ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 1, Spring 1982

FINITE QUOTIENTS AND MULTIPLICITIES IN NILPOTENT GROUPS

BY

P. F. PICKEL[1]

Introduction

Unless otherwise stated, all discrete groups will be finitely generated torsion-free nilpotent groups. If Γ is such a group, there is a unique real nilpotent Lie group N containing Γ as a discrete co-compact subgroup. That is, the coset space is compact and, in addition, has a finite N-invariant measure μ. There is a natural continuous unitary representation U of N on $L^2(N/\Gamma, \mu)$ given by

$$U_s(f)(n\Gamma) = f(s^{-1}n\Gamma), \quad f \in L^2(N/\Gamma, \mu),\ s, n \in N.$$

U is the representation of N induced by the trivial one-dimensional representation of Γ. Since N/Γ is compact, this representation decomposes as a discrete direct sum

$$U = \bigoplus n(\pi, U) \cdot \pi$$

over a countable family of inequivalent irreducible representations π with finite multiplicities $n(\pi, U)$.

Denote the set of isomorphism classes of finite quotients of Γ by $\mathscr{F}(\Gamma)$. Two groups have the same set of finite quotients if and only if their profinite completions are isomorphic [7]. While the finite quotients do not determine Γ up to isomorphism (see Section 3 below), there can be only finitely many groups with the same set of finite quotients [7]. L. Auslander has suggested that the multiplicities $n(\pi, U)$ might fill in the gap. That is, that the finite quotients with the multiplicities $n(\pi, U)$ might determine Γ up to isomorphism. (When the multiplicities corresponding to the profinite completion are computed, they are all one or zero [6].) In this paper, we give some examples showing that this conjecture does hold in certain cases. The multiplicities in these examples have relations with classical arithmetic constructions.

1. Generalities on multiplicities

(Much of this section is taken from [6, Section 2].) The nilpotent Lie group N has a nilpotent Lie algebra $\mathscr{N}$. The exponential map $\exp: \mathscr{N} \to N$ is a homeo-

Received April 20, 1980.

[1] Research partially supported by the National Science Foundation.

morphism with inverse log: $N \to \mathcal{N}$. Suppose $\{g_1, \ldots, g_k\}$ is a Malcev basis for Γ (that is, each element g in Γ can be written uniquely as

$$g = g_1^{n(1)} g_2^{n(2)} \cdots g_k^{n(k)}$$

for integers $n(i)$). Let $x_i = \log(g_i)$. Then $\{x_1, \ldots, x_k\}$ is a basis for $\mathcal{N}$, with respect to which the structure constants of $\mathcal{N}$ are rational numbers. Let $\mathcal{N}_Q$ denote the rational subspace of $\mathcal{N}$ spanned by $\{x_1, \ldots, x_n\}$. $N_Q = \exp(\mathcal{N}_Q)$ is the divisible hull (Malcev completion) of Γ. A subgroup N_0 of N will be called rational if its Lie algebra $\mathcal{N}_0$ is a rational subspace of $\mathcal{N}$ (i.e., $\mathcal{N}_0 = \mathbf{R} \cdot (\mathcal{N}_0 \cap \mathcal{N}_Q)$). Let $\hat{N}$ denote the set of equivalence classes of irreducible unitary representations of N. We will often write π meaning the equivalence class of π.

Let π be in $\hat{N}$ and suppose that W is a one-dimensional rational normal subgroup of N such that $\pi(W) = 1$. Let $N/W = N_0$ and let $\varphi\colon N \to N_0$ be the natural projection. There is a unique element π_0 in $\hat{N}_0$ such that $\pi_0(\varphi(n)) = \pi(n)$ for all n in N. Let Γ_0 denote $\Gamma \cdot W/W$, a discrete co-compact subgroup of N_0 and let U_0 be the representation of N_0 induced by the trivial representation of Γ_0.

LEMMA 1 [6, Lemma 2.2a]. $n(\pi, U) = n(\pi_0, U_0)$.

