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COEFFICIENTS OF BLOCH AND LIPSCHITZ FUNCTIONS[1]

BY

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(Dedicated to the memory of David L. Williams)

We present a new approach to problems concerning the coefficients $(a_n)_n$ of Bloch and Lipschitz functions $f(z) = \sum_0^\infty a_n z^n$ (on the unit disk). The approach is based on a new characterization of Bloch functions which is due to Van Casteren [13]. We use an obvious generalization of his result.

Some of our results are not new, but the proofs we give are more elegant and straightforward than the existing ones. One of the principal advantages of our approach is that it makes it possible to show the extent to which the results are best-possible. To a large extent, our new results are of this nature.

After completing our work, we became aware of unpublished work of A. L. Shields and D. L. Williams which overlaps considerably with ours. We thank them for graciously suggesting that we publish our version.

Before stating our principal result we need some terminology. A vector space S of complex sequences $(a_n)_{n=0}^\infty$ (pointwise operations) is called *solid* if $(a_n)_n \in S$ and $|a'_n| \leq |a_n|$ for all $n \geq 0$ imply $(a'_n)_n \in S$. A function f analytic on the unit disc belongs to the class Λ_α $(0 \leq \alpha \leq 1)$ if

$$(0.1) \qquad |f'(z)| \leq c(1/(1-|z|))^{1-\alpha}$$

for some constant $c > 0$. For $0 < \alpha \leq 1$, Λ_α is the usual Lipschitz class, while Λ_0 is the Bloch space (usually denoted B).

0.2 THEOREM. *The smallest solid (linear) sequence space containing the sequence of coefficients $(a_n)_n$ of every function $f(z) = \sum_{n=0}^\infty a_n z^n$ in Λ_α $(0 \leq \alpha < 1)$ is the sequence space defined by the condition*

$$\sum_{j=n}^{2n} |a_j|^2 = O(n^{-2\alpha}).$$

In fact, given $(a_n)_n$ satisfying this condition, there exists $\sum a'_n z^n \in \Lambda_\alpha$ with $|a_n| \leq |a'_n|$ for all n.

To get a feeling for this result it may help to consider H^p $(2 \leq p < \infty)$. There, the corresponding solid space is l^2, the square-summable series. This will come as no surprise to those familiar with the standard theorem on random power

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series [5, p. 228], [14, Section V.8]. Thus, in a strong sense, finding the smallest solid sequence space containing the coefficients of every function in a class is the same as finding the strongest growth condition the absolute values of the coefficients must satisfy. It is the next best thing to a characterization of the coefficients of the functions in the class (which is often impossible to give).

In Section 1, most of the terminology we use is defined. Our first step is to transfer problems about the coefficients of functions in Λ_α $(0 \leq \alpha \leq 1)$ to problems about the coefficients of the derivatives of functions in these classes. We are thus led to examine the coefficients of functions $g(z) = \sum_0^\infty b_n z^n$ satisfying a growth condition

$$|g(z)| \leq c\psi(1/(1 - |z|)) \tag{0.3}$$

where $\psi(x) = x^\alpha$, c a constant. It is trivial to translate results back to the original setting.

We actually work in somewhat greater generality and consider coefficients of harmonic functions on the disk satisfying a growth condition like (0.3) where $\psi(x)$ is an increasing function on $[1, \infty)$. This is done partly to illustrate the power of our technique and partly because it involves very little extra work.

We apply the result of Van Casteren (suitably generalized) many times. The power of his result is that it enables us to reduce the problems we consider to problems about coefficients of bounded functions. This is usually sufficient to solve our problems because bounded functions are (comparatively) well understood.

Our main result is Theorem 1.9 (c) (it implies Theorem 0.2). We use an imaginative result of de Leeuw, Kahane and Katznelson [4] (or, more precisely a substantial improvement of it due to Kisiliakov [7]). Their result identifies the smallest solid sequence space containing the power series coefficients of all bounded analytic functions on the disk.

