ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1981

SOME REMARKS ON THE HOMOLOGY GROUPS OF WREATH PRODUCTS

BY

JÜRGEN TAPPE

In [1], N. Blackburn determined the Schur multiplier of the regular wreath product, using generators and relations. Blackburn's method was extended to the permutational wreath product of finite groups by E. W. Read in [4]. In this paper we consider a projective resolution for an arbitrary permutational wreath product $G \int K$, and we easily obtain direct summands of $H_n(G \int K)$ in any dimension n. In particular we obtain the Schur multiplier of $G \int K$, and if K is a finite p-group and $G \int K$ a regular wreath product, the homology groups $H_n(G \int K)$ for $n \leq p - 1$.

For regular wreath products $G \int K$, the group $A \otimes ZK$, where A is a right G-module becomes a $G \int K$-module. In [1], Blackburn determined $H_1(G \int K, A \otimes ZK)$, again using generators and relations. An easy homological argument gives a formula for $H_n(G \int K, A \otimes ZK)$ for all n.

1. Statement of results

Let G and K be groups and I a K-set. By G^* we denote the (restricted) direct product of G over I, i.e. G^* consists of all families $(g_i)_{i \in I}$, where $g_i \in G$ and $g_i = 1$ for almost all i. By

$$k^{-1}((g_i)_{i \in I})k := (g_{ki})_{i \in I}$$

we obtain an action of K on G^*, and by $G \int K$ we denote the semi-direct product of G and K, called the (permutational) wreath product of G and K over I. If $I = K$, and if K acts on itself by left multiplication, we call $G \int K$ the regular wreath product.

For all integers $k \geq 1$, let I_k denote the set of all subsets of order k of I, which obviously is a K-set. Let M be an orbit of K on I_k, represented by $J = \{j_1, \ldots, j_k\}$, let $A(J)$ be the stabilizer of J and $B(J)$ the subgroup of $A(J)$ fixing all j_n. The group $C(M) = A(J)/B(J)$ is uniquely determined (up to conjugation), and it can be regarded as a subgroup of the symmetric group on $\{1, 2, \ldots, k\}$. For $c \in C(M)$, we denote by $\varepsilon(c)$ the sign of the permutation induced by c on $\{1, \ldots, k\}$. We call M a trivial orbit, if $C(M) = 1$.

If L is a group and N and L-module, we denote by N_L the quotient $N/[L, N]$. Let $P = (P_i, \partial_i)$ be a projective resolution for G, and assume $P_0 = ZG$ (the

Received May 15, 1979.

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integral group ring of G). By R we denote the chain complex defined by $R_0 = 0$, $R_i = (P_i)_G$ for $i \geq 1$. Let Q be a projective resolution for K, and for all finite subsets J of I we denote by $S(J)$ the chain complex defined by $S(J)_i := (Q_i)_{A(J)}$ for all i.

As $G \int K$ is a semi-direct product, $H_n(K, Z)$ is a direct summand of $H_n(G \int K, Z)$ for all n. In the following we shall see that one obtains a direct summand of $H_n(G \int K)$ (we drop the Z) for each orbit of K on I_k, having a nice form for trivial orbits. In particular we obtain the following results:

THEOREM 1. *Let M be an orbit of K on I_k represented by the subset J of I, and let $R^{(k)}$ denote the k-fold (total) tensor complex of R. Then we have:*

(i) *The group $(\otimes^k H_1(G))/T(M)$ is a direct summand of $H_k(G \int K)$, where $T(M)$ is generated by the elements*

$$\bigotimes_{i=1}^{k} x_i - \varepsilon(c) \bigotimes_{i=1}^{k} x_{ci}, \qquad x_i \in H_1(G), \qquad c \in C(M).$$

(ii) *If M is a trivial orbit, $H_n(R^{(k)} \otimes S(J))$ is a direct summand of $H_n(G \int K)$ for all n.*

The groups $H_n(R^{(k)} \otimes S(J))$ can be computed by an iterated application of the Künneth Formula. For $k = 1$, i.e. $I_k = I$, all orbits are trivial, and we obtain for each orbit M of K on I with $j \in M$ and stabilizer $A(j)$ that

$$\bigoplus_{r,s} H_r(G) \otimes H_s(A(j)) \oplus \bigoplus_{t,m} \operatorname{Tor}(H_t(G), H_m(A(j))),$$

where $r + s = n$, $r \geq 1$, $s \geq 0$, $t + m = n - 1$, $t \geq 1$, $m \geq 0$, is a direct summand of $H_n(G \int K)$. If we restrict our attention to dimension 2, Theorem 1 applied on the orbits of length 2, and the remarks above on the orbits of length 1 yield the direct summands of $H_2(G \int K)$ in the following corollary. The proof of Theorem 1 shows that no other summands occur.