Thus, in computing multiplicities $n(\pi, U)$, we may first factor out by the largest rational normal subgroup on which π is trivial. Since π is irreducible, the action of the center Z of N is given by a character of Z. That is, there is a linear functional f defined on the Lie algebra $\mathcal{Z}$ of the center such that for any $\bar{v}$ in the representation space of π and any z in $\mathcal{Z}$,

$$\exp(z)\bar{v} = \exp(2\pi i f(z))\bar{v}.$$

This means that the representation must be trivial on a subgroup of co-dimension one (at least) in Z (corresponding to the kernel of f). If the representation occurs in U, it must be trivial on $\Gamma \cap Z$, so the kernel of f is a rational subspace of $\mathcal{Z}$. Thus in computing $n(\pi, U)$, we may restrict to the situation where the center of N is one-dimensional. For the remainder of this section, we will assume that N has one-dimensional center.

Now let $Z_2(N)$ denote the second center of the group N (with one-dimensional center) and let W be a two-dimensional rational subgroup of $Z_2(N)$ containing Z. Let N_0 be the centralizer of W in N. N_0 is rational in N, N_0 had co-dimension one in N and $N = N_0 \rtimes S$ (semidirect product) for some rational one-dimensional subgroup S of N.

LEMMA 2. [6, Lemma 2.3]. *Let π be an irreducible representation of N which is non-trivial on $\mathcal{Z}$. Then π is induced by some irreducible representation π_0 of N_0. The set of all representations of N_0 which induce π coincides with the orbit of π_0 in $\hat{N}_0$ under N, i.e.*

$$\{(\pi_0)^x \mid x \in N\}. \quad ((\pi_0)^x = (\pi_0)^y \textit{ if and only if } x = y \bmod N_0.)$$

Now suppose π in $\hat{N}$ is nontrivial on Z (still one-dimensional) and choose π_0 in $\hat{N}_0$ which induces π. Let U_0 be the representation of N_0 induced by the trivial representation of $\Gamma_0 = \Gamma \cap N_0$. Let A' be the set of λ_0 in $\hat{N}_0$ which do not vanish on Z and for which $n(\lambda_0, U_0) > 0$. A' is acted on by Γ. We let A be a subset of A' which meets each orbit under Γ in exactly one element. Then we have

$$(*) \qquad n(\pi, U) = \sum n(\lambda_0, U_0)$$

where the sum is over all λ_0 in A for which $\lambda_0 \in (\pi_0)^N$ [6, p. 153].

Using Lemma 1 and formula $(*)$, we may compute the multiplicities $n(\pi, U)$ by reducing eventually to the abelian case where all non-zero multiplicities are 1.

2. Nilpotent groups of class 2

Suppose now that Γ and N are nilpotent of class 2 and that π is in $\hat{N}$. We wish to find $n(\pi, U)$. If π is trivial on N', then π gives a character of N/N' and $n(\pi, U) = 1$ or 0 depending on whether π is trivial on Γ. Thus the only multiplicities which give significant information about the structure of Γ are the multiplicities of those π in $\hat{N}$ which give a non-trivial action for N'.

Now consider a representation π in $\hat{N}$ which is not trivial on $N' \leq Z$, the center of N. Then π induces a non-trivial character of Z:

$$\pi(z)v = e^{2\pi i f(z)}v \text{ (v any element of the representation space)}$$

where f is a linear function on Z written additively. If $n(\pi, U) \neq 0$, π is trivial on Γ so the kernel of f is a rational subspace of Z. Using Lemma 1, we factor out by this subspace to obtain the situation: $N' = Z$ is one-dimensional and the action of $N' = Z$ via π is given by $f\colon Z \to \mathbf{R}$ as above. The restriction of f to $\Gamma \cap Z = Z_\Gamma$ sends Z_Γ to the integers since π is trivial on Γ. The composition

$$\Phi(x, y) = f([\bar{x}, \bar{y}])$$

of f with commutation thus defines a nondegenerate alternating bilinear form from Γ/Z_Γ to $\mathbf{Z}$ ($\bar{x}$, $\bar{y}$ denote any preimage of x, y resp.). By the structure theorem for such forms [2, Theorem 5.1.1], there is a basis

$$\{x_1, \ldots, x_{2n}\}$$

of Γ/Z_Γ and integers $d_1, \ldots, d_n$ with d_i dividing d_{i+1} such that

$$\Phi(x_{2i-1}, x_{2i}) = d_i$$

and $\Phi(x_i, x_j) = 0$ otherwise. By using $(*)$ and induction on n, one may show that

$$n(\pi, U) = d_1 \cdot d_2 \cdot \cdots \cdot d_n$$

(or see [1, Section 1.6]).