Our results can be applied to various problems concerning multipliers. Thus our results easily imply results in [2], for instance.

1. Statement of the results

Let **C** denote the complex numbers and D the unit disk.

1.1 DEFINITION. Let $\psi\colon [1, \infty) \to [1, \infty)$ be a monotone increasing function. If $\psi(x)$ does not increase to ∞, assume $\psi(x) \equiv 1$. For technical reasons we always assume that ψ satisfies

$$\psi(2x) \leq c\psi(x) \tag{*}$$

for all $x > 1$ (some $c > 0$). Then $h^\infty(\psi)$ denotes the space of all complex-valued functions $u(z)$ harmonic in D satisfying the growth condition

$$|u(z)| \leq c\psi(1/(1 - |z|))$$

for all $|z| < 1$ (some $c \geq 0$). The smallest allowable value of c is denoted $\|u\|_\psi$.

The space of all analytic functions in $h^\infty(\psi)$ is denoted $A^\infty(\psi)$.

1.2 *Remark.* For $\psi = 1$, $A^\infty(\psi) = H^\infty$, the set of bounded analytic functions on D. For $\psi(x) = x^{1-\alpha}$ $(0 \le \alpha \le 1)$, $A^\infty(\psi)$ is the set of all $g = f'$ with $f \in \Lambda_\alpha$.

The spaces $A^\infty(\psi)$ and $h^\infty(\psi)$ were considered in [8], [11], [12].

1.3 *Notation.* For $u(z) = \sum_{-\infty}^{\infty} b_j r^{|j|} e^{ij\theta}$ $(z = re^{i\theta})$ harmonic in the unit disk D, we write

$$(s_n u)(re^{i\theta}) = \sum_{j=-n+1}^{n-1} c_j r^{|j|} e^{ij\theta},$$

$$(\sigma_n u)(re^{i\theta}) = \left(\frac{1}{n}\right) \sum_{j=0}^{n-1} (s_j u)(z)$$

$$= \left(\frac{1}{n}\right) \sum_{-n+1}^{n-1} \left(1 - \frac{|j|}{n}\right) c_j r^{|j|} e^{ij\theta}$$

and $\|u\|_\infty = \sup\{|u(z)| \mid |z| < 1\}$. Here $n \ge 1$.

The well-known inequality $\|\sigma_n u\|_\infty \le \|u\|_\infty$ [6, p. 17] will be used frequently. Also, the relation $\sigma_n u = \sigma_n s_N u$ $(N \ge n)$ will be needed.

Here now is our generalization of the result of Van Casteren [13]. We defer the proof to Section 2.

1.4 THEOREM. *If u is a harmonic function on the unit disk, then $u \in h^\infty(\psi)$ if and only if $\|\sigma_n u\|_\infty \le c\psi(n)$ for all $n \ge 1$ and some constant $c \ge 0$.*

1.5 COROLLARY. *For f analytic on the unit disk, $f \in \Lambda_\alpha$ $(0 \le \alpha \le 1)$ if and only if $\|\sigma_n(f')\|_\infty \le cn^{1-\alpha}$ for all $n \ge 1$ and some constant $c \ge 0$.*

1.6 *Remark.* It is well known [6, p. 23] that $f \in H^p$ if and only if $\|\sigma_n f\|_p = O(1)$.

1.7 DEFINITION. Let S be a linear space of sequences (with pointwise operations). Then S is called *solid* if $(\lambda_n a_n)_n \in X$ whenever $(a_n)_n \in X$ and $(\lambda_n)_n$ is a bounded sequence of scalars.

Notice that the term "solid" applies to spaces of doubly infinite sequences (as well as singly infinite) but we reserve the term for linear sequence spaces.