COROLLARY. *Let r be the number of orbits of K on I, $(A(j))$ a complete system of stabilizers of K on I, t the number of trivial and s the number of non-trivial orbits on I_2. Then the Schur multiplier $H_2(G \int K)$ is the direct sum of $H_2(K)$, r copies of $H_2(G)$, the direct sum over all $H_1(G) \otimes H_1(A(j))$, t copies of $H_1(G) \otimes H_1(G)$, and s copies of $(H_1(G) \otimes H_1(G))/T$, where T is generated by the elements $x \otimes y + y \otimes x$, $x, y \in H_1(G)$.*

The Schur multiplier of an arbitrary semi-direct product with a finite, abelian normal subgroup of odd order was determined by L. Evens in [2].

THEOREM 2. *Let T be a finite p-group and $G \int T$ the regular wreath product.*

Then we have:

$$\text{(i)} \qquad H_n(G \int T) \simeq H_n(T) \oplus \bigoplus_{k=1}^{p-1} \left(\bigoplus^{m(k)} H_n(R^{(k)}) \right) \quad \textit{for all } n \leq p-1.$$

where

$$m(k) = \binom{|T|}{k} \cdot \frac{1}{|T|}.$$

(ii) *$H_n(G^*)$ is an induced T-module for $1 \leq n \leq p-1$, i.e. the terms E^2_{rs}, $1 \leq r \leq p-1$, $1 \leq s$ of the corresponding homology spectral sequence vanish, and $H_n(G \int T)$ is the direct sum of $H_n(T)$ and $H_0\,(T, H_n(G^*))$ for $1 \leq n \leq p$.*

In [1], the order of $H_2(P_n)$ was determined, where P_n is the n-fold wreath product of the cyclic group of order p. Theorem 2 and an easy induction argument yield the following result:

COROLLARY. *For all primes $p > 3$, the group $H_3(P_{n+1})$ is elementary abelian of rank*

$$\left(\frac{(p-1)(p-2)}{3} \sum_{i=1}^{n} i^3 \right) + (n+1) + \left(\frac{(p-1)^2}{2} \sum_{k=2}^{n} k \sum_{j=1}^{k-1} j^2 \right).$$

Let $G \int K$ be a regular wreath product, and A a right G-module. Then $A \otimes ZK$ is a $G \int K$-module by the following definition:

$$(a \otimes m^{-1})(g_k)_{k \in K} := (ag_m) \otimes m^{-1}, \qquad (a \otimes m)k := a \otimes (mk).$$

The homology groups of $A \otimes ZK$ are given by the following theorem (cf. [1] for the result in dimension 1):

THEOREM 3. *Let $G^* = G \times L$, where L is the direct product of $|K| - 1$ copies of G. Then*

$$H_n(G \int K, A \otimes ZK) \simeq \bigoplus_{i,j} H_i(G, A) \otimes H_j(L) \oplus \bigoplus_{k,m} \operatorname{Tor}(H_k(G, A), H_m(L)),$$

where $i + j = n$, $k + m = n - 1$.

2. Proofs

We start the proofs of the theorems above with the construction of a projective resolution for the base group G^*. For an arbitrary subset N of I we denote by $G(N)$ the subgroup of G^* which consists of all systems $(g_i)_{i \in I}$ with $g_m = 1$ for all $m \in N$, i.e. $G(N)$ is the direct product of copies of G built over the complement of N in I. Let P be the resolution for G given in Section 1. Then we define

$$P^*_n := \bigoplus_{N,f} \left(ZG(N) \otimes \left(\bigotimes_{m \in N} P_{f(m)} \right) \right),$$

where N runs through all finite subsets of I, $f\colon N \to Z$, $f(m) \geq 1$ for all $m \in N$, and $\sum_{m \in N} f(m) = n$. Now we are going to define homomorphisms $\partial_n^*\colon P_n^* \to P_{n-1}^*$. Let $[$ be a linear ordering on I, i.e. any two elements can be compared, and we assume that $i \,[\, j$ implies $i \neq j$. For $x \in ZG(N)$, $p_{f(m)} \in P_{f(m)}$ we define

$$\partial_n^*\left(x \otimes \left(\bigotimes_{m \in N} p_{f(m)}\right)\right) := \sum_{r \in N} (-1)^{s(r)} x \otimes \left(\bigotimes_{m [r} p_{f(m)}\right) \otimes \partial(p_{f(r)}) \otimes \left(\bigotimes_{r [s} p_{f(s)}\right),$$

where $\partial = \partial_{f(r)}\colon P_{f(r)} \to P_{f(r)-1}$, and $s(r)$ is the number of $m \in N$ with $m \,[\, r$ and $f(m)$ odd. In the definition of ∂_n^* above it can happen that $\partial(p_{f(r)})$ lies in P_0 (if $f(r) = 1$). As $P_0 = ZG$ we can identify

$$ZG(L) \otimes \left(\bigotimes_{m \in L} P_{f(m)}\right) \quad \text{and} \quad ZG(N) \otimes \left(\bigotimes_{m \in N} P_{f(m)}\right),$$