Thus we see, in the case where Γ is of class two and π in $\hat{N}$ is non-trivial on

N', that $n(\pi, U)$ depends only on the linear functional induced on N' by π. We define a map from the dual of $N' \cap \Gamma = I_\Gamma$ (the isolator of Γ' in N) to the integers as follows: Given a linear functional on I_Γ, extend it to a linear map $f: N' \to \mathbf{R}$. Use this map to define a representation π_f of N induced from $e^{2\pi i f(z)}$. The original linear functional is then sent to $n(\pi_f, U)$. We will call this map $\mu_\Gamma: I_\Gamma^* \to \mathbf{Z}$.

DEFINITION. We say that two finitely generated torsion free nilpotent groups $\Gamma(1)$ and $\Gamma(2)$ of class two have the same multiplicities if there is an isomorphism φ of $I_{\Gamma(1)}$ to $I_{\Gamma(2)}$ such that

$$\mu_{\Gamma(2)} = \mu_{\Gamma(1)} \cdot \varphi^*.$$

That is, φ^* takes multiplicities for $\Gamma(1)$ to multiplicities for $\Gamma(2)$.

3. The examples of Grunewald–Scharlau, generalities

Let R be the ring of integers in the number field K and let I be a fractional ideal of K. We form the matrix group

$$G(K, I) = \left\{ \begin{pmatrix} 1 & r & m \\ 0 & 1 & m' \\ 0 & 0 & 1 \end{pmatrix} \middle| \; r \in R, \quad m, m' \in I \right\}.$$

THEOREM 3 [3, Lemmas 2 and 3]. *$G(K, I)$ is isomorphic with $G(F, J)$ if and only if there is an isomorphism $\varphi: K \to F$ which takes the ideal class of I to the ideal class of J.*

Clearly center $(G(K, I)) = (G(K, I))'$

$$= \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad m \in I$$

which is isomorphic to I as an abelian group and $G(K, I)/G(K, I)'$ is isomorphic to $R \oplus I$ as abelian groups. The commutator induces a map

$$[\ ,\ ]: (R \oplus I) \oplus (R \oplus I) \to I$$

given by $[(r, m), (r', m')] = rm' - r'm$. Suppose now that $f: I \to \mathbf{Z}$ is a nontrivial $\mathbf{Z}$-linear map. Then

$$\Phi = f \circ [\ ,\ ]: (R \oplus I) \oplus (R \oplus I) \to \mathbf{Z}$$

gives an alternating bilinear form which is nondegenerate since we are working in a field. If we choose integral bases $\{w_1, \ldots, w_n\}, \{\sigma_1, \ldots, \sigma_n\}$ for R and I, resp., then the matrix A of Φ has the form

$$\begin{bmatrix} 0 & M \\ -M^T & 0 \end{bmatrix} \quad \text{where } m_{ij} = f(w_i \sigma_j).$$

If E is an elementary $2n \times 2n$ matrix and EA gives a row operation on M, then AE^T gives the corresponding column operation on $(-M^T)$. To obtain the invariants of Φ, we perform elementary row and column operations on M and the corresponding operations on $(-M^T)$ to obtain

$$PAP^T = \begin{bmatrix} 0 & \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \\ \begin{bmatrix} -d_1 & & \\ & \ddots & \\ & & -d_n \end{bmatrix} & 0 \end{bmatrix}$$

where

$$P = \begin{bmatrix} Q & 0 \\ 0 & S \end{bmatrix}.$$

Thus

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = QMS^T$$

and

$$d_1 \cdot d_2 \cdot \cdots d_n = \det \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = \det (QMS^T) = |\det M|$$

since $\det Q = \pm 1 = \det S$.