We will frequently use the observation that a space of harmonic (or analytic) functions on D may be considered as a sequence space via the correspondence

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} b_j r^{|j|} e^{ij\theta} \leftrightarrow (b_j)_{j=-\infty}^{\infty}$$

1.8 THEOREM. (a) *If $u \in h^\infty(\psi)$, then*

$$\left(\sum_{|j|\le n} |b_j|^2\right)^{1/2} = O(\psi(n)). \tag{1.9}$$

(b) *The smallest solid space containing $h^\infty(\psi)$ (resp. $A^\infty(\psi)$) is the space of sequences $(b_j)_{-\infty}^{\infty}$ (resp. $(b_j)_{j=-\infty}^{\infty}$) satisfying* (1.9).

Proof of (a). It follows easily from 1.4 that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |(\sigma_{2n}u)(e^{i\theta})|^2\,d\theta &= \sum_{|j|\le 2n}\left(1-\frac{|j|}{2n}\right)^2 |b_j|^2\\ &\le \|\sigma_{2n}u\|_\infty^2\\ &= O(\psi(2n)^2)\end{aligned}$$

Restrict j to the range $|j| \le n$ in the summation, observe that $1 - |j|/(2n) \ge \frac{1}{2}$ for $|j| \le n$ and apply (*) to get the result. We defer the proof of (b).

Note that Theorem 0.2 follows easily from Theorem 1.8.

1.10 THEOREM. (a) *Let $u(re^{i\theta}) = \sum_{-\infty}^{\infty} b_j r^{|j|} e^{ij\theta}$ be harmonic in D and satisfy $b_j \ge 0$ for all j. Then $u \in h^\infty(\psi)$ if and only if*

$$\sum_{|j|\le n} |b_j| = O(\psi(n)). \tag{1.11}$$

(b) *The largest solid subspace of $h^\infty(\psi)$ (resp. $A^\infty(\psi)$) is the space of sequences $(b_j)_{-\infty}^{\infty}$ (resp. $(b_j)_0^{\infty}$) satisfying* (1.11).

Proof. Use Theorem 1.4.

For the case $\psi(x) = x$ this theorem is known [2].

1.12 THEOREM. *Let $u(re^{i\theta}) = \sum_{-\infty}^{\infty} b_j r^{|j|} e^{i\theta} \in h^\infty(\psi)$. Then*

(a) $|b_n| = O(\psi(n))$,
(b) $\sum_{|j|\le n} |b_j| = O(\psi(n)\sqrt{n})$,
(c) $\|s_n u\|_\infty = O(\psi(n)\log n)$,
(d) $\|s_n u\|_\psi = O(\log n)$.

Furthermore, if $\psi \not\equiv 1$, these results are best possible in the sense that there exist four functions in $A^\infty(\psi)$ which achieve the required growth for infinitely many values of n.

Proof. Part (a) follows trivially from (1.8), while (1.9) and the Cauchy-Schwarz inequality imply (b). Part (c) follows from 1.4, the observation

$$s_{n+1}u = s_{n+1}[2\sigma_{2n}u - \sigma_n u]$$

and the fact that the nth Dirichlet kernel has L^1 norm at most $\log n$ [14, Section II.12]. Part (d) can be deduced from (c) and Theorem 1.4.

To show that (a) is best possible (for $\psi \not\equiv 1$), choose an increasing sequence $(n_j)_{j=1}^{\infty}$ of positive integers satisfying

$$\sum_{j=1}^{k} \psi(n_j) \leq c\psi(n_{k+1}) \tag{1.13}$$

for all $k \geq 1$ ($c > 0$ a constant). For $\psi(x) = x^a$ ($a > 0$), $n_j = 2^j$ will do. Then, by (1.10) (a),

$$\sum_{j=1}^{\infty} b_j z^{n_j} \in A^{\infty}(\psi) \quad \text{if and only if} \quad |b_j| = O(\psi(n_j)).$$

In particular, $\sum_1^{\infty} \psi(n_j)z^{n_j} \in A^{\infty}(\psi)$.

To show that (b) is best possible, choose an increasing sequence $(n_j)_{j=1}^{\infty}$ satisfying (1.13) and, in addition, $n_{j+1} \geq 2n_j$, $\psi(n_{j+1}) \leq c_1\psi(n_j)$ (for some constant $c_1 > 0$). This requires (*). Set $m_j = n_{j+1} - n_j$.