if $L = N \cup \{k\}$, $k \notin N$ and $f(k) = 0$. We can regard $P^* = (P_n^*, \partial_n^*)$ as a direct limit of the tensor complexes, built over all finite subsets of I. Obviously, P^* is a projective resolution for G^*, and we obtain

$$Z \otimes_{G^*} P_n^* = (P_n^*)_{G^*} = \bigoplus_{N, f} \bigotimes_{m \in N} (P_{f(m)})_G,$$

where N and f are as above. Now we can use the fact that $P_0 = ZG$, which implies $(P_0)_G = Z$ and $(P_1)_G$ is mapped onto zero. Hence, for each subset N, the term $\bigoplus_f \bigotimes_{m \in N} (P_{f(m)})_G$ yields a direct summand of the chain complex $Z \oplus_{G^*} P^*$, which for a non-empty set N of order k is isomorphic to $R^{(k)}$. Let $\tilde{Z}$ be the chain complex with $\tilde{Z}_0 = Z$ and $\tilde{Z}_n = 0$ for $n \neq 0$. Then we have

$$Z \otimes_{G^*} P^* \simeq \tilde{Z} \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{N \in I_k} R^{(k)}.$$

Let $h \in K$. We define

$$h^{-1}\left(x \otimes \left(\bigotimes_{m \in N} p_{f(m)}\right)\right) := (-1)^{t(h)} (h^{-1} x h) \otimes \left(\bigotimes_{m \in N} p_{f(hm)}\right),$$

(the element $h^{-1}xh$ belongs to $Z(h^{-1}G(N)h) = ZG(h^{-1}N)$) where $t(h)$ is the number of all pairs $k, m \in N$, $k \,[\, m$, $hm \,[\, hk$, $f(k)$ and $f(m)$ odd. This definition yields an action of $G \int K$ on P^*. If Q is a projective resolution for K, there is a natural diagonal action of $G \int K$ on $P^* \otimes Q$, so that the latter is a projective resolution for $G \int K$. Similar resolutions for wreath products of finite groups were studied by Nakaoka in [3]. It is easy to see that

$$Z \otimes_{G \int K} (P^* \otimes Q) \simeq (Z \otimes_{G^*} P^*) \otimes_K Q$$

holds. Now the considerations above yield the desired direct summands of the homology groups of $G \int K$. The term $\tilde{Z} \otimes_K Q$ yields the well-known direct summand $H_n(K)$ of $H_n(G \int K)$. For each positive integer k and each orbit M of

K on I_k we obtain the homology of the subcomplexes having the form $(\bigoplus_{N \in M} R^{(k)}) \otimes_K Q$. Here we have to keep in mind that the permutation action of K on the tensor complexes $R^{(k)}$ is endowed with a sign. In general, it does not seem to be easy to evaluate the homology of such a subcomplex. Its first non-zero term is $(\bigoplus_{N \in M} (\otimes^k (P_1)_G)) \otimes_K Q_0$, having dimension k. As the whole term is mapped onto zero, it is not very difficult to see that it yields the summand of $H_k(G \int K)$ given in Theorem 1(i). The result for trivial orbits (Theorem 1(ii)), where we have trivial action of K on the terms $R^{(k)}$, follows immediately.

Theorem 2(i) is also an easy consequence of the considerations above. For $H_n(G \int K)$, $n \leq p-1$ we only have to consider the subcomplexes for $k \leq p-1$. As $G \int K$ is regular, K has $m(k)$ orbits of I_k and acts regularly on each orbit, finishing the proof of 2(i).

The action of K on $H_n(G^*)$ can be derived from the action of K on $Z \oplus_{G^*} (P^* \otimes Q)$, which leads to $(Z \otimes_{G^*} P^*) \otimes_K Q$. Now we obtain that the desired action can also be described by the action of K on $Z \otimes_{G^*} P$. As mentioned above, K regularly permutes the direct summands of $Z \otimes_{G^*} P^*$ for all $k \leq p-1$, which proves that $H_n(G^*)$ is an induced K-module.

We finish this section with a study of the module $A \otimes ZK$. The G-module A can be regarded as a G^*-module by the following definition:

$$a(g, m) := ag \quad \text{for all } a \in A,\ g \in G,\ m \in L.$$

Then $A \otimes_{G^*} Z(G \int K)$ and $A \otimes ZK$ are isomorphic as $G \int K$-modules. Hence, we have

$$H_n(G \int K, A \otimes ZK) \simeq H_n(G^*, A).$$

Let P be a projective resolution for G and X a projective resolution for L. Then we have $A \otimes_{G^*} (P \otimes X) \simeq (A \otimes_G P) \otimes (Z \otimes_L X)$, and Theorem 3 follows from the Künneth Formula.

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Rheinisch-Westfälische Technische Hochschule Aachen
Aachen, Germany