Now let $\{w_1, \ldots, w_n\}$ be an integral basis of R and let $\{\sigma_1, \ldots, \sigma_n\}$, $\{\tau_1, \ldots, \tau_n\}$ be integral bases of I. There are integers γ_{ij}^k so that

$$w_i \sigma_j = \sum_{k=1}^{n} \gamma_{ij}^k \tau_k \quad \text{for all } i \text{ and } j.$$

If $f\colon I \to \mathbf{Z}$ is given by $\tau_i \to c_i$, then $f(w_i\sigma_j) = \sum_{k=1}^n \gamma_{ij}^k c_k$. By Section 2 and the calculation above, the multiplicity $n(\pi_f, U)$ of the irreducible representation π_f induced from $e^{2\pi i f(x)}$ on I is

$$(\dagger) \qquad n(\pi_f, U) = \left| \det \left(\sum_{k=1}^{n} \gamma_{ij}^k c_k \right) \right|.$$

Thus the multiplicity map μ_G is a form of degree n in the n variables $c_1, \ldots, c_n$ and two such groups have the same multiplicities if and only if the forms are equivalent. Note that the form depends only on the ideal class of I since we may choose $\{\eta\sigma_1, \eta\sigma_2, \ldots, \eta\sigma_n\}$ and $\{\eta\tau_1, \ldots, \eta\tau_n\}$ as bases of ηI and obtain the same form. The equivalence class of the form does not depend on the choice of bases.

A change in $\{w_i\}$ or $\{\sigma_j\}$ multiplies by $|\det|$ of the change of basis matrix which is 1. A change in $\{\tau_i\}$ gives a form equivalent via the change of basis matrix.

There is a second, well-known, way of obtaining a form from the ideal I. Given an integral basis $\{\sigma_1, \ldots, \sigma_n\}$ of I, $N(c_1\sigma_1 + \cdots + c_n\sigma_n)$ is an integer divisible by the norm $N(I)$ of I. Thus

$$g(c_1, \ldots, c_n) = N(c_1\sigma_1 + \cdots + c_n\sigma_n)/N(I)$$

gives a form of n-th degree in n variables whose equivalence class does not depend on the choice of basis and depends only on the class of I. Let us see how this form may be calculated in a way similar to the above multiplicity formula. Use the bases of the previous paragraph and write $\tau_i = \sum \alpha_{ij} w_j$. Then

$$w_i\sigma_j = \sum_{k=1}^{n} \gamma_{ij}^k \tau_k = \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{ij}^k \alpha_{kl} w_l$$

so

$$w_i\left(\sum_{j=1}^{n} c_j\sigma_j\right) = \sum_{j=1}^{n} c_j(w_i\sigma_j) = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} c_j\gamma_{ij}^k\alpha_{kl}w_l$$

and

$$N\left(\sum_{j=1}^{n} c_j\sigma_j\right) = \det\left(\sum_{j=1}^{n} \sum_{k=1}^{n} c_j\gamma_{ij}^k\alpha_{kl}\right) = \det\left(\sum_{j=1}^{n} c_j\gamma_{ij}^k\right) \det(\alpha_{kl}).$$

But $\det(\alpha_{kl}) = N(I)$ so we have finally

$$g(c_1, \ldots, c_n) = N(c_j\sigma_j)/N(I) = \det\left(\sum_{j=1}^{n} c_j\gamma_{ij}^k\right).$$

We shall see below that, in a special case, these two forms are equivalent. It would be interesting to know if the forms are equivalent in general.

4. Examples of Grunewald–Scharlau, quadratic field case

We now restrict to the case where the field K is a quadratic number field $Q(\sqrt{m})$ (m square-free) and I is an ideal. In the case

$$R = \{a + b\sigma \mid a, b \in \mathbf{Z}\}$$

where

$$\sigma = \sqrt{m} \text{ for } m \equiv 2, 3 \pmod 4 \quad \text{and} \quad \sigma = \frac{1+\sqrt{m}}{2} \text{ for } m \equiv 1 \pmod 4.$$

(Many of the results which follow may be found in [5, Chapter VIII].) I has a basis

$$\{ua, u(\sigma + e)\}$$

where u, a and e are integers and $N(\sigma + e) = ka$ for some integer k. Then we have

$$1 \cdot ua = ua$$

$$1 \cdot u(\sigma + e) = u(\sigma + e)$$

$$\sigma \cdot ua = ua\sigma = a(u(\sigma + e)) - e(ua)$$

$$\sigma \cdot u(\sigma + e) = \begin{cases} -(ua)k + e(u(\sigma + e)), & m \equiv 2, 3, \\ -(ua)k + (e + 1)(u(\sigma + e)), & m \equiv 1. \end{cases}$$