It is known that there exists a sequence $(\varepsilon_j)_0^{\infty}$ in $\{-1, 1\}$ with the property that the polynomials

$$P_m(z) = \frac{(\sum_{j=0}^{m} \varepsilon_j z^j)}{\sqrt{m+1}}$$

satisfy $\|P_m\|_{\infty} \leq 5$ (see [9]).

The function $g(z) \in A^{\infty}(\psi)$ is of the form

$$g(z) = c\sum_{j=1}^{\infty} \psi(n_j)z^{n_j}P_{m_j}(z)$$

(where $c > 0$ is some constant). To show $g \in A^{\infty}(\psi)$ observe that

$$\|s_{n_{j+1}}g\|_{\infty} \leq \text{const. } \psi(n_j)$$

and Theorem 1.4 now implies $g \in A^{\infty}(\psi)$.

We do not give complete details for the construction required to prove that (c) is best possible. It is somewhat similar to (b) but involves the following lemma.

1.14 LEMMA. *There exists a constant $c > 0$ and a sequence $(P_n(z))_n$ of polynomials with*

(i) *degree of P_n equal to $2n$,*
(ii) $\|P_n\|_{\infty} = 1$,
(iii) $\|s_n P_n\|_{\infty} \geq c \log (n + 1)$,

for all $n \geq 1$.

Choose $(n_j)_j$ as for (b) but with $n_{j+1} \geq 4n_j$. Set

$$g(z) = \sum_{j=1}^{\infty} \psi(n_j)z^{n_j}P_{n_j}(z)$$

(where the P_{n_j} come from Lemma 1.14). Now $g \in A^{\infty}(\psi)$ can be shown as in (b) and $\|s_n g\|_{\infty}$ achieves the required rate of growth for $n = 2n_j$.

This example works for (d) also because, when $n = n_j$ is large,

$$\begin{aligned} 2\|\sigma_{2n}s_n g\|_\infty &\geq \|2\sigma_{2n}s_n g - \sigma_n s_n g\|_\infty - \|\sigma_n s_n g\|_\infty \\ &= \|s_n g\|_\infty - \|\sigma_n g\|_\infty \\ &\geq \text{const. } \psi(n) \log n \\ &\geq \text{const. } \psi(2n) \log 2n. \end{aligned}$$

1.15 *Remarks.* (1) For $\psi \equiv 1$, the "O" in (a) and (b) of Theorem 1.12 can be replaced by "o" so that (a) and (b) are not best possible in the case $\psi \equiv 1$. For $\psi \equiv 1$, it is known [3] that there exists $u \in h^\infty(\psi)$ with $\|s_n u\|_\infty > \log n$ for all n. On the other hand if u can be extended to be continuous on $\bar{D}$, $\|s_n u\|_\infty = o(\log n)$ is known [14, Section II.11]. For $\psi \equiv 1$, we do not know if (c) is best possible for $A^\infty(\psi) = H^\infty$.

(2) For $\psi(x) = x$, the fact that 1.12 (c) is best possible answers a question posed by Van Casteren [13].

1.16 THEOREM. (a) *If $g(z) = \sum_0^\infty b_n z^n \in A^\infty(\psi)$ and $|b_n|$ increases with n, then*

$$|b_n| = O(\psi(n)/\sqrt{n}). \tag{1.17}$$

(b) *There exists a sequence $(\varepsilon_n)_{n=0}^\infty$ in $\{-1, 1\}$ such that, if $(b_n)_n$ is an increasing sequence of positive numbers satisfying (1.17), then*

$$g(z) = \sum_{n=0}^\infty \varepsilon_n b_n z^n \in A^\infty(\psi).$$

Proof. (*a*) is easy to deduce from 1.12(b) and (*). The sequence $(\varepsilon_n)_{n=0}^\infty$ for (b) is the same one referred to in the proof of Theorem 1.12. By the technique of summation by parts one can show that $\|s_n g\|_\infty = O(\psi(n))$. Since $\|\sigma_n g\|_\infty \leq \|s_n g\|_\infty$, Theorem 1.4 yields the result.

