of Aliprantis and Langford [2] it is σ-complete and being order separable means that it is complete. Also, by Lemma 4, $B_1(\Omega)$ is closed under pointwise convergence. By Theorem 23.24 of Aliprantis and Burkinshaw [1], $B_1(\Omega)$ is universally complete. By Theorem 23.23 of Aliprantis and Burkinshaw [1], it is Riesz isomorphic to R^x and since it is order separable, X must be countable.

If X is a topological space, then X is *realcompact* if it is homeomorphic to a closed subset of a product of real lines. If X is realcompact, then for every real Riesz homomorphism φ on $C(X)$ there is a number c and a point x of X such that $\varphi(f) = cf(x)$.

THEOREM 8. *If X is a realcompact topological space and α is an ordinal, $0 < \alpha \leq \omega_1$, then every positive operator on $B_\alpha(C(X))$ to a function space preserves pointwise convergence of nets.*

Proof. If X is realcompact every real valued Riesz homomorphism on $C(X)$ is fixed. Thus for every positive linear functional φ on $B_\alpha(C(X))$ there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and a number sequence $c_1, c_2, \ldots, c_n$ such that $\varphi(f) = \sum_{i=1}^{n} c_i f(x_i)$, by Theorem 1. The theorem follows.

Example 9. Theorem 8 is false if pointwise convergence is replaced by order convergence. Let X be the space of all ordinals $\alpha \leq \omega_1$, the first uncountable ordinal, with the interval topology. As X is compact, it is realcompact. Also if f is in $C(X)$ there is an $\alpha_0 < \omega_1$ such that $f(\alpha) = f(\omega_1)$ for $\alpha > \alpha_0$. The same is true for all f in $B_1(C(X))$. Take the collection of all f in $B_1^+(C(X))$ such that $f(\omega_1) = 1$ and order it pointwise downward. The resulting net order converges to θ but the positive linear functional defined by $\varphi(f) = f(\omega_1)$ has the value 1 for each point of the net.

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