If the action of a linear functional on I is given by $ua \to x$ and $u(\sigma + e) \to y$, then by formula (†) of the previous section, the multiplicity of the induced representation is

$$\left| \det \begin{pmatrix} x & y \\ ay - ex & -kx + ry \end{pmatrix} \right| = | -kx^2 + 2rxy - ay^2 |$$
$$= kx^2 - 2rxy + ay^2$$
$$= \mu_I$$

where $r = e$ if $m \equiv 2, 3 \pmod 4$ and $r = e + 1$ if $m \equiv 1 \pmod 4$. Two groups $G(K, I)$ and $G(K, J)$ have the same multiplicities if and only if the corresponding binary quadratic forms are integrally equivalent.

We now obtain the second type of form for I:

$$g_I(x, y) = N(uax + u(\sigma + e)y)/N(I) = ax^2 + 2rxy + ky^2.$$

The form obtained in this way is equivalent (under $x \to y$, $y \to -x$) to the form obtained above. Now g_I is equivalent to g_J if and only if the ideal class of J is the ideal class of I or I^c, the conjugate of I under the Galois automorphism [5, Theorem 64]. By Theorem 3 above, this holds if and only if $G(K, I)$ is isomorphic to $G(K, J)$.

To finish up we need a result on quadratic field Grunewald–Scharlau groups with the same finite quotients.

LEMMA 4. *If two quadratic field Grunewald-Scharlau groups $G(K, I)$ and $G(F, J)$ have the same finite quotients, then $K = F$.*

Proof. Let R be the ring of integers in K and let S be the ring of integers in F. Since $G(K, I)$ has the same finite quotients as $G(K, R)$, it is sufficient to consider $G(K, R)$ and $G(F, S)$. Since $G(K, R)$ and $G(F, S)$ have the same finite quotients, their p-adic completions must be the same for each rational prime p. We have a basis $\{l_1, \sigma_1, l_2, \sigma_2\}$ for $G(K, R)/(G(K, R))'$ and a basis $\{l_3, \sigma_3\}$ for $(G(K, R))'$. Under the isomorphism taking $G(K_{(p)}, R_{(p)})$ to $G(F_{(p)}, S_{(p)})$ we have

$$\left.\begin{array}{ll} l_1 \to (\alpha_{11}, \alpha_{12}), & \sigma_1 \to (\alpha_{21}, \alpha_{22}), \\ l_2 \to (\alpha_{31}, \alpha_{32}), & \sigma_2 \to (\alpha_{41}, \alpha_{42}), \end{array}\right\} \text{ modulo } (G(F_{(p)}, S_{(p)}))'$$

$$l_3 \to \beta_1, \qquad \sigma_3 \to \beta_2,$$

where α_{ij} and β_i are elements of $S_{(p)}$. Since the map is a homomorphism and $[l_1, \sigma_1] = \mathrm{id}$, $\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = 0$. Now one of α_{11}, α_{12} is not zero say α_{11}. Then if $\alpha_{12} = 0$, then $\alpha_{22} = 0$ and if $\alpha_{12} \neq 0$ then $\alpha_{22} \neq 0$ and

$$\frac{\alpha_{21}}{\alpha_{11}} = \frac{\alpha_{22}}{\alpha_{12}} = \delta$$

so in any case we have

$$\alpha_{11} = \alpha_{11}, \quad \alpha_{12} = \mu\alpha_{11}, \quad \alpha_{21} = \delta\alpha_{11}, \quad \alpha_{22} = \alpha_{11}\,\delta\mu$$

(δ and μ in $S_{(p)}$). Similarly we have

$$\alpha_{31} = \alpha_{31}, \quad \alpha_{32} = \alpha_{31}\tau, \quad \alpha_{41} = \alpha_{31}\gamma, \quad \alpha_{42} = \alpha_{31}\gamma\tau.$$