1.18 THEOREM. (a) *If $(b_n)_{n=0}^\infty$ is a sequence satisfying*

$$\left(\sum_{j=0}^n |b_j|^2\right)^{1/2} = O\left(\frac{\psi(n)}{\sqrt{\log n}}\right)$$

then $\sum_{n=0}^\infty e^{i\theta_n} b_n z^n \in A^\infty(\psi)$ for almost every choice of a sequence $(e^{i\theta_n})_n$ of unimodular complex numbers.

(b) *If $\liminf_{n\to\infty} \psi(n)/\sqrt{\log n} > 0$ and $(\eta_n)_0^\infty$ is a sequence of positive numbers increasing to ∞, then there exists a sequence $(b_n)_{n=0}^\infty$ such that*

$$\left(\sum_{j=0}^n |b_j|^2\right)^{1/2} < \eta_n \frac{\psi(n)}{\sqrt{\log n}} \quad (\textit{for all } n \geq 2)$$

while $\sum_{n=0}^\infty e^{i\theta} b_n z^n$ does not belong to $A^\infty(\psi)$ for any choice of unimodular numbers $(e^{i\theta_n})_n$.

For $\psi(x) = x$, part (a) is to be found in [1] and part (b) improves a result in [1]. The proof of (a) is immediate when one uses Theorem 1.4 and the Lemma of Salem and Zygmund used in [1] (Lemma 3.2) (see also [10]).

The proof of part (b) is constructive and we sketch the details. The basic building blocks of the construction are the polynomials $P_n(z) = \sum_{j=1}^n z^{2^j}$. One makes use of the fact that

$$\left\| \sum_{j=1}^{n} c_j z^{2^j} \right\|_\infty \geq \frac{1}{4} \sum_{j=1}^{n} |c_j|.$$

Let $(n_j)_j$ be a sequence of positive integers which increases so rapidly that

$$\sum_{j=1}^{k-1} (\eta'_j)^2 \psi(n_j)^2 (\log n_j)^{-1} \leq (\eta'_k)^2 \psi(n_k)^2 (\log n_k)^{-1},$$

$$\sum_{j=1}^{k-1} \eta'_j \psi(n_j) \leq 10^{-2} \eta'_k \psi(n_k),$$

$$n_k \geq 4n_{k-1},$$

for all $k \geq 2$ (where η'_j denotes η_{n_j}). The required sequence $(b_n)_n$ is given by $b_n = \eta'_k \psi(n_k)(\log n_k)^{-1}$, when $n = n_k + 2^r$, $0 \leq r \leq \log n_k$, and $b_n = 0$ for all n not of this form. Now, every function $g(z) = \sum e^{i\theta_n} b_n z^n$ satisfies

$$\|\sigma_n g\|_\infty \geq c_1 \eta'_k \psi(n_k) \geq c_2 \eta'_k \psi(4n_k),$$

for $n = 4n_k$. Hence $g \notin A^\infty(\psi)$, by Theorem 1.4.

1.19 *Remark.* For $\psi \not\equiv 1$, one can define spaces $h_0(\psi)$, $A_0(\psi)$ by using the growth condition $|u(z)| = o(\psi(1/(1 - |z|))$. Many of our results (including 1.4) carry through in this setting, if one replaces "O" by "o" throughout.

2. Proofs of the main results

We begin with some technical lemmas.