Now

$$[l_1, l_2] = l_3 \quad \text{so} \quad \alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31} = \alpha_{11}\alpha_{31}(\tau - \mu) = \beta_1,$$

$$[l_1, \sigma_2] = \sigma_3 \quad \text{so} \quad \alpha_{11}\alpha_{42} - \alpha_{12}\alpha_{41} = \alpha_{11}\alpha_{31}(\tau - \mu)\gamma = \beta_1\gamma = \beta_2,$$

$$[\sigma_1, l_2] = \sigma_3 \quad \text{so} \quad \alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31} = \alpha_{11}\alpha_{31}(\tau - \mu)\,\delta = \beta_1\delta = \beta_2.$$

Therefore $\gamma = \delta$ and

$$[\sigma_1, \sigma_2] = \sigma_3^2 = ml_3,\ m \equiv 2, 3 \pmod 4 \quad \text{or} \quad \sigma_3 + \frac{m-1}{4}\, l_3,\ m \equiv 1 \pmod 4)$$

so $\alpha_{21}\alpha_{42} - \alpha_{22}\alpha_{41} = \alpha_{11}\alpha_{31}(\tau - \mu)\,\delta\gamma = \beta_1\,\delta^2$ equals

$$m\beta_1, \quad m \equiv 2, 3 \pmod 4)$$

or

$$\beta_2 + \frac{m-1}{4}\,\beta_1 = \left(\delta + \frac{m-1}{4}\right)\beta_1, \quad m \equiv 1 \pmod 4).$$

In either case δ in $S_{(p)}$ satisfies the minimal polynomial of $\sigma \in R$. By comparing the possibilities for p-adic completions of quadratic number fields [8, p. 248], the presence of δ in $S_{(p)}$ forces K to be isomorphic to F.

THEOREM 5. *If $G(K, I)$ and $G(F, J)$ are quadratic field Grunewald–Scharlau groups with the same finite quotients and the same multiplicities, then $G(K, I)$ and $G(K, J)$ are isomorphic.*

Proof. By Lemma 4, $K = F$. Then by the first part of the section the groups must be isomorphic.

5. Some directions for further research

In order to extend Theorem 5 to all Grunewald-Scharlau groups, one would need to know the relation between ideals with equivalent forms in general number fields. One would also need an analogue of Lemma 4. This seems to require being able to identify number fields by their p-adic completions for rational primes p.

In a more general direction: It is not clear exactly how one should define the concept of having the same multiplicities for groups of class greater than two. While representations induced from N' are still the important ones, the action of N on N' now comes into play.

As a further test, we propose the following question: If Γ and Λ satisfy $\Gamma \times \mathbf{Z} \cong \Lambda \times \mathbf{Z}$ (direct product with an infinite cyclic group) then should Γ and Λ have the same multiplicities? Examples exist of non-isomorphic groups of class 3 for which the above equation holds. These groups have the same finite quotients so if they also have the same multiplicities, they would furnish a counterexample to Auslander's suggestion in class three. (This cannot happen in class two [4].)

REFERENCES

1. L. AUSLANDER, *Lecture Notes on Nil-Theta Functions*, Regional Conference Ser. Math. No. 34, Amer. Math. Soc., Providence, R. I.
2. N. BOURBAKI, *Éléments de Mathématique*, Livre XXIV, Hermann, Paris.
3. FRITZ J. GRUNEWALD and RUDOLF SCHARLAU, *A note on finitely generated torsion-free nilpotent groups of class* 2, J. Algebra, vol. 58 (1979), pp. 162–175.
4. R. HIRSHON, *Some cancellation theorems with applications to nilpotent groups*, J. Austral. Math. Soc., vol. 23 (1977), pp. 147–165.
5. B. W. JONES, *The arithmetic theory of quadratic forms*, Math. Assoc. Amer., Washington, D.C., 1967.
6. C. C. MOORE, *Decomposition of unitary representations defined by discrete subgroups of nilpotent groups*, Ann. of Math., vol. 82 (1965), pp. 146–182.
7. P. F. PICKEL, *Finitely generated nilpotent groups with isomorphic finite quotients*, Trans. Amer. Math. Soc., vol. 160 (1971), pp. 327–341.
8. E. WEISS, *Algebraic number theory*, McGraw-Hill, N.Y., 1963.

POLYTECHNIC INSTITUTE OF NEW YORK
BROOKLYN, NEW YORK