2.1 Lemma. *Let $\psi\colon [1, \infty) \to [1, \infty)$ be a monotone increasing function. Then the following are equivalent conditions on ψ:*

(i) *ψ satisfies (*).*

(ii) *There exist constants a, $\alpha > 0$ such that $\psi(x)x^{-a} \leq \alpha\psi(y)y^{-a}$ for all $x \geq y \geq 1$.*

(iii) *There exists $\beta > 0$ such that $\sum_{n=1}^\infty n\psi(n)r^n \leq [\beta/(1 - r)^2]\psi(1/(1 - r))$ for $0 \leq r < 1$.*

Proof. It is not hard to show (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follows by an argument in [12, Lemma 1]. Since we really only need to know that (i) $\Rightarrow$ (iii), we omit the proof that (iii) $\Rightarrow$ (i).

2.2 LEMMA. *Let* $K_{n,r}(\theta) = \sum_{k=-n+1}^{n-1} (1 - |k|/n)e^{ik\theta}r^{-|k|}$. *There exists* $M > 0$ *such that, for* $r = 1 - 1/n$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_{n,r}(\theta)|\, d\theta \le M$$

for all $n \ge 1$.

Proof. Observe that, for $r = 1 - 1/n$,

$$\begin{aligned}
\tfrac{1}{2}|K_{n,r}(\theta)| &\le \left|\operatorname{Re}\left\{\left(\frac{1}{n}\right)\left(\frac{e^{i\theta}}{r}\right)^{n}\sum_{0}^{n-1}(n-k)(re^{-i\theta})^{n-k}\right\}\right| + 1 \\
&= |\operatorname{Re}\{(1/n)r^{-n}e^{in\theta}[(n+1)(re^{-i\theta})^{n+1}/(1-re^{-i\theta}) \\
&\qquad + re^{-i\theta}(1 - r^{n+1}e^{-i(n+1)\theta})/(1-re^{-i\theta})^2]\}| + 1 \\
&\le |\operatorname{Re}\{(r(n+1)/n)e^{i\theta}/(1-re^{i\theta})\}| + 2/(n|1-re^{i\theta}|^2) + 1 \\
&\le 2|\cos\theta - r|/|1-re^{i\theta}|^2 + 2/(n|1-re^{i\theta}|^2) + 1
\end{aligned}$$

Using Taylor's theorem, one finds

$$\begin{aligned}
|K_{n,r}(\theta)| &\le c/(n|1-re^{i\theta}|^2) + c\theta^2/[1 + r^2 - 2r\cos\theta] \\
&\le c/(n|1-re^{i\theta}|^2) + c/[(1-\pi^2/12)r]
\end{aligned}$$

The last step requires the inequality

$$1 - \cos\theta \ge \theta^2/2 - \theta^4/4! \ge (\theta^2/2)[1 - \pi^2/12],$$

valid for $|\theta| \le \pi$. Now integration yields the estimate required.

Proof of Theorem 1.4. We adapt the proof used by Van Casteren [13] in the analytic case (for $\psi(x) = x$). First observe the elementary formula

$$u(rz) = (1-r)^2 \sum_{n=1}^{\infty} nr^{n-1}(\sigma_n u)(z) \tag{2.3}$$

(valid for u harmonic in D, $0 \le r < 1$ and $|z| < 1$). If u satisfies

$$\|\sigma_n u\|_\infty = O(\psi(n)),$$

then part (iii) of Lemma 2.1 and (2.3) immediately imply $u \in h^\infty(\psi)$.

On the other hand, if $u \in h^\infty(\psi)$,

$$(\sigma_n u)(z) = \left(\frac{1}{2\pi}\right)\int_0^{2\pi} u(zre^{i\theta})K_{n,r}(\theta)\, d\theta, \tag{2.4}$$

where $K_{n,r}(\theta)$ is as in Lemma 2.2. Taking $r = 1 - 1/n$ and applying Lemma 2.2 one easily deduces $\|\sigma_n u\|_\infty = O(\psi(n))$.

We remark that the proof is even simpler in the analytic case since one can avoid the use of Lemma 2.2 by using the elementary formula

$$(2.5)\qquad n(\sigma_n g)(z) = \left(\frac{1}{2\pi i}\right)\int_{|w|=r} g(zw)(1-w)^{-2}w^{-n}\,dw,$$

valid for g analytic in D, $n \geq 1$, $|z| \leq 1$ and $0 < r < 1$.

2.6 *Remark.* (1) It seems that the assumption (*) on the increasing function ψ is necessary for the validity of Theorem 1.4. If ψ increases so rapidly that $\limsup_{n\to\infty} \psi(n)^{1/n} > 1$, then there is a subsequence $(n_j)_j$ of the positive integers so that $\psi(n)^{1/n} > 1 + \varepsilon > 1$ on the subsequence. For such $(n_j)_j$, any gap series $g(z) = \sum a_j z^{n_j}$ converging in D satisfies $\|\sigma_n g\|_\infty = O(\psi(n))$. However, such a g need not be in $A^\infty(\psi)$. [In fact, by an elementary construction, one can show that given $(n_j)_j$, any sequence $(r_j)_j$ in $[0, 1)$ tending to 1 and any sequence $(x_j)_j$ in $\mathbf{R}$, there exists $g(z) = \sum a_j z^{n_j}$ converging in D with $g(r_j) \geq x_j$.]

On the other hand, if $\limsup \psi(n)^{1/n} \leq 1$ and $x\psi(x)$ is convex, define $g(z) = \sum_0^\infty b_n z^n$ by $b_0 = \psi(1)$ and

$$b_n = (n+1)\psi(n+1) - n\psi(n) + (n-1)\psi(n-1)$$

for $n \geq 1$. Then $\|\sigma_n g\|_\infty = \psi(n)$ and (by (2.3))

$$g(r) = (1-r)^2 \sum_{n=1}^{\infty} n r^{n-1}\psi(n).$$

Hence, by the implication (iii) $\Rightarrow$ (i) of Lemma 2.1 (which we did not prove), if ψ fails to satisfy (*), then $g \notin A^\infty(\psi)$.

(2) If ψ satisfies (*) and the formal power series $g(z) = \sum^\infty b_n z^n$ satisfies $\|\sigma_n g\|_\infty = O(\psi(n))$, then the series converges for $|z| < 1$. The details of some of the arguments in Section one required this remark.

It remains now to prove Theorem 1.8(b). The proof is based on the following result of Kisliakov [7] which is an improvement of the result in [4].

2.7 THEOREM. *If $(b_n)_{n=0}^\infty$ is a sequence with $\sum_0^\infty |b_n|^2 < \infty$, then there exists a function $g(z) = \sum_{n=0}^\infty b'_n z^n$ with a uniformly convergent power series satisfying*

$$|b'_n| \geq |b_n| \quad \text{and} \quad \left\|\sum_{k=0}^{n} b_k z^k\right\|_\infty \leq B\left(\sum_{k=0}^{\infty} |b_k|^2\right)^{1/2}$$

(where $B > 0$ is an absolute constant).

2.8 COROLLARY. *If $(b_j)_{j=m}^n$ is given $(0 < m < n)$, then there exists a polynomial $p(z) = \sum_{j=m}^n b'_j z^j$ satisfying*

$$|b'_j| \geq |b_j| \text{ for } m \leq j \leq n \quad \text{and} \quad \|p\|_\infty \leq B\left(\sum_m^n |b_j|^2\right)^{1/2}$$

(where $B > 0$ is the absolute constant from Theorem 2.7).

Proof of Theorem 1.8(b). Suppose $(b_n)_{n=0}^{\infty}$ is a sequence satisfying

$$\sum_{j=0}^{n} |b_j|^2 \le c^2 \psi(n)^2$$

for all $n \ge 1$ (some constant $c \ge 0$). Since the case $\psi \equiv 1$ is disposed of by Theorem 2.7, assume $\psi \not\equiv 1$. Using (*), one can find an increasing sequence $(n_k)_{k=1}^{\infty}$ with $n_1 = 1$, $\sum_{j=1}^{k} \psi(n_j) \le \psi(n_{k+1})$ and $\psi(n_{k+1}) \le c_1 \psi(n_k)$ for all $k \ge 1$).

By Corollary 2.8, for each $k \ge 1$, one can find polynomials

$$P_k(z) = \sum_{n_k \le j < n_{k+1}} b'_j z^j$$

satisfying

$$\|P_k\|_\infty \le B\left(\sum_{n_k \le j < n_{k+1}} |b_j|^2\right)^{1/2} \quad \text{and} \quad |b'_j| \ge |b_j| \text{ for } n_k \le j < n_{k+1}.$$

Now define $g(z) = \sum_0^\infty b'_j z^j$. Then, if $m = n_{k+1} - 1$, $s_m g = \sum_{l=1}^{k} P_k(z)$. Therefore,

$$\begin{aligned}
\|s_m g\|_\infty &\le \sum_{l=1}^{k} \|P_l\|_\infty \\
&\le B \sum_{l=1}^{k} \left(\sum_{j \ge n_l} |b_j|^2\right)^{1/2} \\
&\le Bc \sum_{l=1}^{k} \psi(n_l) \\
&\le Bc\psi(n_{k+1}) \\
&\le Bcc_1 \psi(n_k).
\end{aligned}$$

Hence $\|\sigma_r g\|_\infty \le Bcc_1 \psi(r)$ for $r \le 1$. By Theorem 1.4, $g \in A^\infty(\psi)$. The desired result is now immediate.

2.9 *Remarks.* (1) The function g constructed in the above-proof satisfies $\|g\|_\psi < B_\psi c$, where B_ψ is a constant depending only on ψ and

$$c = \sup_{n<1} \left(\sum_{j=0}^{n} |b_j|^2\right)^{1/2} \Big/ \psi(n).$$

(2) The harmonic case of Theorem 1.8(b) follows immediately from the analytic case (of course). However, the harmonic case is easier to prove in the sense that one can carry out a modified version of the above construction (for harmonic functions) using only the result of de Leeuw, Kahane and Katznelson [4] (i.e., without appealing to Kisliakov [7]). The modified construction is more complicated, however.

If ψ is such that $h^\infty(\psi)$ is invariant under harmonic conjugation (e.g., if $\psi(x) = x^a$, $a > 0$), then the analytic case follows from the harmonic case. A. L. Shields and D. L. Williams have recently obtained necessary and sufficient conditions on ψ for $h^\infty(\psi)$ to be conjugation-invariant.

We close with a result on multipliers. First, we need a definition.

2.10 DEFINITION. If S_1, S_2 are sequence spaces, then $(\lambda_n)_n$ is called a multiplier from S_1 to S_2 if $(\lambda_n a_n)_n \in S_2$ whenever $(a_n)_n \in S_1$.

Notice that, if S_2 is a solid sequence space, then the multipliers from S_1 to S_2 are the same as the multipliers from the smallest solid space containing S_1 to S_2 (see [2]). Using this remark, Theorem 1.8(b) easily implies the following result, which includes a result in [2].

2.11 PROPOSITION. *Suppose $\psi(x) = x^a$, $a > 0$ (or more generally that ψ satisfies $\sum_{j=0}^n \psi(2^j) \le c\psi(2^n)$ for all $n \ge 1$, some $c > 0$). Then a sequence $(\lambda_n)_n$ is a multiplier from $A^\infty(\psi)$ (regarded as a sequence space) to l^p $(0 < p \le \infty)$ if and only if*

$$\sum_{n=0}^{\infty} \left[\sum_{j \in I_n} |\lambda_j|^{2p/(2-p)}\right]^{(2-p)/2} \psi(2^n)^p < \infty \quad (\textit{if } 0 < p < 2),$$

$$\sum_{n=0}^{\infty} \left(\sum_{j \in I_n} |\lambda_j|^p\right) \psi(2^n)^p < \infty \quad (\textit{if } 2 \le p < \infty),$$

$$\sup_n |\lambda_n| \psi(n) < \infty \quad (\textit{if } p = \infty)$$

(where $I_n = [2^n, 2^{n+1})$).

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